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PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

LONDON.

(A.)

FOR THE YEAR MDCCCLXXXIX.

VOL. 180.

LONDON:
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Printers in Ordinary to Her Majesty.
MDCCCXC.

ADVERTISEMENT.

The Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions* take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind: the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions;* which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body,

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upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

List of Institutions entitled to receive the Philosophical Transactions or Proceedings of the Royal Society.

Institutions marked A are entitled to receive Philosophical Transactions, Series A, and Proceedings.
" " B " " " " Series B, and Proceedings.
" " AB " " " " Series A and B, and Proceedings.
" " *p* " " Proceedings only.

America (Central).
Mexico.
p. Sociedad Científica "Antonio Alzate."

America (North). (See United States.)

America (South).
Buenos Ayres.
AB. Museo Nacional.
Caracas.
B. University Library.
Cordova.
AB. Academia Nacional de Ciencias.
Rio de Janeiro.
p. Observatorio.

Australia.
Adelaide.
p. Royal Society of South Australia.
Brisbane.
p. Royal Society of Queensland.
Melbourne.
p. Observatory.
p. Royal Society of Victoria.
AB. University Library.
Sydney.
p. Linnean Society of New South Wales.
AB. Royal Society of New South Wales.
AB. University Library.

Austria.
Agram.
p. Societas Historico-Naturalis Croatica.
Brünn.
AB. Naturforschender Verein.
Gratz.
AB. Naturwissenschaftlicher Verein für Steiermark.
Hermannstadt.
p. Siebenbürgischer Verein für die Naturwissenschaften.

Austria (continued).
Innsbruck.
AB. Das Ferdinandeum.
p. Naturwissenschaftlich - Medicinischer Verein.
Klausenburg.
AB. Az Erdélyi Muzeum. Das siebenbürgische Museum.
Prague.
AB. Königliche Böhmische Gesellschaft der Wissenschaften.
Schemnitz.
p. K. Ungarische Berg- und Först-Akademie.
Trieste.
B. Museo di Storia Naturale.
p. Società Adriatica di Scienze Naturali.
Vienna.
p. Anthropologische Gesellschaft.
AB. Kaiserliche Akademie der Wissenschaften.
p. K.K. Geographische Gesellschaft.
AB. K.K. Geologische Reichsanstalt.
B. K.K. Zoologisch-Botanische Gesellschaft.
B. Naturhistorisches Hof-Museum.
p. Œsterreichische Gesellschaft für Meteorologie.

Belgium.
Brussels.
B. Académie Royale de Médecine.
AB. Académie Royale des Sciences.
B. Musée Royal d'Histoire Naturelle de Belgique.
p. Observatoire Royale.
p. Société Malacologique de Belgique.
Ghent.
AB. University.
Liége.
AB. Société des Sciences.
p. Société Géologique de Belgique.

Belgium (continued).

Louvain.

AB. L'Université.

Canada.

Hamilton.

p. Scientific Association.

Montreal.

AB. McGill College.

p. Natural History Society.

Ottawa.

AB. Geological Survey of Canada.

AB. Royal Society of Canada.

Toronto.

p. Canadian Institute.

AB. University.

Cape of Good Hope.

A. Observatory.

AB. South African Library.

Ceylon.

Colombo.

B. Museum.

China.

Shanghai.

p. China Branch of the Royal Asiatic Society.

Denmark.

Copenhagen.

AB. Kongelige Danske Videnskabernes Selskab.

England and Wales.

Birmingham.

AB. Free Central Library.

AB. Mason College.

p. Philosophical Society.

Bristol.

p. Merchant Venturers' School.

Cambridge.

AB. Philosophical Society.

p. Union Society.

Cooper's Hill.

AB. Royal Indian Engineering College.

Dudley.

p. Dudley and Midland Geological and Scientific Society.

Essex.

p. Essex Field Club.

Greenwich.

A. Royal Observatory.

Kew.

B. Royal Gardens.

Leeds.

p. Philosophical Society.

AB. Yorkshire College.

Liverpool.

AB. Free Public Library.

p. Historic Society of Lancashire and Cheshire.

England and Wales (continued).

Liverpool (continued).

p. Literary and Philosophical Society.

A. Observatory.

AB. University College.

London.

AB. Admiralty.

p. Anthropological Institute.

B. British Museum (Nat. Hist.).

A. Chemical Society.

p. "Electrician," Editor of the.

B. Entomological Society.

AB. Geological Society.

AB. Geological Survey of Great Britain.

p. Geologists' Association.

AB. Guildhall Library.

A. Institution of Civil Engineers.

A. Institution of Mechanical Engineers.

A. Institution of Naval Architects.

p. Iron and Steel Institute.

B. Linnean Society.

AB. London Institution.

p. London Library.

A. Mathematical Society.

p. Meteorological Office.

p. Odontological Society.

p. Pharmaceutical Society.

p. Physical Society.

p. Quekett Microscopical Club.

p. Royal Asiatic Society.

A. Royal Astronomical Society.

B. Royal College of Physicians.

B. Royal College of Surgeons.

p. Royal Engineers (for Libraries abroad, six copies).

AB. Royal Engineers. Head Quarters Library.

p. Royal Geographical Society.

p. Royal Horticultural Society.

p. Royal Institute of British Architects.

AB. Royal Institution of Great Britain.

B. Royal Medical and Chirurgical Society.

p. Royal Meteorological Society.

p. Royal Microscopical Society.

p. Royal Statistical Society.

AB. Royal United Service Institution.

AB. Society of Arts.

p. Society of Biblical Archæology.

p. Standard Weights and Measures Department.

AB. The Queen's Library.

AB. The War Office.

AB. University College.

p. Victoria Institute.

B. Zoological Society.

England and Wales (continued).

- Manchester.
  - AB. Free Library.
  - AB. Literary and Philosophical Society.
  - *p.* Geological Society.
  - AB. Owens College.
- Netley.
  - *p.* Royal Victoria Hospital.
- Newcastle.
  - AB. Free Library.
  - *p.* North of England Institute of Mining and Mechanical Engineers.
  - *p.* Society of Chemical Industry (Newcastle Section).
- Norwich.
  - *p.* Norfolk and Norwich Literary Institution.
- Oxford.
  - *p.* Ashmolean Society.
  - AB. Radcliffe Library.
  - A. Radcliffe Observatory.
- Penzance.
  - *p.* Geological Society of Cornwall.
- Plymouth.
  - B. Marine Biological Association.
  - *p.* Plymouth Institution.
- Richmond.
  - A. "Kew" Observatory.
- Salford.
  - *p.* Royal Museum and Library.
- Stonyhurst.
  - *p.* The College.
- Swansea.
  - AB. Royal Institution.
- Woolwich.
  - AB. Royal Artillery Library.

Finland.

- Helsingfors.
  - *p.* Societas pro Fauna et Flora Fennica.
  - AB. Société des Sciences.

France.

- Bordeaux.
  - *p.* Académie des Sciences.
  - *p.* Faculté des Sciences.
  - *p.* Société de Médecine et de Chirurgie.
  - *p.* Société des Sciences Physiques et Naturelles.
- Cherbourg.
  - *p.* Société des Sciences Naturelles.
- Dijon.
  - *p.* Académie des Sciences.
- Lille.
  - *p.* Faculté des Sciences.

France (continued).

- Lyons.
  - AB. Académie des Sciences, Belles-Lettres et Arts.
- Marseilles.
  - *p.* Faculté des Sciences.
- Montpellier.
  - AB. Académie des Sciences et Lettres.
  - B. Faculté de Médecine.
- Paris.
  - AB. Académie des Sciences de l'Institut.
  - *p.* Association Française pour l'Avancement des Sciences.
  - *p.* Conservatoire des Arts et Métiers.
  - *p.* Cosmos (M. L'ABBÉ VALETTE).
  - AB. Dépôt de la Marine.
  - AB. École des Mines.
  - AB. École Normale Supérieure.
  - AB. École Polytechnique.
  - AB. Faculté des Sciences de la Sorbonne.
  - AB. Jardin des Plantes.
  - A. L'Observatoire.
  - *p.* Revue Internationale de l'Électricité.
  - *p.* Revue Scientifique (Mons. H. DE VARIGNY).
  - *p.* Société de Biologie.
  - AB. Société d'Encouragement pour l'Industrie Nationale.
  - AB. Société de Géographie.
  - *p.* Société de Physique.
  - B. Société Entomologique.
  - AB. Société Géologique.
  - *p.* Société Mathématique.
  - *p.* Société Météorologique de France.
- Toulouse.
  - AB. Académie des Sciences.
  - A. Faculté des Sciences.

Germany.

- Berlin.
  - A. Deutsche Chemische Gesellschaft.
  - A. Die Sternwarte.
  - *p.* Gesellschaft für Erdkunde.
  - AB. Königliche Preussische Akademie der Wissenschaften.
  - A. Physikalische Gesellschaft.
- Bonn.
  - AB. Universität.
- Bremen.
  - *p.* Naturwissenschaftlicher Verein.
- Breslau.
  - *p.* Schlesische Gesellschaft für Vaterländische Kultur.
- Brunswick.
  - *p.* Verein für Naturwissenschaft.
- Carlsruhe. See Karlsruhe.
- Danzig.
  - AB. Naturforschende Gesellschaft.

Germany (continued).

Dresden.
- *p*. Verein für Erdkunde.

Emden.
- *p*. Naturforschende Gesellschaft.

Erlangen.
- AB. Physikalisch-Medicinische Societät.

Frankfurt-am-Main.
- AB. Senckenbergische Naturforschende Gesellschaft.
- *p*. Zoologische Gesellschaft.

Frankfurt-am-Oder.
- *p*. Naturwissenschaftlicher Verein.

Freiburg-im-Breisgau.
- AB. Universität.

Giessen.
- AB. Grossherzogliche Universität.

Görlitz.
- *p*. Naturforschende Gesellschaft.

Göttingen.
- AB. Königliche Gesellschaft der Wissenschaften.

Halle.
- AB. Kaiserliche Leopoldino - Carolinische Deutsche Akademie der Naturforscher.
- *p*. Naturwissenschaftlicher Verein für Sachsen und Thüringen.

Hamburg.
- AB. Naturwissenschaftlicher Verein.

Heidelberg.
- *p*. Naturhistorisch-Medizinische Gesellschaft.
- AB. Universität.

Jena.
- AB. Medicinisch-Naturwissenschaftliche Gesellschaft.

Karlsruhe.
- A. Grossherzogliche Sternwarte.

Kiel.
- A. Sternwarte.
- AB. Universität.

Königsberg.
- AB. Königliche Physikalisch - Ökonomische Gesellschaft.

Leipsic.
- *p*. Annalen der Physik und Chemie.
- A. Astronomische Gesellschaft.
- AB. Königliche Sächsische Gesellschaft der Wissenschaften.

Magdeburg.
- *p*. Naturwissenschaftlicher Verein.

Marburg.
- AB. Universität.

Germany (continued).

Munich.
- AB. Königliche Bayerische Akademie der Wissenschaften.
- *p*. Zeitschrift für Biologie.

Münster.
- AB. Königliche Theologische und Philosophische Akademie.

Rostock.
- AB. Universität.

Strasburg.
- AB. Universität.

Tübingen.
- AB. Universität.

Würzburg.
- AB. Physikalisch-Medicinische Gesellschaft.

Holland. (See NETHERLANDS.)

Hungary.

Pesth.
- *p*. Königl. Ungarische Geologische Anstalt.
- AB. Á Magyar Tudós Társaság. Die Ungarische Akademie der Wissenschaften.

India.

Bombay.
- AB. Elphinstone College.

Calcutta.
- AB. Asiatic Society of Bengal.
- AB. Geological Museum.
- *p*. Great Trigonometrical Survey of India.
- AB. Indian Museum.
- *p*. The Meteorological Reporter to the Government of India.

Madras.
- B. Central Museum.
- A. Observatory.

Roorkee.
- *p*. Roorkee College.

Ireland.

Armagh.
- A. Observatory.

Belfast.
- AB. Queen's College.

Cork.
- *p*. Philosophical Society.
- AB. Queen's College.

Dublin.
- A. Observatory.
- AB. National Library of Ireland.
- B. Royal College of Surgeons in Ireland.
- AB. Royal Dublin Society.
- AB. Royal Irish Academy.

Galway.
- AB. Queen's College.

Italy.

Bologna.
- AB. Accademia delle Scienze dell' Istituto.

Catania.
- AB. Accademia Gioenia di Scienze Naturali.

Florence.
- *p.* Biblioteca Nazionale Centrale.
- AB. Reale Museo di Fisica.

Milan.
- AB. Reale Istituto Lombardo di Scienze, Lettere ed Arti.
- AB. Società Italiana di Scienze Naturali.

Naples.
- AB. Società Reale, Accademia delle Scienze.
- B. Stazione Zoologica (Dr. DOHRN).

Padua.
- *p.* University.

Palermo.
- A. Circolo Matematico.
- AB. Consiglio di Perfezionamento (Società di Scienze Naturali ed Economiche).
- A. Reale Osservatorio.

Pisa.
- *p.* Società Toscana di Scienze Naturali.

Rome.
- *p.* Accademia Pontificia de' Nuovi Lincei.
- A. Osservatorio del Collegio Romano.
- AB. Reale Accademia dei Lincei.
- *p.* R. Comitato Geologico d' Italia.
- AB. Società Italiana delle Scienze.

Turin.
- *p.* Laboratorio di Fisiologia.
- AB. Reale Accademia delle Scienze.

Venice.
- *p.* Ateneo Veneto.
- AB. Reale Istituto Veneto di Scienze, Lettere ed Arti.

Japan.

Tokiô.
- AB. Imperial University.

Yokohama.
- *p.* Asiatic Society of Japan.

Java.

Batavia.
- AB. Bataviaasch Genootschap van Kunsten en Wetenschappen.

Buitenzorg.
- *p.* Jardin Botanique.

Malta.
- *p.* Public Library.

Mauritius.
- *p.* Royal Society of Arts and Sciences.

Netherlands.

Amsterdam.
- AB. Koninklijke Akademie van Wetenschappen.
- *p.* K. Zoologisch Genootschap 'Natura Artis Magistra.'

Delft.
- *p.* École Polytechnique.

Haarlem.
- AB. Hollandsche Maatschappij der Wetenschappen.
- *p.* Musée Teyler.

Leyden.
- AB. University.

Luxembourg.
- *p.* Société des Sciences Naturelles.

Rotterdam.
- AB. Bataafsch Genootschap der Proefondervindelijke Wijsbegeerte.

Utrecht.
- AB. Provinciaal Genootschap van Kunsten en Wetenschappen.

New Zealand.

Wellington.
- AB. New Zealand Institute.

Norway.

Bergen.
- AB. Bergenske Museum.

Christiania.
- AB. Kongelige Norske Frederiks Universitet.

Tromsoe.
- *p.* Museum.

Trondhjem.
- AB. Kongelige Norske Videnskabers Selskab.

Nova Scotia.

Windsor.
- *p.* King's College Library.

Portugal.

Coimbra.
- AB. Universidade.

Lisbon.
- AB. Academia Real das Sciencias.
- *p.* Secção dos Trabalhos Geologicos de Portugal.

Russia.

Dorpat.
- AB. Université.

Kazan.
- AB. Imperatorsky Kazansky Universitet.

Kharkoff.
- *p.* Section Médicale de la Société des Sciences Expérimentales, Université de Kharkow.

Kieff.
- *p.* Société des Naturalistes.

Russia (continued).
Moscow.
AB. Le Musée Publique.
B. Société Impériale des Naturalistes.
Odessa.
p. Société des Naturalistes de la Nouvelle-Russie.
Pulkowa.
A. Nikolai Haupt-Sternwarte.
St. Petersburg.
AB. Académie Impériale des Sciences.
AB. Comité Géologique.
p. Compass Observatory.
A. L'Observatoire Physique Central.

Scotland.
Aberdeen.
AB. University.
Edinburgh.
p. Geological Society.
p. Royal College of Physicians (Research Laboratory).
p. Royal Medical Society.
A. Royal Observatory.
p. Royal Physical Society.
p. Royal Scottish Society of Arts.
AB. Royal Society.
Glasgow.
AB. Mitchell Free Library.
p. Philosophical Society.

Servia.
Belgrade.
p. Académie Royale de Serbie.

Spain.
Cadiz.
A. Observatorio de San Fernando.
Madrid.
p. Comisión del Mapa Geológico de Espãna.
AB. Real Academia de Ciencias.

Sweden.
Gottenburg.
AB. Kongl. Vetenskaps och Vitterhets Samhälle.
Lund.
AB. Universitet.
Stockholm.
A. Acta Mathematica.
AB. Kongliga Svenska Vetenskaps-Akademie.
AB. Sveriges Geologiska Undersökning.
Upsala.
AB. Universitet.

Switzerland.
Basel.
p. Naturforschende Gesellschaft.

Switzerland (continued).
Bern.
AB. Allg. Schweizerische Gesellschaft.
p. Naturforschende Gesellschaft.
Geneva.
AB. Société de Physique et d'Histoire Naturelle.
AB. Institut National Genevois.
Lausanne.
p. Société Vaudoise des Sciences Naturelles.
Neuchâtel.
p. Astronomische Mittheilungen (Professor R. WOLF).
p. Société des Sciences Naturelles.
Zürich.
AB. Das Schweizerische Polytechnikum.
p. Naturforschende Gesellschaft.

Tasmania.
Hobart.
p. Royal Society of Tasmania.

United States.
Albany.
AB. New York State Library.
Annapolis.
AB. Naval Academy.
Baltimore.
AB. Johns Hopkins University.
Berkeley.
p. University of California.
Boston.
AB. American Academy of Sciences.
B. Boston Society of Natural History.
A. Technological Institute.
Brooklyn.
AB. Brooklyn Library.
Cambridge.
AB. Harvard University.
Chapel Hill (N.C.).
p. Elisha Mitchell Scientific Society.
Charleston.
p. Elliott Society of Science and Art of South Carolina.
Chicago.
AB. Academy of Sciences.
Davenport (Iowa).
p. Academy of Natural Sciences.
Madison.
p. Wisconsin Academy of Sciences.
Mount Hamilton (California).
A. Lick Observatory.
New Haven (Conn.).
AB. American Journal of Science.
AB. Connecticut Academy of Arts and Sciences.
New York.
p. American Geographical Society.

United States (continued).

New York (continued).

- *p*. American Museum of Natural History.
- *p*. New York Academy of Sciences.
- *p*. New York Medical Journal.
- *p*. School of Mines, Columbia College.

Philadelphia.

- AB. Academy of Natural Sciences.
- AB. American Philosophical Society.
- *p*. Franklin Institute.
- *p*. Wagner Free Institute of Science.

St. Louis.

- *p*. Academy of Science.

Salem (Mass.).

- *p*. Essex Institute.
- AB. Peabody Academy of Science.

United States (continued).

San Francisco.

- AB. California Academy of Sciences.

Washington.

- *p*. Department of Agriculture.
- A. Office of the Chief Signal Officer.
- AB. Patent Office.
- AB. Smithsonian Institution.
- AB. United States Coast Survey.
- *p*. United States Commission of Fish and Fisheries.
- AB. United States Geological Survey.
- A. United States Naval Observatory.

West Point (N Y.)

- AB. United States Military Academy.

ADJUDICATION of the MEDALS of the ROYAL SOCIETY for the year 1889, by the PRESIDENT and COUNCIL.

The COPLEY MEDAL to the Rev. GEORGE SALMON, D.D., F.R.S., for his various Papers on subjects of Pure Mathematics, and for the valuable Mathematical Treatises of which he is the Author.

A ROYAL MEDAL to WALTER HOLBROOK GASKELL, F.R.S., for his Researches in Cardiac Physiology, and his important Discoveries in the Anatomy and Physiology of the Sympathetic Nervous System.

A ROYAL MEDAL to THOMAS EDWARD THORPE, F.R.S., for his Researches on Fluorine Compounds, and his Determination of the Atomic Weights of Titanium and Gold.

The DAVY MEDAL to WILLIAM HENRY PERKIN, F.R.S., for his Researches on Magnetic Rotation in relation to Chemical Constitution.

The Bakerian Lecture, "A Magnetic Survey of the British Isles for the Epoch January 1, 1886," was delivered by Professor A. W. RÜCKER, F.R.S., and Professor T. E. THORPE, F.R.S.

The Croonian Lecture, "Les Inoculations Préventives," was delivered by Dr. E. ROUX.

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PHILOSOPHICAL TRANSACTIONS.

I. *On the Mechanical Conditions of a Swarm of Meteorites, and on Theories of Cosmogony.*

By G. H. Darwin, *LL.D., F.R.S., Fellow of Trinity College and Plumian Professor in the University of Cambridge.*

Received July 12,—Read November 15, 1888.

Mr. Lockyer writes in his interesting paper on Meteorites* as follows :—

" The brighter lines in spiral nebulæ, and in those in which a rotation has been set up, are in all probability due to streams of meteorites with irregular motions out of the main streams, in which the collisions would be almost *nil.* It has already been suggested by Professor G. Darwin (' Nature,' vol. 31, 1884–5, p. 25)—using the gaseous hypothesis—that in such nebulæ ' the great mass of the gas is non-luminous, the luminosity being an evidence of condensation along lines of low velocity, according to a well known hydrodynamical law. From this point of view, the visible nebula may be regarded as a luminous diagram of its own stream-lines.' "

The whole of Mr. Lockyer's paper, and especially this passage in it, leads me to make a suggestion for the reconciliation of two apparently divergent theories of the origin of planetary systems.

The nebular hypothesis depends essentially on the idea that the primitive nebula is a rotating mass of fluid, which at successive epochs becomes unstable from excess of rotation, and sheds a ring from the equatorial region.

The researches of Roche† (apparently but little known in this country) have imparted to this theory a precision which was wanting in Laplace's original exposition, and have rendered the explanation of the origin of the planets more perfect.

But notwithstanding the high probability that some theory of the kind is true,‡ the acceptance of the nebular hypothesis presents great difficulties.

Sir William Thomson long ago expressed to me his opinion that the most probable origin of the planets was through a gradual accretion of meteoric matter, and the researches of Mr. LOCKYER afford actual evidence in favour of the abundance of meteorites in space.

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* ' Nature,' Nov. 17, 1887. The paper itself is in ' Roy. Soc. Proc.,' Nov. 15, 1887 (No. 259, p. 117).

† ' Montpellier, Acad. Sci. Mém.'

‡ [The very remarkable photograph of the nebula in Andromeda, exhibited to the Royal Astronomical Society by Mr. Isaac Roberts on December 6, 1888, affords something like a proof of the substantial truth of the nebular hypothesis.—G. H. D. December 19, 1888.]

But the very essence of the nebular hypothesis is the conception of fluid pressure, since without it the idea of a figure of equilibrium becomes inapplicable. Now, at first sight, the meteoric condition of matter seems absolutely inconsistent with a fluid pressure exercised by one part of the system on another. We thus seem driven either to the absolute rejection of the nebular hypothesis, or to deny that the meteoric condition was the immediate antecedent of the Sun and Planets. M. FAYE has taken the former course, and accepts as a necessary consequence the formulation of a succession of events quite different from that of the nebular hypothesis.* I cannot myself find that his theory is an improvement on that of LAPLACE, except in regard to the adoption of meteorites, for he has lost the conception of the figure of equilibrium of a rotating mass of fluid.

The object of this paper is to point out that by a certain interpretation of the meteoric theory we may obtain a reconciliation of these two orders of ideas, and may hold that the origin of stellar and planetary systems is meteoric, whilst retaining the conception of fluid pressure.

According to the kinetic theory of gases, fluid pressure is the average result of the impacts of molecules. If we imagine the molecules magnified until of the size of meteorites, their impacts will still, on a coarser scale, give a quasi-fluid pressure. I suggest then that the fluid pressure essential to the nebular hypothesis is, in fact, the resultant of countless impacts of meteorites.

The problems of hydrodynamics could hardly be attacked with success, if we were forced to start from the beginning and to consider the cannonade of molecules. But when once satisfied that the kinetic theory will give us a gas, which, in a space containing some millions of molecules, obeys all the laws of an ideal non-molecular gas filling all space, we may put the molecules out of sight and treat the gas as a plenum.

In the same way, the difficulty of tracing the impacts of meteorites in detail is insuperable; but, if we can find that such impacts give rise to a quasi-fluid pressure on a large scale, we may be able to trace out many results by treating an ideal plenum. LAPLACE'S hypothesis implies such a plenum, and it is here maintained that this plenum is merely the idealisation of the impacts of meteorites.

As a bare suggestion this view is worth but little, for its acceptance or rejection must turn entirely on numerical values, which can only be obtained by the consideration of some actual system. It is obvious that the solar system is the only one about which we have sufficient knowledge to afford a basis for discussion. This paper is accordingly devoted to a consideration of the mechanics of a swarm of meteorites, with special numerical application to the solar system.

The investigation has entailed a considerable amount of mathematical analysis;

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* 'Sur l'Origine du Monde,' Paris, GAUTHIER-VILLARS, 1884; 'Annuaire pour l'an 1885, Bureau des Longitudes,' p. 757.

there is, however, no analysis in §§ 1 and 2. The reader who only wishes to know the arguments and results, without a consideration of the mathematical details, is therefore recommended, after reading §§ 1 and 2, to pass on to the Summary.

§ 1. *On the Effective Elasticity of Meteorites in Collision.*

When two meteoric stones meet with planetary velocity, the stress between them during impact must generally be such that the limits of true elasticity are exceeded; and it may be urged that a kinetic theory is inapplicable unless the colliding particles are highly elastic. It may, however, I think, be shown that the very greatness of the velocities will impart what virtually amounts to an elasticity of a high order of perfection.

It appears, *a priori*, probable that, when two meteorites clash, a portion of the solid matter of each is volatilised, and Mr. LOCKYER considers the spectroscopic evidence conclusive that it is so. There is, no doubt, enough energy liberated on impact to volatilise the whole of both bodies, but only a small portion of each stone will undergo this change.

A rough numerical example will show the kind of quantities with which we are here dealing.

It will appear hereafter that the mean velocity of a meteorite may be at the least about 5 kilometres a second; and, accordingly, the mean relative velocity of a pair would then be about 7 kilometres a second.* Hence, if two stones, weighing a kilogramme, move each with a velocity of $3\frac{1}{2}$ kilometres per second directly towards one another, the energy liberated at the moment of impact is $2 \times \frac{1}{2} \times 10^3 (3\frac{1}{2} \times 10^5)^2$ or 12×10^{13} ergs.

Now JOULE's equivalent is $4{\cdot}2 \times 10^7$ ergs; hence, the energy liberated is about 3 million calories.

It is quite uncertain how much of each stone would be volatilised; but, if it were 3 grammes, there would be a million calories of energy applied to each gramme.

The melting temperature of iron is about 1500 degrees Centigrade, and the mean specific heat of iron may be about $\frac{1}{7}$.† Hence, about 300 calories are required to raise a gramme of iron from absolute zero to melting point. I do not know the latent heat of the melting of iron, but for platinum it is 27, and the latent heat of volatilisation of mercury is 62. Hence, about 400 or 500 calories suffice to raise a gramme of iron from absolute zero to volatilisation. It is clear, then, that there is energy enough, not only to volatilise the iron, but also to render the gas incandescent; and the same would be true if the mass operated on by the energy were 30 grammes instead of 3.

It must necessarily be obscure as to how a small mass of solid matter *can* take up a very large amount of energy in a small fraction of a second, but spectroscopic evidence seems to show that it does so; and, if so, we have what is virtually a violent explosive introduced between the two stones.

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* If v be the velocity of mean square, $v \sqrt{2}$ is the square root of the mean square of relative velocity.

† 'Physikalisch-Chemische Tabellen.' LANDOLT and BÖRNSTEIN.

In a direct collision each stone is probably shattered into fragments, like the splashes of lead when a bullet hits an iron target. But direct collision must be a comparatively rare event. In glancing collisions the velocity of neither body is wholly arrested, the concentration of energy is not so enormous (although probably still sufficient to effect volatilisation), and, since the stones rub past one another, more time is allowed for the matter round the point of contact to take up the energy; thus, the whole process of collision is much more intelligible. The nearest terrestrial analogy is when a cannon-ball bounds off the sea. In glancing collisions fracture will probably not be very frequent.

From these arguments, it is probable that, when two meteorites meet, they attain an effective elasticity of a high order of perfection; but there is, of course, some loss of energy at each collision. [It must, however, be admitted that on collision the deflection of path is rarely through a very large angle. But a succession of glancing collisions would be capable of reversing the path; and, thus, the kinetic theory of meteorites may be taken as not differing materially from that of gases.*]

Perhaps the most serious difficulty in the whole theory arises from the fractures which must often occur. If they happen with great frequency, it would seem as if the whole swarm of meteorites would degrade into dust. We know, however, that meteorites of considerable size fall upon the Earth; and, unless Mr. LOCKYER has misinterpreted the spectroscopic evidence, the nebulæ do now consist of meteorites. Hence, it would seem as if fracture was not of very frequent occurrence. It is easy to see that, if two bodies meet with a given velocity, the chance of fracture is much greater if they are large, and it is possible that the process of breaking up will go on only until a certain size, dependent on the velocity of agitation, is reached, and will then become comparatively unimportant.

When the volatilised gases cool, they will condense into a metallic rain, and this may fuse with old meteorites whose surfaces are molten. A meteorite in that condition will certainly also pick up dust. Thus, there are processes in action tending to counteract subdivision by fracture and volatilisation. The mean size of meteorites probably depends on the balance between these opposite tendencies. If this is so, there will be some fractures, and some fusions, but the mean mass will change very slowly with the mean kinetic energy of agitation. This view is, at any rate, adopted in the paper as a working hypothesis. It was not, however, possible to take account of fracture and fusion in the mathematical investigation, but the meteorites are treated as being of invariable mass.

§ 2. *On the Velocity of Agitation of Meteorites, and on its Secular Change.*

The velocity with which the meteorites move is derived from their fall from a great distance towards a centre of aggregation. In other words, the potential energy of their mutual attraction when widely dispersed becomes converted, at least partially, into kinetic energy. When the condensation of a swarm is just beginning, the mass of the aggregation towards which the meteorites fall is small; and, thus, the new bodies arrive at the aggregation with small velocity. Hence, initially, the kinetic energy is small, and the volume of the sphere within which hydrostatic ideas are (if anywhere) applicable is also small.

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* Added Nov. 16, 1888.

As more and more meteorites fall in that volume is enlarged, and the velocity with which they reach the aggregation is increased. Finally, the supply of meteorites in that part of space begins to fail, and the imperfect elasticity of the colliding bodies brings about a gradual contraction of the swarm.

I do not now attempt to trace the whole history of a swarm, but the object of the paper is to examine its mechanical condition at an epoch when the supply of meteorites from outside has stopped, and when the velocities of agitation and distribution of meteorites in space have arranged themselves into a sub-permanent condition, only affected by secular changes. This examination will enable us to understand, at least roughly, the secular change in the velocity and in the distribution of the meteorites as the swarm contracts, and will throw light on other questions.

§ 3. *Formulæ for Mean Square of Velocity, Mean Free Path, and Interval between Collisions.*

We have to investigate whether, when the solar system consisted of a swarm of meteorites, the velocities and encounters could have been such that the mechanics of the system can be treated as subject to the laws of hydrodynamics. The formulæ which form the basis of this discussion will now be considered.

For the sake of simplicity, the meteorites will, in the first instance, be treated as spheres of uniform size.

The sum of the masses of the meteorites is equal to that of the Sun, for the planets only contribute a negligible mass. If M_0 be the Sun's mass, and m that of a meteorite, their number is M_0/m.

If, at each encounter between two meteorites, there were no loss of energy, the sum of the kinetic energies of all the meteorites would be equal to the potential energy lost in the concentration of the swarm from a condition of infinite dispersion, until it possessed its actual arrangement. In such a computation the rotational energy of the system is negligible.

Suppose the Sun's mass to be concentrated from infinite dispersion until it is arranged in the form of a homogeneous sphere of radius a and density ρ. Then let the sphere be cut up into as many equal spaces as there are meteorites, and let the matter in each space be concentrated into a meteorite. When the number of meteorites is large, the potential energy lost in the first process is very great compared with that lost in the subsequent partial condensation into meteorites.* Thus, the energy lost in the partial condensation is negligible.

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* It depends, in fact, on the square of the ratio of the diameter $2a$ to the linear dimension of one of the equal spaces.

If μ be the attractional constant, the lost energy of condensation is well known to be $\frac{3}{5}\mu M_0^2/a$. But on the hypothesis that there is no loss of energy at each encounter, this must be equal to the sum of the kinetic energies of all the meteorites. If, therefore, v^2 be the mean square of velocity of a meteorite, we must have $\frac{1}{2} M_0 v^2 = \frac{3}{5}\mu M_0^2/a$, so that $v^2 = \frac{6}{5} \mu M_0/a$.

But homogeneity of density and uniformity of kinetic energy of agitation are impossible; for the meteor-swarm must be much condensed towards its centre, so that we have largely underestimated the lost potential energy of the system. Also, the velocity of agitation must decrease towards the outside, or else the swarm would extend to infinity. Besides this, the partial conversion of molar into molecular energy, which must take place on each encounter, has been neglected.

We shall see below reason for believing that throughout a large central volume the mean square of velocity of agitation is nearly uniform, and that outside of this region it falls off.

Suppose, then, that M is the mass and a the radius of that portion of the swarm in which the square of velocity of agitation is uniform; let v_0^2 be that square of velocity, and let it be defined by reference to the potential of M at distance a, so that

$$v_0^2 = \beta^2 \frac{\mu M}{a}, \quad \text{. . . . .} \quad (1)$$

where β is a coefficient for which a numerical value will be found below.

The square of velocity of agitation outside of the radius a is to be denoted by v^2, and subsequent investigation will be necessary to evaluate v^2 in terms of v_0^2.

If we denote by a_0 the Earth's distance from the Sun, and by u_0 the Earth's velocity in its orbit, we have

$$u_0^2 = \mu \frac{M_0}{a_0}. \quad \text{. . . . .} \quad (2)$$

Whence,

$$v_0 = \beta u_0 \left(\frac{M a_0}{M_0 a}\right)^{\frac{1}{2}}. \quad \text{. . . . .} \quad (3)$$

If in any distribution of meteorites w is the sum of the masses of all the meteorites in unit volume, or the density of the swarm at any point, and if λ be that distance which is called in the kinetic theory of gases "the mean distance between neighbouring molecules," we have

$$\lambda^3 = \frac{m}{w}. \quad \text{. . . . .} \quad (4)$$

Now, the mean density of that part of the swarm in which the kinetic energy of agitation is constant being ρ, we have

$$\rho = \frac{3M}{4\pi a^3}, \quad \text{. . . . .} \quad (5)$$

and

$$\lambda^3 = 4\,\pi\,a^3 . \frac{m}{M} . \frac{\frac{1}{3}\rho}{w}. \quad \text{. . . . .} \quad (6)$$

Suppose that s is "the radius of the sphere of action" of a meteorite, so that when two of them approach so that the distance between their centres is s there is a collision.

Let L and T be the mean free path and mean interval between collisions. Then, since the mean velocity is $v\sqrt{(8/3\pi)}$, we have, according to the kinetic theory of gases,*

$$L = \frac{\lambda^3}{\pi s^2\sqrt{2}}, \qquad T = \frac{L}{v}\sqrt{\tfrac{3}{8}\pi}. \quad \ldots \ldots \quad (7)$$

Then, on substitution from (4), (5), and (6), we have

$$L = \left[\frac{a_0{}^3\sqrt{8}}{M_0}\right]\left(\frac{a}{a_0}\right)^3 \frac{M_0}{M} \cdot \frac{\frac{1}{3}\rho}{w} \cdot \frac{m}{s^2}, \qquad T = \left[\frac{a_0{}^3\sqrt{3\pi}}{M_0 u_0}\right]\left(\frac{a}{a_0}\right)^{\frac{7}{2}} \cdot \frac{1}{\beta}\left(\frac{M_0}{M}\right)^{\frac{3}{2}} \frac{v_0}{v} \cdot \frac{\frac{1}{3}\rho}{w} \cdot \frac{m}{s^2}. \quad . \quad (8)$$

Now let

$$\left.\begin{aligned} u_0 &= \left(\frac{\mu M_0}{a_0}\right)^{\frac{1}{2}}, & u &= u_0\left(\frac{a_0}{a}\right)^{\frac{1}{2}} \\ l_0 &= \frac{a_0{}^3\sqrt{8}}{M_0}, & l &= l_0\left(\frac{a}{a_0}\right)^3 \\ \tau_0 &= \frac{a_0{}^3\sqrt{3\pi}}{M_0 u_0}, & \tau &= \tau_0\left(\frac{a}{a_0}\right)^{\frac{7}{2}} \end{aligned}\right\}, \quad \ldots \quad \ldots \quad (9)$$

and we have

$$\left.\begin{aligned} v &= \beta u\left(\frac{M}{M_0}\right)^{\frac{1}{2}} \\ L &= l \cdot \left(\frac{M_0}{M}\right) \cdot \frac{\frac{1}{3}\rho}{w} \cdot \frac{m}{s^2} \\ T &= \tau \cdot \frac{1}{\beta}\left(\frac{M_0}{M}\right)^{\frac{3}{2}} \cdot \frac{v_0}{v} \cdot \frac{\frac{1}{3}\rho}{w} \cdot \frac{m}{s^2} \end{aligned}\right\} \quad \ldots \ldots \quad (10)$$

We now proceed to calculate u_0, l_0, τ_0, and also $2a_0/l_0$, using the centimetre-gramme-second system of units.

The Sun's mass may be taken as 315,511 times that of the Earth, and the Earth as $6{\cdot}14 \times 10^{27}$ grammes†; hence

$$M_0 = 10^{33{\cdot}28718} = 1{\cdot}9372 \times 10^{33} \text{ grammes.}$$

The attractional constant and the Earth's mean distance from the Sun are

$$\mu = \frac{648}{10^{10}}, \qquad a_0 = 1{\cdot}487 \times 10^{13} \text{ cm.}$$

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* MEYER, 'Kinetische Theorie der Gase.'

† Here and elsewhere I generally use EVERETT's 'Units and Physical Constants.'

With these values

$$\left.\begin{aligned} u_0 &= 10^{6\cdot46323} = 2{,}905{,}600 \text{ cm. per sec.} \\ l_0 &= 10^{6\cdot68129} = 4{,}800{,}600 \text{ cm.} \\ \tau_0 &= 10^{0\cdot25366} = 1\cdot79334 \text{ sec.} \\ \frac{2a_0}{l_0} &= 10^{6\cdot79204} = 6{,}195{,}000 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

The dimensions of l_0 and τ_0 are not those of length and time; but, if meteorites of 1 gramme mass, with sphere of action 1 centimetre, and "velocity of mean square" of agitation equal to the Earth's velocity in its orbit, have density of distribution equal to one-third of the mean density of the sphere M, then l_0, τ_0 will be the mean free path and time, as stated in centimetres and seconds. We may thus regard l_0, τ_0 as a length and time, provided care be taken in the subsequent use of the symbols to adhere to the c.g.s. system of units.

§ 4. *On the Equilibrium of a Gas at Uniform Temperature in Concentric Spherical Layers under its own Gravitation.*

It is assumed provisionally that the conditions are satisfied which permit us to regard the swarm of meteorites as a quasi-gas, subject to the laws of hydrostatics.

The solution of this problem, then, becomes a necessary preliminary to the discussion of the kinetic theory of meteorites. The equilibrium of a gas under its own gravitation has been ably discussed by Professor RITTER in one of his series of papers on gaseous planets.* The intrinsic interest of the problem renders an independent solution valuable. Suppose, then, that a mass M_1 of gas is enclosed in a spherical envelope of radius a_1, and is in equilibrium in concentric spherical layers. Let v_1^2, the mean square of the velocity of agitation of the gaseous molecules, be defined by reference to the potential of the mass M_1 at the radius a_1, so that

$$v_1^2 = \beta_1^2 \frac{\mu M_1}{a_1},$$

where β_1^2 is a numerical coefficient, and μ is the attractional constant.

Let p and w be the pressure and density of the gas at radius r, and k the modulus of elasticity, so that

$$p = k w,$$

$$k = \tfrac{1}{3} v_1^2 = \tfrac{1}{3} \beta_1^2 \frac{\mu M_1}{a_1}.$$

---

* "Untersuchungen über die Höhe der Atmosphäre und die Constitution gasförmiger Weltkörper," 'WIEDEMANN'S Annalen' (New Series), vol. 16, 1882, p. 166. A very elegant solution of part of my problem has also been given by Mr. G. W. HILL in the 'Annals of Mathematics,' vol. 4, No. 1, p. 19 (February, 1888). Mr. HILL'S paper only reached my hands after my own calculations had been completed, and I therefore adhere to my own less elegant method. Mr. HILL has obviously not seen M. RITTER'S papers.

Then the equation for the hydrostatic equilibrium of the gas is

$$\frac{r^2}{w}\frac{dp}{dr} + 4\pi\mu\int_0^r wr^2\,dr = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

It is obvious that $\frac{-r^2}{\mu w}\frac{dp}{dr}$ is equal to the whole mass enclosed inside radius r, and this relation will hold however the equation be transformed, provided we do not multiply the equation by any factor.

In consequence of the relation between p and w this may be written

$$k\left[r^2\frac{d}{dr}\log w + \frac{4\pi\mu}{k}\int_0^r wr^2\,dr\right] = 0.$$

If ρ_1 be the mean density of the mass M_1, we have

$$4\pi\mu = \frac{3\mu M_1}{\rho_1 a_1^3} = \frac{9k}{\beta_1^2 a_1^2 \rho_1}.$$

Hence, we may write the equation (12) in the form

$$ka_1\left[\frac{r^2}{a_1}\frac{d}{dr}\log w + \frac{9}{\beta_1^2}\int_0^r \frac{w}{\rho_1}\frac{r^2}{a_1^3}\,dr\right] = 0.$$

Now, let

$$x_1 = \frac{a_1}{r}, \qquad \frac{w}{\rho_1} = \tfrac{1}{9}\beta_1^2 e^{y_1},$$

and the equation becomes

$$\tfrac{1}{3}\beta_1^2\mu M_1\left[-\frac{dy_1}{dx_1} + \int_{x_1}^{\infty}\frac{e^{y_1}}{x_1^4}\,dx_1\right] = 0. \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

By differentiation we obtain the equation

$$\frac{d^2y_1}{dx_1^2} + \frac{e^{y_1}}{x_1^4} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

It is obvious from (13) that $\frac{1}{3}\beta_1^2 M_1\,dy_1/dx_1$ is the mass enclosed inside radius a/x_1, and therefore $\frac{1}{3}\beta_1^2\,dy_1/dx_1$ is equal to unity when $x = 1$.

A general analytical solution of (14) does not seem to be attainable, and recourse must be had to numerical processes. Although this is an equation of the second degree, and its general solution must involve two arbitrary constants, we shall see (as pointed out by M. Ritter) that the general solution, as applicable to our problem, may be deduced from one single numerical solution. M. Ritter proceeds by a graphical method, which he has worked with surprising accuracy. I shall therefore adopt an analytical and numerical method, which, although laborious, is susceptible of greater accuracy.

Whatever be the arrangement of the gas, the density at the centre must have some value. I therefore start with a central density ω, corresponding to the value η of y_1, so that

$$\frac{\omega}{\rho_1} = \tfrac{1}{9}\beta_1^{\,2}e^{\eta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

For the sake of brevity the suffixes 1 will be now omitted from the various symbols, to be reaffixed later when they are required.

At the centre, where x is infinite, dy/dx, d^2y/dx^2, &c., are all zero, and we put $y = \eta$.

Let $\xi = e^{\eta}/x^2$, and let us assume

$$\begin{aligned} y &= \eta + v \\ &= \eta - A_1\xi + A_2\xi^2 - A_3\xi^3 + \ldots \,. \end{aligned}$$

Now, the differential equation (14) to be satisfied is

$$x^2\frac{d^2y}{dx^2} = -\frac{e^y}{x^2} = -\xi e^v.$$

But

$$x^2\frac{d^2y}{dx^2} = -2\,.\,3A_1\xi + 4\,.\,5A_2\xi^2 - 6\,.\,7A_3\xi^3 + \ldots\,,$$

and by expanding e^v we obtain

$$\begin{aligned} -\xi e^v = -\xi + A_1\xi^2 &- (A_2 + \tfrac{1}{2}A_1^{\,2})\,\xi^3 + (A_3 + A_1A_2 + \tfrac{1}{2.3}A_1^{\,3})\,\xi^4 \\ &- (A_4 + A_1A_3 + \tfrac{1}{2}A_2^{\,2} + \tfrac{1}{2}A_1^{\,2}A_2 + \tfrac{1}{2.3.4}A_1^{\,4})\,\xi^5 \\ &+ (A_5 + A_1A_4 + A_2A_3 + \tfrac{1}{2}A_1^{\,2}A_3 + \tfrac{1}{2}A_1A_2^{\,2} + \tfrac{1}{2.3}A_1^{\,3}A_2 + \tfrac{1}{2.3.4.5}A_1^{\,5})\,\xi^6 - \ldots \end{aligned}$$

By equating coefficients in these two series, I find

$$A_1 = \tfrac{1}{6}, \qquad A_2 = \tfrac{1}{120}, \qquad A_3 = \tfrac{1}{1890}, \qquad A_4 = \tfrac{61}{1{,}632{,}960},$$

$$A_5 = \tfrac{629}{224{,}532{,}000}, \qquad A_6 = \tfrac{34\cdot07383\ldots}{156\times10^6}, \quad \text{\&c.},$$

and

$$\log A_1 = 9\cdot2218487, \qquad \log A_2 = 7\cdot9208188, \qquad \log A_3 = 6\cdot7235382,$$

$$\log A_4 = 5\cdot5723543, \qquad \log A_5 = 4\cdot4473723, \qquad \log A_6 = 3\cdot3392964\,;$$

whence, by extrapolation,

$$\log A_7 = 2\cdot243, \qquad \log A_8 = 1\cdot13.$$

In M. RITTER's paper, already referred to, he takes a certain function u as equal to 1·031, when the radius is unity. Now, RITTER's function u is equal in my notation to

$\frac{1}{2}(w/\rho) \div \frac{1}{9}\beta^2 a^2/r^2$ or $\frac{1}{2}e^y/x^2$. It follows, therefore, that Ritter takes the surface value of $y = \log_e 2{\cdot}062$. But he intends the central density to be 100 times the surface density; hence, to take the same solution, we must have e^η equal to 100 times 2·062. Therefore, his value of η would be

$$\eta = \log_e 100 + \log_e 2{\cdot}062,$$

or

$$\eta = 5{\cdot}3288465.$$

Now, as I want to make a comparison between my solution and his, I start with this value of η. The only object attained by the choice of this particular value is that the two solutions become easily comparable. It will be seen below that the value of η does not make the central density exactly 100 times the surface density, but only satisfies that condition approximately. In Ritter's graphical treatment of the problem this value 100 is the exact datum, whereas in my method we start with an exact value of η, and proceed to find the ratio of central to surface density.

With this value of η (whence $\log_{10} e^\eta = 2{\cdot}3142888$) I find the following series for y:—

$$y = 5{\cdot}3288465 - \frac{34{\cdot}3667}{x^2} + \frac{354{\cdot}321}{x^4} - \frac{4638{\cdot}79}{x^6} + \frac{67{,}532{\cdot}0}{x^8} - \frac{1{,}044{,}280}{x^{10}}$$
$$+ \frac{16{,}789{,}000}{x^{12}} - \frac{2{\cdot}77 \times 10^8}{x^{14}} + \frac{4{\cdot}4 \times 10^9}{x^{16}} - \ldots; \quad . \quad . \quad (16)$$

and, by differentiation, the series for dy/dx is obvious.

This series will be very accurate from $x = \infty$ to about $x = 8$. Thus, when $r/a = {\cdot}1$, or $x = 10$, we have

$$y = 5{\cdot}016558, \qquad \frac{dy}{dx} = +{\cdot}0568910,$$

and even the far less convergent series for d^2y/dx^2 gives $-{\cdot}0150891$, agreeing with $-e^y/x^4$ to the last place of decimals. When $r/a = {\cdot}125$, or $x = 8$, we have*

$$y = 4{\cdot}863925, \qquad \frac{dy}{dx} = {\cdot}101168,$$

whence,

$$\frac{d^2y}{dx^2} = -{\cdot}031624,$$

with y correct to four, and probably to five, places of decimals, and dy/dx probably correct to four places of decimals. This is amply sufficient for our purpose. Indeed, accuracy of this order would be altogether pedantic, were it not that the errors accumulate.

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* Even when $x = 5$, I find from this series $y = 4{\cdot}342$, which lies very near to $y = 4{\cdot}332$, found below. But the series for dy/dx is useless.

We cannot, then, rely on this method of procedure beyond the region included between $x = \infty$ and $x = 8$, and must now make a new departure.

Since

$$\frac{d^2y}{dx^2} = -\frac{e^y}{x^4},$$

$$\log\left(-\frac{d^2y}{dx^2}\right) = y - 4\log x;$$

therefore,

$$\frac{d^3y}{dx^3} = \frac{d^2y}{dx^2}\left(\frac{dy}{dx} - \frac{4}{x}\right). \quad \text{. . . . . (17)}$$

Now, let

$$A_n = \frac{1}{n!}\frac{d^ny}{dx^n},$$

where, after differentiation, x is put equal to c, a constant.

Then (17) may be written—

$$A_3 = \frac{2!}{3!}A_2\left(A_1 - \frac{4}{c}\right). \quad \text{. . . . . (18)}$$

Now, it is clear that

$$\frac{d^pA_n}{dc^p} = \frac{n+p!}{n!}A_{n+p}$$

Hence, differentiating (18) $n - 3$ times, we have

$$\frac{n!}{3!}A_n = \sum_{q=0}^{q=n-3} \frac{n-3!}{n-3-q!\,q!}\,\frac{2!}{3!}\,\frac{n-q-1!}{2!}A_{n-q-1}\left\{\frac{q+1!}{1!}A_{q+1} + (-)^q\frac{4(q!)}{c^{q+1}}\right\},$$

or

$$A_n = \frac{1}{n.n-1.n-3}\sum_{q=0}^{q=n-3}(n-q-1)(n-q-2)A_{n-q-1}\left\{(q+1)A_{q+1} + (-)^q\frac{4}{c^{q+1}}\right\},$$

or

$$A_n = \frac{1}{n.n-1.n-3}\left\{2.1A_2\left[(n-2)A_{n-2} + (-)^n\frac{4}{c^{n-2}}\right]\right.$$
$$\left. + 3.2A_3\left[(n-3)A_{n-3} - (-)^n\frac{4}{c^{n-3}}\right] + \ldots\right\}. \quad \text{(19)}$$

Now, if, for a given value of x, viz., c, we know y, or A_0, and dy/dx, or A_1, then we can compute A_2 from the formula $-\frac{1}{2}e^{A_0}c^{-4}$; and, by the formula (18), viz. :—

$$A_3 = \frac{1}{3.2.1}\left\{2.1A_2\left[A_1 - \frac{4}{c}\right]\right\},$$

A_3 may be computed.

Afterwards, A_4, A_5, &c., may be computed by successive applications of (19). This being so,

$$y = A_0 + A_1(x-c) + A_2(x-c)^2 + \dots, \qquad . \ . \ . \ . \ . \quad (20)$$

$$\frac{dy}{dx} = A_1 + 2A_2(x-c) + 3A_3(x-c)^2 + \dots .$$

In these series x may have any value, provided the series converges adequately. The convergence may be much improved by an artifice, which, however, I unfortunately did not discover until most of the computations were completed. Let us add and subtract $\log 2x^2$ on the right-hand side of (20).

Now,

$$\log_e 2x^2 = \log_e 2c^2 + 2\log_e\left[1 + \frac{x-c}{c}\right]$$

$$= \log_e 2c^2 + 2\frac{x-c}{c} - \tfrac{2}{2}\frac{(x-c)^2}{c^2} + \tfrac{2}{3}\frac{(x-c)^3}{c^3} - \dots .$$

If, then, we write

$$B_0 = A_0 - \log_e 2c^2, \qquad B_1 = A_1 - \frac{2}{c}, \qquad B_2 = A_2 + \frac{2}{2c^2}, \qquad B_3 = A_3 - \frac{2}{3c^3} \text{ \&c.},$$

we have

$$y = \log_e 2x^2 + B_0 + B_1(x-c) + B_2(x-c)^2 + \dots, \qquad . \ . \ . \quad (21)$$

a more convergent series than that with the A's.

The simplest way of computing the B's appears to be by first computing the A's.

The process for obtaining the numerical solution is then as follows:—

We have the values of y, dy/dx, $\frac{1}{2}dy^2/dx^2$ when $x = 8$, that is to say, of A_0, A_1, A_2 when $c = 8$. From these the successive A's and B's are computed, and the resulting series gives the values of y and dy/dx when x is 5 or $r = \cdot 2$. Starting from this point a new series gives the result when $r = \cdot 3$, another series gives the values for $r = \cdot 4$, and so on. Later in the calculation several values may be computed from one formula.*

When the computation has been carried out to $r = a$, we have reached the spherical envelope, but that envelope may be replaced by another at any more remote distance from the centre. Thus, the integration may be pursued for values of x less than unity, and when the lower limit is zero the envelope is at infinity.

If we write

$$\log u = B_0 + B_1(x-c) + B_2(x-c)^2 + \dots,$$

---

* If the series be carried as far as B_8, several steps may be included in one series. For example, the first series, when $c = 8$, may be pushed even as far as $r = \cdot 4$ without serious error; for it gives $y = 2\cdot 960$ instead of the true value $2\cdot 965$, and $dy/dx = \cdot 957$, instead of the true value $\cdot 944$. I have not, however, been satisfied with this degree of accuracy.

we have

$$e^{y} = u \,.\, 2x^{2},$$

and

$$\frac{w}{\rho} = \frac{2}{9}\frac{\beta^2 a^2}{r^2} \cdot u.$$

But it may easily be seen that $2\beta^2a^2/9r^2$ is a particular solution of the problem; hence, u is a factor by which the particular solution is to be multiplied to obtain the general solution. The function u is given by

$$u = \frac{1}{2}\frac{e^{y}}{x^2} = -\frac{1}{2}x^2\frac{d^2y}{dx^2}. \quad \dots\dots\dots \quad (22)$$

A table of the values of u is given below, showing how the general solution shades off into the particular solution. This function, u, is also tabulated by **Ritter**, and I made use of its value, when $x = 1$, to determine the value of η, with which the integration is to begin. I find, however (see Table I.), that, when $x = 1$, $u = 1{\cdot}0063$, in place of 1·031, as given by him.

The last row in Table I. gives the ratio of the central density ω to w, the density at the distance r; this ratio is equal to $e^{\eta-\eta_1}$.

The following Table gives the results of the computation, and the suffix 1 is reintroduced in the several symbols.

I.—Table of Results.

$\frac{1}{x_1} = \frac{r}{a_1} =$	0	·1	·125	·2	·3	·4	·5	·6	·7	·8	·9	1·0
$y_1 =$	5·328845	5·01656	4·8639	4·3317	3·6064	2·9653	2·4235	1·9672	1·5794	1·2458	·9553	·6995
$\frac{dy_1}{dx_1} =$	0·0	·05689	·10117	·2967	·6217	·9442	1·2404	1·5098	1·7569	1·9867	2·2030	2·4087
$\frac{d^2y_1}{dx_1^2} =$	0·0	−·01509	−·03162	−·1217	−·2984	−·4967	−·7053	−·9267	−1·1647	−1·4237	−1·7055	−2·0126
$u = -\frac{1}{2} x_1^2 \frac{d^2y_1}{dx_1^2} =$	0·0	·75445	1·0120	1·5215	1·6577	1·5521	1·4107	1·2871	1·1888	1·1123	1·0528	1·0063
$\frac{\varpi}{w} = e^{\eta - y_1} =$	1·0000	1·3666	. .	2·7105	5·598	10·63	18·29	28·84	42·50	59·32	79·33	102·45

$\frac{1}{x_1} = \frac{r}{a_1} =$	1·0	1·1	1·2	1·5	2·0	2·5	3·0	$\infty \left(\frac{1}{x_1}\right)$
$y_1 =$	·6995	·4717	·2672	−·241	−·864	−1·328	−1·699	$-\infty \ (\log 2x_1^2)$
$\frac{dy_1}{dx_1} =$	2·4087	2·6062	2·7971	3·342	4·202	5·139	6·038	$+\infty \left(\frac{2}{x_1}\right)$
$\frac{d^2y_1}{dx_1^2} =$	−2·0126	−2·3466	−2·7089	−3·976	−6·744	−10·35	−14·82	$-\infty \left(-\frac{2}{x_1^2}\right)$
$u = -\frac{1}{2} x_1^2 \frac{d^2y_1}{dx_1^2} =$	1·0063	·9697	·9406	·884	·843	·828	·823	1·0
$\frac{\varpi}{w} = e^{\eta - y_1} =$	102·45	128·7	157·8	263	489	778	1127	$\infty \left(\frac{1}{2} \frac{e^{\eta}}{x_1^2}\right)$

It will be noticed that u rises from zero to a maximum of about 1·66, falls to a minimum of about ·82, and then rises to unity.

Since $\frac{1}{3}\beta_1^2\, dy_1/dx_1 = 1$ when $x_1 = 1$, we have $\frac{1}{3}\beta_1^2 = 1/2{\cdot}4087 = {\cdot}4152$.

M. RITTER has ·4143 for this constant, which he calls m.

It appears from the Table that the density at the centre is $102\frac{1}{2}$ times as great as that where $r = a_1$. M. RITTER's solution is intended to make that ratio exactly 100, but this solution shows that we ought to have started with a slightly different value of η to obtain that result.

In the general solution of the differential equation $d^2y/dx^2 = -e^y/x^4$ the two arbitrary constants may be taken to be the values of y and dy/dx when x is infinite. Now, we have taken arbitrarily $y = 5{\cdot}329$ when x is infinite, and the physical conditions of the problem imply that dy/dx is zero when x is infinite. For if dy/dx had any positive or negative value different from zero, it would mean that at the centre there was a nucleus of infinitely small dimensions, but of finite positive or negative mass. Now, a_1 is that distance from the centre at which the density is 1/102·45 of the central density; hence, we may regard a_1 as the arbitrary constant of the solution. Whatever be the elasticity of the gas, we may always take as our unit of length that distance from the centre of the nebula at which the density has fallen to 1/102·45 of its central value. Hence, the above table gives the general solution of the problem, subject, however, to the condition that there is no central nucleus.

If we view the nebula from a very great distance, a_1 appears very small, and thus the solution of the problem becomes $y = \log 2x^2$. It is easy to verify that this is a particular algebraic solution of the differential equation, as is pointed out by RITTER in his paper.* I found this solution very useful in a preliminary consideration of the problem treated in this paper.

The next point which we have to consider is the form which the solution will take, if, instead of taking a_1 as the unit of length, we take any other value.

The density at any distance and the elasticity are to remain unchanged, but are to be referred to new constants.

Thus, w, r, v^2 remain unchanged, but are to be referred to M, ρ, β^2, a, instead of to M_1, ρ_1, β_1^2, a_1.

Now, since w remains unchanged,

$$\tfrac{1}{9}\beta^2\rho\, e^y = \tfrac{1}{9}\beta_1^2\rho_1 e^{y_1},$$

and, since v^2 remains unchanged,

$$\beta^2\rho\, a^2 = \beta_1^2\rho_1 a_1^2.$$

Also

$$x = \frac{a}{r} = x_1\frac{a}{a_1}\,.$$

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* I have made use of this solution in a paper in the 'Proceedings of the Royal Society,' Dec. 3, 1883, and it has also been referred to in a paper by Sir W. THOMSON, 'Phil. Mag.,' vol. 23, p. 287. Sir W. THOMSON's paper covers much the same ground as some of M. RITTER's earlier papers, but was written by him independently and in ignorance of them.

From these relations it is clear that

$$y = y_1 - 2 \log \frac{a_1}{a},$$

and

$$M = \tfrac{1}{3} \beta_1^2 M_1 \frac{dy_1}{dx_1} \left(x_1 = \frac{a_1}{a} \right).$$

Then, since $\frac{1}{3} \beta^2 \, dy/dx = 1$, when $x = 1$, and since $dy = dy_1$ and $dx = dx_1 \, a/a_1$, it follows that

$$\tfrac{1}{3} \beta^2 = \frac{1}{x_1 \, dy_1/dx_1}, \text{ when } x_1 = \frac{a_1}{a}. \quad \ldots\ldots \quad (23)$$

This relationship has been already used for determining β_1^2.

It is obvious also that

$$\frac{\rho}{\rho_1} = \frac{x_1^3 \, dy_1/dx_1, \text{ when } x_1 = a_1/a}{x_1^3 \, dy_1/dx_1, \text{ when } x_1 = 1}.$$

Therefore,

$$\frac{w}{\rho} = \frac{\frac{1}{3} e^{y_1}}{x_1^3 \, dy_1/dx_1, \text{ when } x_1 = a_1/a}. \quad \ldots\ldots\ldots \quad (24)$$

If w_0 be the density when $r = a$, we have

$$\frac{w_0}{\rho} = \frac{\frac{1}{3} e^{y_1}, \text{ when } x_1 = a_1/a}{x_1^3 \, dy_1/dx_1} = -\tfrac{1}{3} \cdot \frac{x_1 \, d^2 y_1/dx_1^2}{dy_1/dx_1}, \text{ when } x_1 = a_1/a. \quad . \quad (25)$$

If p_0 be the pressure when $r = a$, we have

$$p_0 = \tfrac{1}{3} v^2 w_0 = \tfrac{4}{9} \pi \mu a^2 \rho^2 \, . \, \beta^2 \frac{w_0}{\rho}.$$

If, therefore, we write $P = \frac{4}{3} \pi \mu a^2 \rho^2$,

$$\frac{p_0}{P} = \tfrac{1}{3} \beta^2 \, . \, \frac{w_0}{\rho} = - \frac{d^2 y_1/dx_1^2}{(dy_1/dx_1)^2}, \text{ when } x_1 = a_1/a. \quad \ldots\ldots \quad (26)$$

By (26) we are able to find how the pressure on an envelope of given radius a varies with the variation of the temperature of a given mass M of gas contained in it. By means of the formulæ (23), (25), (26), we are now able to obtain from the original solution any number of other ones; for, after the changes have been effected in the notation, we may proceed to magnify or diminish all the various values of a until they are of one size, and we shall thus obtain the solution for a gas at any temperature whatever.

I shall now proceed to give a table of results when the standard radius a, which may be conveniently called the boundary, is placed successively infinitely near the centre, where $r = 0 \times a_1$, at $r = \cdot 1 \times a_1$, $r = \cdot 2 \times a_1$, and so on. The first line of entries gives the various values of $\frac{1}{3}\beta^2$ (computed from (23)), on which the elasticity of the gas depends; the second line gives w_0/ρ (computed from (25)), or the ratio of the boundary density to the mean density of all inside of it; the third line gives p_0/P (computed from (26)), by which we trace the variations of pressure at the boundary.

TABLE II.

Value of a by reference to former solution $\Big\} \frac{a}{a_1} =$	0	·1	·2	·3	·4	·5	·6	·6264	·7	·8
$\frac{\frac{1}{3}v^2}{\mu M/a} = \frac{1}{3}\beta^2\left[= \frac{1}{x_1\, dy_1/dx_1}\right] =$	∞	1·7577	·6741	·4826	·4236	·4031	·3974	·3972	·3984	·4027
$\frac{w_0}{\rho}\left[= \frac{-\frac{1}{3}x_1\, d^2y_1/dx_1^2}{dy_1/dx_1}\right] =$	1·0000	·8841	·6838	·5333	·4383	·3791	·3410	$\frac{1}{3}$	·3158	·2986
$\frac{p_0}{P}\left[= \frac{-d^2y_1/dx_1^2}{[dy_1/dx_1]^2}\right] =$	∞	4·662	1·383	·772	·557	·458	·407	·397	·377	·361

Value of a by reference to former solution $\Big\} \frac{a}{a_1} =$	·9	1·0	1·25	1·5	2·0	2·5	3·0	∞
$\frac{\frac{1}{3}v^2}{\mu M/a} = \frac{1}{3}\beta^2\left[= \frac{1}{x_1\, dy_1/dx_1}\right] =$	·4085	·4152	·4325	·449	·476	·487	·497	$\frac{1}{2}$
$\frac{w_0}{\rho}\left[= \frac{-\frac{1}{3}x_1\, d^2y_1/dx_1^2}{dy_1/dx_1}\right] =$	·2867	·2785	·2676	·264	·267	·269	·273	$\frac{1}{3}$
$\frac{p_0}{P}\left[= \frac{-d^2y_1/dx_1^2}{[dy_1/dx_1]^2}\right] =$	·351	·347	·347	·356	·382	·392	·406	$\frac{1}{2}$

The minimum value of w_0/ρ occurs when $a/a_1 = 1·6$ very nearly, for, when $a/a_1 = 1·4$, 1·5, 1·6, I find $w_0/\rho = ·26521$, ·26437, ·26425 respectively.* When $r/a_1 = 1·6$, $y_1 = -·38435$ and $dy_1/dx_1 = 3·5180$. The minimum value of p_0/P occurs when $a/a_1 = 1·1$ very nearly, for, when $a/a_1 = 1·0$, 1·1, 1·2, I find $p_0/P = ·3469$ ·3455, ·3462 respectively.

When w_0/ρ is a minimum, the density at the centre is 381 times that at the boundary, and, when p_0/P is a minimum, the density at the centre is 129 times that at the boundary. M. RITTER finds the pressure to be a minimum when this ratio is 258, instead of 129. As this corresponds to $a/a_1 = 1·5$, this discrepancy between our solutions is not so large as might be expected from the great discrepancy between these results, and I cannot but think that my result is more accurate than his.

The minimum value of $\frac{1}{3}\beta^2$ occurs when $a/a_1 = ·6264$, and its value is ·39723. This value makes the surface density exactly one-third of the mean density, for $\frac{1}{3}\beta^2$ is a minimum when $x_1\, dy_1/dx_1$ is a maximum, and this occurs when $x_1 d^2y_1/dx_1^2 + dy_1/dx_1 = 0$; and, when this relationship is satisfied, $w_0/\rho = \frac{1}{3}$.

It is interesting to note that in this case β^2 is very nearly equal to $\frac{6}{5}$, so that the total internal kinetic energy of agitation of the sphere of gas at minimum temperature limited by the radius a is $\frac{1}{2}(\frac{6}{5}\mu M/a)M = \frac{3}{5}\mu M^2a$ *very nearly*. Now, the energy lost in the concentration of a homogeneous sphere M from a condition of infinite dispersion is *exactly* $\frac{3}{5}\mu M^2/a$. It might, therefore, be suspected that ·39723 is only an approximation to $\frac{2}{5}$, which may be the rigorous value. But my numerical calculations were carried out with so much care that I find it almost impossible to believe that there is an error as large as 3 in the third place of decimals, or, indeed, any error at all in the third figure. Moreover, it would be expected that, if this very simple relationship is rigorously correct, it would be possible to prove it rigorously, just as it is rigorously shown above that $w_0/\rho = \frac{1}{3}$; but I am unable to find any analytical relationships by which the minimum value of $\frac{1}{3}\beta^2$ can be deduced. If my arithmetical process be correctly carried out, then we ought to find that, when $r = {\cdot}6264$, dy_1/dx_1 should be equal to $-x_1 d^2y_1/dx_1^2$ or e^{y_1}/x_1^3. Now, I find that, when $r = {\cdot}6264$, $dy_1/dx_1 = 1{\cdot}57703$ and $e^{y_1}/x_1^3 = 1{\cdot}5770$, so that the two agree to four places of decimals. I conclude, therefore, that the true minimum of $\frac{1}{3}\beta^2$ is ·3972.*

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* Mr. HILL finds that the minimum value of w_0/ρ approximates to $\frac{4}{15}$, or ·2667. The agreement between our results is satisfactory.

It will be observed that, as a/a_1 increases to infinity, $\frac{1}{3}\beta^2$ terminates by being equal to $\frac{1}{2}$. M. RITTER has found that it rises above $\frac{1}{2}$, and oscillates about that value an indefinite number of times with diminishing amplitude, gradually settling down to $\frac{1}{2}$ as a/a_1 becomes infinite. The values in the preceding table are not, however, carried far enough to exhibit these oscillations of $\frac{1}{3}\beta^2$. A consequence of this result is that there are a number of modes of equilibrium of a gas at a given temperature, provided that the temperature lies within certain narrow limits. This very remarkable conclusion is rendered more intelligible by Mr. HILL's treatment than by M. RITTER's.

This point has, however, no bearing on the present investigation.

In any one of the solutions comprised in Table II. we may complete the table of densities by the formula (24), viz.,

$$\frac{w}{\rho} = \frac{\frac{1}{3}e^{y_1}}{x_1^3\, dy_1/dx_1\, (x_1 = a_1/a)},$$

and I shall later proceed to do this in the one case which has interest for our present problem, namely, where the temperature is a minimum, so that a/a_1 is ·6264. The full numerical results may be more conveniently given hereafter, and it will only be now necessary to indicate how they are to be computed.

When, for example, $r = {\cdot}1 \times a_1$, $r/a = {\cdot}1/{\cdot}6264 = {\cdot}1596$; thus, our equidistant values of the density and other functions will proceed by multiples of $\cdot1596\,a$ up to $\cdot9578\,a$, and the limit of the isothermal sphere is where $r = a$.

When the temperature is a minimum $\frac{1}{3}\beta^2 = {\cdot}39723$, and we have $w_0 = \frac{1}{3}\rho$; therefore, $w/w_0 = w/\frac{1}{3}\rho$, and, therefore, if $y_{1,0}$ be the value of y_1, when $r = \cdot6264\,a_1$,

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* This is confirmed by Mr. HILL. His equation $s = z$ is equivalent to $x_1^2 d^2y_1/dx_1^2 + x_1 dy_1/dx_1 = 0$, and it appears from his tables that $s = z = 2{\cdot}517$. Now, $s = 3/\beta^2$, and the reciprocal of $2{\cdot}517$ is ·397.

$w/\frac{1}{3}\rho = e^{y_1 - y_{1,0}}$. Thus, for example, at the centre, $w/\frac{1}{3}\rho$ is 32·14, and when $r = {\cdot}4789\,a$ it is 5·7417.

The proportion of the mass M which is included in radius a/x is $\frac{1}{3}\beta^2\,dy/dx = \frac{1}{3}\beta^2 a_1\,dy_1/a\,dx_1 = \frac{\cdot 39723}{\cdot 6264}\,dy_1/dx_1$. Hence, the masses may be computed.

At any part of the isothermal sphere gravity g is to be found from

$$g = \tfrac{1}{3}\beta^2\mu M\frac{dy}{dx}\cdot\frac{x^2}{a^2};$$

or, expressing g in terms of G gravity at the surface, we have, since $G = \mu M/a^2$,

$$\frac{g}{G} = \tfrac{1}{3}\beta^2 x^2\frac{dy}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

The angular velocity of a body moving in a circular orbit at any part of the nebula, and its linear velocity v are also easily to be found.

§ 5. *On an Atmosphere in Convective Equilibrium.*

I shall now suppose that a sphere of gas of mass M at minimum temperature is bounded by an atmosphere in convective equilibrium, with continuity of temperature and density at the sphere of discontinuity of radius a. Let v_0^2 be the mean square of velocity of agitation in the isothermal sphere, and v^2 that at any other radius r. Then throughout the isothermal sphere $v^2 = v_0^2$, but in the layer outside v^2 gradually decreases to zero.

Let w_0 be the density and p_0 the pressure at radius a, and w, p the same things at radius r.

Then, if the ratio of the two specific heats be that deduced from the simple kinetic theory of gases, without any allowance for intra-molecular vibrations, we have that ratio equal to $\frac{5}{3}$.

Hence,

$$p = p_0\left(\frac{w}{w_0}\right)^{\frac{5}{3}},$$

and

$$\tfrac{1}{3}v^2 = p_0\frac{w^{\frac{2}{3}}}{w_0^{\frac{5}{3}}} = \tfrac{1}{3}v_0^2\cdot\left(\frac{w}{w_0}\right)^{\frac{2}{3}},$$

also

$$\frac{dp}{w} = \tfrac{5}{3}\cdot\frac{p_0}{w_0^{\frac{5}{3}}}w^{-\frac{1}{3}}\,dw = \tfrac{5}{2}\cdot\tfrac{1}{3}v_0^2\,d\left(\frac{w}{w_0}\right)^{\frac{2}{3}}.$$

Now, the equation for the hydrostatic equilibrium of the layer is

$$\frac{r^2}{w}\frac{dp}{dr} + \mu M + 4\pi\mu\int_a^r wr^2\,dr = 0. \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

Let

$$x = \frac{a}{r}, \qquad z = \frac{v^2}{{v_0}^2} = \left(\frac{w}{w_0}\right)^{\frac{2}{3}},$$

and we have

$$\frac{r^2\,dp}{w\,dr} = -\tfrac{5}{2}\,.\,\tfrac{1}{3}\,{v_0}^2\,a\,\frac{dz}{dx},$$

$$\mu M = \frac{{v_0}^2\,a}{\beta^2},$$

$$4\pi\mu a^3 = \frac{3\mu M}{\rho} = \frac{\mu M}{w_0}, \text{ since } w_0 = \tfrac{1}{3}\,\rho \text{ rigorously,}$$

$$= \frac{{v_0}^2 a}{\beta^2 w_0}.$$

Hence, our equation is

$$\mu M\left\{-\tfrac{5}{6}\,\beta^2\frac{dz}{dx} + 1 + \int_x^1 \frac{z^{\frac{3}{2}}}{x^4}\,dx\right\} = 0. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

It is obvious the $\frac{5}{6}\,\beta^2\,M\,dz/dx$ is the whole mass (expressed in terms of the mass of the isothermal sphere) enclosed inside of radius a/x. The differential equation to be satisfied is

$$\tfrac{5}{6}\,\beta^2\frac{d^2z}{dx^2} + \frac{z^{\frac{3}{2}}}{x^4} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

We have seen in the last section that $\frac{1}{3}\,\beta^2 = \cdot 39723$, and, hence, $\frac{5}{6}\beta^2 = \cdot 99308$.

This equation is not so easy to solve as that in the last section, and I have not succeeded in finding the general law of the coefficients in an expansion. Nevertheless it is easy to find a series which will do all that is required.

Let c be any value of x for which we know z and dz/dx, and let

$$\xi = x - c.$$

Assume

$$z = z_0\,\{1 + A_1\xi + A_2\xi^2 + A_3\xi^3 + \ldots\}.$$

Then, if the suffix 0 indicates the value of a symbol when $x = c$ and $\xi = 0$, we have

$$z_0 = z_0,$$

$$\left(\frac{dz}{dx}\right)_0 = A_1 z_0,$$

$$\left(\frac{d^2z}{dx^2}\right)_0 = 2A_2 z_0.$$

But

$$\left(\frac{d^2z}{dx^2}\right)_0 = -\frac{6}{5\beta^2}\,.\,\frac{{z_0}^{\frac{3}{2}}}{c^4},$$

and

$$2A_2 z_0 = -\frac{6}{5\beta^2}\cdot\frac{z_0^{\frac{3}{2}}}{c^4}, \qquad \text{or} \qquad A_2 = -\frac{3}{5\beta^2}\frac{z_0^{\frac{3}{2}}}{c^4},$$

so that, if z_0 and $(dz/dx)_0$ are known, A_2 is known.

The differential equation (30) which we have to satisfy is

$$\tfrac{5}{6}\beta^2(\xi + c)^4 \frac{d^2 z}{d\xi^2} = -z^{\frac{3}{2}},$$

or

$$\frac{1}{A_2}\left(\frac{\xi}{c} + 1\right)^4 \frac{d^2(z/z_0)}{d\xi^2} = 2\left(\frac{z}{z_0}\right)^{\frac{3}{2}}.$$

Now, by expansion,

$$\begin{aligned}
2\left(\frac{z}{z_0}\right)^{\frac{3}{2}} = 2 &+ 3A_1\xi + 3\left[A_2 + \tfrac{1}{4}A_1^2\right]\xi^2 + 3\left[A_3 + \tfrac{1}{2}A_1A_2 - \tfrac{1}{24}A_1^3\right]\xi^3 \\
&+ 3\left[A_4 + \tfrac{1}{2}(A_1A_3 + \tfrac{1}{2}A_2^2) - \tfrac{1}{8}A_1^2A_2 + \tfrac{1}{64}A_1^4\right]\xi^4 \\
&+ 3\left[A_5 + \tfrac{1}{2}(A_1A_4 + A_2A_3) - \tfrac{1}{8}(A_1^2A_3 + A_1A_2^2) + \tfrac{1}{16}A_1^3A_2 - \tfrac{1}{128}A_1^5\right]\xi^5 \\
&+ 3\left[A_6 + \tfrac{1}{2}(A_1A_5 + A_2A_4 + \tfrac{1}{2}A_3^2) - \tfrac{1}{8}(A_1^2A_4 + 2A_1A_2A_3 + \tfrac{1}{3}A_2^3)\right. \\
&\qquad \left. + \tfrac{1}{16}(A_1^3A_3 + \tfrac{3}{2}A_1^2A_2^2) - \tfrac{5}{128}A_1^4A_2 + \tfrac{7}{1536}A_1^6\right]\xi^6 \\
&+ 3\left[A_7 + \tfrac{1}{2}(A_1A_6 + A_2A_5 + A_3A_4) - \tfrac{1}{8}(A_1^2A_5 + 2A_1A_2A_4 + A_1A_3^2 + A_2^2A_3)\right. \\
&\qquad + \tfrac{1}{16}(A_1^3A_4 + 3A_1^2A_2A_3 + A_1A_2^3) - \tfrac{5}{128}(A_1^4A_3 + 2A_1^3A_2^2) \\
&\qquad \left. + \tfrac{1}{256}A_1^5A_2 - \tfrac{3}{1024}A_1^7\right]\xi^7 + \dots \qquad (31)
\end{aligned}$$

And

$$\begin{aligned}
\frac{1}{A_2}\left(\frac{\xi}{c}+1\right)^4\frac{d^2}{d\xi^2}(z/z_0) = 2 &+ \left(3\,.\,2\,\frac{A_3}{A_2} + \frac{4}{c}\,.\,2\,.\,1\right)\xi + \left(4\,.\,3\,.\,\frac{A_4}{A_2} + \frac{4}{c}\,.\,3\,.\,2\,\frac{A_3}{A_2} + \frac{6}{c^2}\,.\,2\,.\,1\right)\xi^2 \\
&+ \left(5\,.\,4\,\frac{A_5}{A_2} + \frac{4}{c}\,.\,4\,.\,3\,\frac{A_4}{A_2} + \frac{6}{c^2}\,.\,3\,.\,2\,\frac{A_3}{A_2} + \frac{4}{c^3}\,.\,2\,.\,1\right)\xi^3 \\
&+ \left(6\,.\,5\,\frac{A_6}{A_2} + \frac{4}{c}\,.\,5\,.\,4\,\frac{A_5}{A_2} + \frac{6}{c^2}\,.\,4\,.\,3\,\frac{A_4}{A_2} + \frac{4}{c^3}\,.\,3\,.\,2\,\frac{A_3}{A_2} + \frac{1}{c^4}\,.\,2\,.\,1\right)\xi^4 \\
&+ \left(7\,.\,6\,\frac{A_7}{A_2} + \frac{4}{c}\,.\,6\,.\,5\,\frac{A_6}{A_2} + \frac{6}{c^2}\,.\,5\,.\,4\,\frac{A_5}{A_2} + \frac{4}{c^3}\,.\,4\,.\,3\,\frac{A_4}{A_2} + \frac{1}{c^4}\,.\,3\,.\,2\,\frac{A_3}{A_2}\right)\xi^5 \\
&+ \dots \qquad (32)
\end{aligned}$$

By equating the coefficients in (31) and (32) we are able to determine the A's. The law of the series (32) is obvious, and sufficient of the series (31) is written down to enable us to find A_9. We can, however, obtain a good approximation to higher coefficients, because the coefficients in (31) become relatively unimportant.

We now begin the solution with

$$c = 1, \qquad z_0 = 1, \qquad \left(\frac{dz}{dx}\right)_0 = \frac{1}{\frac{5}{6}\beta^2} = 1{\cdot}0070, \qquad \left(\frac{d^2z}{dx^2}\right)_0 = -\frac{1}{\frac{5}{6}\beta^2} = -1{\cdot}0070.$$

Hence,

$$A_1 = 1{\cdot}0070, \qquad A_2 = -{\cdot}5035,$$

whence I compute

$$A_3 = +{\cdot}41782, \quad A_4 = -{\cdot}30068, \quad A_5 = +{\cdot}16175, \quad A_6 = -{\cdot}01306, \quad A_7 = -{\cdot}1333,$$
$$A_8 = +{\cdot}266, \quad A_9 = -{\cdot}378, \quad A_{10} = +{\cdot}48, \quad A_{11} = -{\cdot}6.$$

With these coefficients I find

$$\left.\begin{array}{llllll} \frac{r}{a} = \frac{12}{11}, & \frac{12}{10}, & \frac{12}{9}, & \frac{12}{8}, & \frac{12}{7}, & \frac{12}{6} \\ z = {\cdot}9123 & {\cdot}8160 & {\cdot}7089 & {\cdot}5887 & {\cdot}4525 & {\cdot}2982 \end{array}\right\} \quad . \quad . \quad (33)$$

Then, evaluating $x^{-4}z^4$, and combining the several values by the rules for integration of the calculus of finite differences, I find

$$\left.\begin{array}{llllll} \frac{r}{a} = \frac{12}{11}, & \frac{12}{10}, & \frac{12}{9}, & \frac{12}{8}, & \frac{12}{7}, & \frac{12}{6} \\ \frac{5}{6}\beta^2\frac{dz}{dx} = \ldots & 1{\cdot}21 & 1{\cdot}35 & 1{\cdot}527 & 1{\cdot}729 & 1{\cdot}9513 \end{array}\right\} \quad . \quad . \quad (34)$$

When $r = 2$, we begin a new series with

$$c = \tfrac{1}{2}, \qquad z_0 = {\cdot}2982, \qquad A_1 = \left(\frac{1}{z}\frac{dz}{dx}\right)_0 = \frac{1{\cdot}9513}{{\cdot}9907 \times {\cdot}2982} = +6{\cdot}5894,$$
$$A_2 = \left(\frac{1}{2z}\frac{d^2z}{dx^2}\right)_0 = \frac{-z_0^4}{2(\frac{1}{2})^4} = -4{\cdot}3686.$$

From these I compute $A_3 = -2{\cdot}744$, $A_4 = +21{\cdot}365$, $A_5 = -45{\cdot}409$, $A_6 = +9{\cdot}932$, $A_7 = +{\cdot}319$.

It appears that z vanishes when $x - c = -{\cdot}141$ or $x = {\cdot}359$.

It follows, therefore, that four equidistant values of x lying between $r = 2a$ and $r = a/{\cdot}359 = 2{\cdot}786\,a$ correspond to $x - c = 0$, $x - c = -{\cdot}047$, $x - c = -{\cdot}094$, $x - c = -{\cdot}141$.

For the first of these, where $r = 2\,a$, we have $z = {\cdot}2982$, and for the last, where $r = 2{\cdot}786\,a$, $z = 0$; and, when $x - c = -{\cdot}047$, or $r = a/{\cdot}453 = 2{\cdot}208\,a$, I find $z = {\cdot}2031$; and, when $x - c = -{\cdot}094$, or $r = a/{\cdot}406 = 2{\cdot}463\,a$, I find $z = {\cdot}1033$.

Finding $x^{-4}z^{\frac{3}{2}}$ for these four values and combining them by the rules of integration, I find

$$\tfrac{5}{6}\beta^2\frac{dz}{dx} = 2\cdot1767, \quad \text{when } r = 2\cdot786\,a. \qquad \ldots\ldots \qquad (35)$$

We thus see that the mass of the whole system is 2·1767 times the mass of the isothermal nucleus, and its radius is 2·786 times the radius of the nucleus.

The mass of the isothermal nucleus is thus 46 per cent. of the whole. M. RITTER, taking the ratio of the specific heats as $\frac{7}{5}$ instead of $\frac{5}{3}$, says that the proportion is about 40 per cent.

§ 6. *On a Gaseous Sphere in "Isothermal-Adiabatic" Equilibrium.*

M. RITTER calls a sphere, with isothermal nucleus and a layer in convective equilibrium above it, a case of isothermal-adiabatic equilibrium. Since the height of an atmosphere in convective equilibrium depends only on the temperature at the base, and since the isothermal nucleus in our numerical example is at minimum temperature, the thickness of the adiabatic layer is a minimum, and the isothermal nucleus a maximum.

We are now in a position to collect together all the numerical results of the last two sections in a form appropriate for our subsequent investigation. It will be convenient to refer all the densities and masses to the mean density and mass of the isothermal nucleus. Gravity may also be referred to gravity G at the limit of the isothermal nucleus, and velocity to v_0^2, the mean square of velocity of agitation in the isothermal nucleus.

TABLE III.—Isothermal-Adiabatic Sphere.

Radius $\frac{r}{a}$ =	0	·1596	·3193	·4789	·6385	·7982	·9578	1	1·0909	1·2	1·3333	1·5	1·7143	2·0	2·208	2·463	2·786
Square of velocity of agitation $\frac{v^2}{v_0^2}$ =	1	1	1	1	1	1	1	1	·912	·816	·709	·589	·452	·298	·203	·103	0
Density $\frac{w}{\frac{1}{3}\rho}$ =	32·14	23·52	11·86	5·742	3·024	1·759	1·115	1	·871	·737	·597	·452	·304	·163	·092	·033	0
Mass in terms of M =	0	·0361	·1881	·3942	·5988	·7866	·9574	1	..	1·207	1·349	1·527	1·729	1·951	..	..	2·177
Gravity $\frac{g}{G}$ =	0	1·4156	1·8467	1·7188	1·4685	1·2346	1·0436	1	..	·8385	·7589	·6785	·5882	·4878	..	..	·28
Square of velocity of satellite $\frac{v^2}{v_0^2}$ =	0	·1896	·4945	·6908	·7869	·8269	·8388	·8391	..	·844	·849	·854	·846	·818	..	..	·66
$\log\frac{\frac{1}{3}\rho}{w}$ =	8·4929	8·6286	8·9260	9·2410	9·5194	9·7547	9·9529	·0000	·0598	·1325	·2241	·3452	·5166	·7882	1·0385	1·4791	∞
$\log\frac{1}{[w/\frac{1}{3}\rho][v/v_0]}$ =	8·4929	8·6286	8·9260	9·2410	9·5194	9·7547	9·9529	·0000	·0797	·1766	·2988	·4603	·6889	1·0510	1·3846	1·9722	∞
$\log F_1 = \log\left[\frac{3\pi}{8\beta^2}\frac{g/G}{[v^2/v_0^2][w/\frac{1}{3}\rho]}\right]$ =	− ∞	8·7745	9·1872	9·4712	9·6813	9·8413	9·9664	9·9950	..	·1393	·2487	·4019	·6256	·9970	..	..	∞
$\log F_2 = \log\left[\frac{1}{\pi}\frac{[g/G]^{\frac{1}{2}}x^{\frac{1}{2}}}{[v/v_0][w/\frac{1}{3}\rho]}\right]$ =	− ∞	8·6053	8·8098	9·0213	9·2031	9·3523	9·4744	9·5029	..	9·5967	9·6793	9·7909	9·9594	·2475	..	..	∞

§ 7. *On the Kinetic Energy of Agitation and its Distribution in an Isothermal-Adiabatic Sphere of Gas.*

We shall now consider what would be the distribution of kinetic energy in the nebula if each meteorite (or molecule) were to fall from infinity to the neighbourhood where we find it, and were to retain that energy afterwards. This will give the distribution of energy in a swarm of the supposed arrangement of density, if the rate of diffusion of kinetic energy were to be infinitely slow, and if there were no loss of energy through imperfect elasticity.

The square of the velocity of a satellite in a circular orbit is one half of the square of the velocity acquired by the fall from infinity to the distance of the satellite from the centre. If the concentration has proceeded as far as radius r, and if a meteorite falls from infinity to distance r, then, if U be its velocity, and v the velocity in a circular orbit at distance r,

$$\tfrac{1}{2}U^2 = v^2 = \tfrac{1}{3}\beta^2\frac{\mu M}{a}\,.\,x\frac{dy}{dx} = \tfrac{1}{3}v_0^2x\frac{dy}{dx},\text{ in the isothermal sphere,}$$

$$= \tfrac{5}{6}\beta^2\frac{\mu M}{a}\,.\,x\frac{dz}{dx} = \tfrac{5}{6}v_0^2x\frac{dz}{dx},\text{ in the adiabatic layer.}$$

In these formulæ, by the definitions of y and z,

$$y = \log_e\left(\frac{9w}{\beta^2\rho}\right)\text{ in the first, and }z = \left(\frac{w}{w_0}\right)^{\frac{2}{3}}\text{ in the second.}$$

From these formulæ v^2 was computed in Table III. The value of v^2 or $\tfrac{1}{2}U^2$ gives what may be called the theoretical value of the kinetic energy, because it gives us a measure of the amount of redistribution of energy by diffusion and loss of energy by imperfect elasticity, which must take place before the whole system can assume the form of an isothermal adiabatic sphere.

We will now go on to consider the total potential energy lost in condensation.

We have seen that the potential energy lost by the fall of a single meteorite is $\tfrac{1}{3}v_0^2x\,dy/dx$ in the isothermal part, and $\tfrac{5}{6}v_0^2x\,dz/dx$ in the outer part.

Now, in the isothermal part a spherical element of mass is

$$-\tfrac{1}{3}M\beta^2\,.\,\frac{d^2y}{dx^2}\,dx,$$

and the energy lost by its fall is

$$-\tfrac{1}{9}M\beta^2v_0^2\,.\,x\frac{dy}{dx}\frac{d^2y}{dx^2}\,dx.$$

Hence, the whole energy lost in the concentration of the isothermal nucleus is

$$\tfrac{1}{9}M\beta^2 v_0^2 \cdot \int_1^\infty x\,\frac{dy}{dx}\,\frac{d^2y}{dx^2}\,dx.$$

But

$$-\int_1^\infty x\,\frac{dy}{dx}\,\frac{d^2y}{dx^2}\,dx = \int_1^\infty \frac{e^y}{x^3}\,\frac{dy}{dx}\,dx = 3\int_1^\infty \frac{e^y}{x^4}\,dx - e^{y}$$

$$= 3\left(\frac{dy}{dx}\right)_0 - e^{y}$$

$$= \frac{9}{\beta^2} - \frac{9w_0}{\beta^2\rho}.$$

Hence, the energy lost is $Mv_0^2\left(1 - \frac{w_0}{\rho}\right)$. But in an isothermal sphere of minimum temperature $w_0 = \frac{1}{3}\rho$, and thus the total lost energy is $\frac{2}{3}Mv_0^2$.

Again, in the adiabatic layer an element of mass is

$$-\tfrac{5}{6}M\beta^2\,\frac{d^2z}{dx^2}\,dx = +\,M\,\frac{z^{\frac{3}{2}}}{x^4}\,dx,$$

and, therefore, the energy lost by its fall from infinity is

$$\tfrac{5}{6}Mv_0^2 \cdot \frac{z^{\frac{3}{2}}}{x^3}\,\frac{dz}{dx}\,dx,$$

and the whole loss of energy is the integral of this from $x = 1$ to $x = {\cdot}359$. When $x = 1$, $z = 1$, and when $x = {\cdot}359$, $z = 0$. Hence

$$\int_{\cdot 359}^1 \frac{z^{\frac{3}{2}}}{x^3}\,\frac{dz}{dx}\,dx = \tfrac{2}{5} + \tfrac{6}{5}\int_{\cdot 359}^1 \frac{z^{\frac{5}{2}}}{x^4}\,dx.$$

Thus, the whole energy lost in the adiabatic layer is

$$Mv_0^2\left[\tfrac{1}{3} + \int_{\cdot 359}^1 \frac{z^{\frac{5}{2}}}{x^4}\,dx\right].$$

Add this to the energy found before for the isothermal part, and the whole lost energy of the system is found to be

$$Mv_0^2\left[1 + \int_{\cdot 359}^1 \frac{z^{\frac{5}{2}}}{x^4}\,dx\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Now let us evaluate the total kinetic energy existing in the form of agitation of molecules.

In the isothermal part it is clearly $\frac{1}{2}Mv_0^2$. In the adiabatic part it is half the element of mass into the square of velocity of agitation integrated through the layer, that is to say, $\frac{1}{2} . M \frac{z^{\frac{3}{2}}}{x^4} dx \times v^2$, and, since $z = \frac{v^2}{v_0^2}$, we have

$$\tfrac{1}{2}Mv_0^2 \left[1 + \int_{\cdot 359}^{1} \frac{z^{\frac{5}{2}}}{x^4} dx \right]$$

for the total internal kinetic energy of agitation. This is rigorously one-half of the energy lost in concentration.

Hence, if a meteor swarm concentrates into this arrangement of density, one half of the original energy is occupied in vaporising and heating parts of the meteorites on impact, and the other half is retained as kinetic energy of agitation.

I find by quadrature that $\int_{\cdot 359}^{1} \frac{z^{\frac{5}{2}}}{x^4} dx = \cdot 643$. Hence, the potential energy lost in concentration is $Mv_0^2 (1 \cdot 643)$, and that part of it which is retained as energy of agitation is $\frac{1}{2}Mv_0^2 (1 \cdot 643)$. The whole mass of the system is $2 \cdot 1767\, M$, and we may, therefore, write these

$$\cdot 7548\, (2 \cdot 1767\, M)\, v_0^2 \quad \text{and} \quad \tfrac{1}{2} \times \cdot 7548\, (2 \cdot 1767\, M)\, v_0^2.$$

It is clear then that the average mean square of velocity of agitation of the *whole* system is $\cdot 7548\, v_0^2$.* Or, shortly, the average temperature is very nearly $\frac{3}{4}$ of the temperature of the isothermal nucleus.

It follows from this whole investigation that for any given mass of matter, arranged in an isothermal-adiabatic sphere of given dimensions, the actual velocities of agitation are determinable throughout.

§ 8. *On the "Sphere of Action."*

When two meteorites pass near to one another, each will be deflected from its straight path by the attraction of the other. The question arises as to whether the amount of such deflection can be so great that the passage of two meteorites near to one another ought to be estimated as an encounter in the kinetic theory.

We shall now, therefore, find the deflection of two meteorites, moving with the mean relative velocity, when they pass so close as just to graze one another.

The mean square of relative velocity in the isothermal portion is $2v_0^2$, and this may be taken as the square of the velocity at infinity in the relative hyperbola described The angle between the asymptotes of the hyperbola is the deflection due to this sort of encounter.

Let α, ϵ be the semi-axis and eccentricity of the hyperbola. Then, if ϵ be large, the angle between the asymptotes is $1/\epsilon$; and, if $\frac{1}{2}s$ be the radius of either meteorite, the pericentral distance (when they graze) is s. Therefore,

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* M. RITTER gives ·741 in place of ·755, but, as already remarked, he uses a different value for the ratio of the specific heats.

$$s = \alpha (\epsilon - 1).$$

By the law of central orbits

$$2v_0^2 = \frac{\mu m}{\alpha}.$$

Therefore,

$$\epsilon = \frac{2v_0^2 s}{\mu m} + 1.$$

But, since $v_0^2 = \beta^2 \mu M/a$, we have

$$\epsilon = 2\beta^2 \frac{Ms}{ma} + 1.$$

The unity on the right-hand side is negligible, and, since $180/\pi\epsilon$ is the deflection in degrees, that deflection is

$$\frac{180}{2\pi\beta^2} \frac{ma}{Ms} \text{ degrees.}$$

Now, if δ be the density of the body of a meteorite, $m = \frac{1}{6}\pi\delta s^3$, and, therefore, this expression becomes

$$\frac{15^\circ}{\beta^2} \times \frac{\delta a s^2}{M}.$$

Let us find what s must be if the deflection is 10°; we have

$$s = \beta \sqrt{\frac{2M}{3a\delta}}.$$

We may, for a rough evaluation, take β as unity instead of $\sqrt{6/5}$, and suppose a to be equal to the distance of Neptune from the Sun (viz., $4{\cdot}5 \times 10^{14}$ cm.), and, as a very high estimate of the value of δ, let us suppose the density of a meteorite is 10. Then, since the Sun, $M_0 = 2 \times 10^{33}$ grammes, and M is about a half of the Sun's mass, we have

$$s = \left[\frac{2 \times 10^{33}}{3 \times 4{\cdot}5 \times 10^{14} \times 10}\right]^{\frac{1}{2}} = (15 \times 10^{16})^{\frac{1}{2}} = 4 \times 10^{8}.$$

Hence, $m = \frac{1}{6}\pi\delta s^3 = \frac{1}{6}\pi \times 10 \times 64 \times 10^{24} = 3 \times 10^{26}$ grammes, in round numbers.

But the Earth's mass is 6×10^{27} grammes, and therefore the meteorites are one-twentieth of the mass of the Earth.

It follows, therefore, that, with such small masses as those with which the present theory deals, the deflection due to gravity is insensible, and we need only estimate actual impacts as encounters.

Hence, the radius of the sphere of action of a meteorite is identical with the diameter of its body.

§ 9. *On the Criterion for the Applicability of Hydrodynamics to a Swarm of Meteorites.*

The question at issue is to determine within what limits the quasi-gas formed by a swarm of colliding meteorites may be treated as a plenum, subject to the laws of hydrodynamics. The doctrines of the nebular hypothesis depend on the stability of a rotating mass of fluid, and that stability depends on the frequencies of its gravitational oscillations. Now the works of POINCARÉ and others seem to show that instability, at least in a homogeneous fluid, first arises from one of the graver modes of oscillation, and the period of the gravest mode does not differ much from the period of a satellite grazing the surface of the mass of fluid. Then, in order that hydrodynamical treatment should be applicable for the discussion of such questions of stability, the mean free time between collisions must be small compared with the period of such a satellite. Another way of stating this is that the mean free path of a meteorite shall be but little curved, and that the velocity of a meteorite shall be but little changed by gravity in the interval between two collisions. This must be fulfilled not only at the limits of the swarm, but at every point of it. The condition above stated will be satisfied if the space through which a meteorite falls from rest, at any part of the swarm, in the mean interval between collisions is small compared with the mean free path. If this criterion is fulfilled, then, in most respects which we are likely to discuss, the swarm will behave like a gas, and we must at present confine the consideration of the matter to this general criterion.

It would be laborious to determine exactly the space fallen through from rest, because gravity varies as the meteorite falls, but a sufficiently close approximation may be found by taking gravity constant throughout the fall and equal to its value at the point from which the meteorite starts.

We have already denoted by g the value of gravity at any part of the swarm, and have tabulated it in Table III. in terms of G or $\mu M/a^2$.

Now the mean interval is $T = L/(v\sqrt{8/3\pi})$. Hence, if D be the distance fallen in this time,

$$D = \tfrac{1}{2}\, gT^2 = \tfrac{1}{2}\frac{gL^2}{v^2} \cdot \frac{3\pi}{8}\,.$$

But

$$L = l\left(\frac{M_0}{M}\right)\frac{\frac{1}{3}\rho}{w} \cdot \frac{m}{s^2} \quad \text{and} \quad \frac{G}{v_0^2} = \frac{1}{\beta^2 a}\,.$$

Therefore,

$$\frac{D}{L} = \frac{l}{2a} \cdot \frac{M_0}{M} \cdot \frac{m}{s^2} \cdot \left\{\frac{3\pi}{8\beta^2} \cdot \frac{[g/G]}{[v^2/v_0^2][w/\frac{1}{3}\rho]}\right\} = \frac{l}{2a} \cdot \frac{M_0}{M} \cdot \frac{m}{s^2} \cdot F_1\,. \qquad . \quad . \quad . \quad (37)$$

The factor F_1 has been tabulated above in Table III., and it increases from the centre to the outside.

This criterion may be regarded from another point of view, for, if the meteorite be describing a circular orbit about the centre of the swarm, D is the deflection from the straight path in the mean interval between two collisions. Then the criterion is that the deflection shall be small compared with the mean free path.

We may consider the criterion from again another point of view, and state that the arc of circular orbit described in the mean interval shall be a small fraction of the whole circumference.

The linear velocity v in the circular orbit is given by

$$v^2 = g\frac{a}{x} = \frac{v_0^2}{\beta^2 x} \cdot \frac{g}{G} \cdot$$

And the mean interval $T = L/[v\sqrt{8/3\pi}]$. Hence, if A be the arc described with velocity v in time T,

$$A^2 = \frac{3\pi}{8\beta^2} \cdot \frac{L^2}{v^2/v_0^2} \cdot \frac{g}{Gx} = \frac{L^2}{v^2/v_0^2} \cdot \frac{g}{Gx} \text{ nearly, since } \frac{3\pi}{8\beta^3} = \cdot 988.$$

But the whole arc of circumference C is $2\pi a/x$.

Therefore,

$$\frac{A}{C} = \frac{L}{2a} \cdot \frac{1}{\pi} \cdot \frac{[g/G]^{\frac{1}{2}} x^{\frac{1}{2}}}{v/v_0}$$

$$= \frac{l}{2a} \cdot \frac{M_0}{M} \cdot \frac{m}{s^2} \cdot \left\{ \frac{1}{\pi} \cdot \frac{[g/G]^{\frac{1}{2}} x^{\frac{1}{2}}}{[v/v_0][w/\frac{1}{3}\rho]} \right\} = \frac{l}{2a} \cdot \frac{M_0}{M} \cdot \frac{m}{s^2} \cdot F_2 . \quad . \quad . \quad . \qquad (38)$$

The factor F_2 has been tabulated above, in Table III.

§ 10. *On the Density of Meteorites and Numerical Application.*

It is necessary to make assumptions both as to the mass and the density of the meteorites. We have a right to assume, I think, that the density δ is a little less than that of iron, say about 6, and we may put $\frac{4}{3}\pi\delta$ equal to 25. Then we have

$$m = \tfrac{1}{6}\pi\delta s^3 = \tfrac{25}{8}s^3, \text{ and } \frac{m}{s^2} = \tfrac{25}{8}s.$$

There is but little information about the average size of meteorites; but, if we retain the symbol s, it will be easy, by merely shifting the decimal point in the final results, to obtain results for all sizes. Thus, if $s = 1$ cm., $m = 3\frac{1}{8}$ grammes; if $s = 10$ cm., $m = 3\frac{1}{8}$ kilogrammes; if $s = 100$ cm., $m = 3\frac{1}{8}$ tonnes, and if $s = 1000$ cm., $m = 3125$ tonnes. I shall, therefore, keep s in the analytical formulæ, and put it equal to unity in the numerical results.

In the first place, making no assumptions as to the density or masses of the meteorites, we have

$$M_0 = 2{\cdot}1767 \times M, \quad \tfrac{1}{3}\beta^2 = \cdot 39723.$$

Then, by substitution in (10) and (11), we have

$$\left.\begin{aligned} v_0 &= u \times 10^{9{\cdot}86918-10} \\ L &= l \times 10^{0{\cdot}33781} \frac{m/s^2}{w/\frac{1}{3}\rho} \\ T &= \tau \times 10^{0{\cdot}46863} \frac{m/s^2}{[v/v_0]\,[w/\frac{1}{3}\rho]} \\ \frac{D}{L} &= \frac{l}{2a} \times 10^{0{\cdot}33781} \times F_1 \times \frac{m}{s^2} \\ \frac{A}{C} &= \frac{l}{2a} \times 10^{0{\cdot}33781} \times F_2 \times \frac{m}{s^2} \end{aligned}\right\} \quad \ldots\ldots \quad (39)$$

We will now apply this solution to a case which will put the theory to a severe test. Suppose that the limit of the sphere of uniformly distributed energy of agitation is nearly as far as the planet Uranus, so that, say $a = 16a_0$. Then the extreme limit of the swarm is at $44\frac{1}{2}a_0$; but the orbit of the planet Neptune is at $30a_0$, so that the limit is further beyond Neptune than Saturn is from the Sun.

Now, if $a/a_0 = 16$, I find

$$\left.\begin{aligned} u &= 10^{5{\cdot}86117} \text{ cm. per sec.} \\ &= 10^{\cdot 18795} a_0 \text{ per annum,} \\ \tau &= 10^{4{\cdot}46808} \text{ seconds} \\ &= 10^{6{\cdot}96899-10} \text{ years} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (40)$$

Introducing these values in (39) and putting $2\frac{5}{8}s$ for m/s^2, I find

$$\left.\begin{aligned} v &= 1{\cdot}141a_0 \text{ per annum} = 5{\cdot}374 \text{ kilom. per sec.} \\ \frac{L}{a_0} &= 10^{7{\cdot}9540-10} \times \frac{s}{[w/\frac{1}{3}\rho]} \\ T &= 10^{7{\cdot}9325-10} \times \frac{s}{[v/v_0]\,[w/\frac{1}{3}\rho]} \\ \frac{D}{L} &= 10^{6{\cdot}4489-10} s F_1 \\ \frac{A}{C} &= 10^{6{\cdot}4489-10} s F_2 \end{aligned}\right\} \quad \ldots\ldots \quad (41)$$

Now we have in Table III. the logarithms of the several factors, which occur last in these formulæ (41), at various distances from the centre.

It will suffice for our purpose only to take every other value from Table III. The distances from the centre are expressed in terms of the astronomical unit distance, viz., the Earth's mean distance from the Sun. The mean free path is expressed both in the same unit and in kilometres; and the mean intervals between collisions in days. The criteria D/L and A/C are, of course, pure numbers. Table IV., as it stands, is applicable to meteorites weighing $3\frac{1}{8}$ grammes, but by shifting the decimal point one place to the right in the last four rows of entries it becomes kilogrammes, one more and it becomes tonnes, and another, thousands of tonnes, and so on.

IV.—Table of Results.

The meteorites weigh $3\frac{1}{8}$ grammes, and have the density of iron. The swarm extends to $44\frac{1}{2}a_0$, a_0 being Earth's distance from Sun.

	Sun.	Asteroids.	Saturn.			Uranus.		Neptune.	
Distance from centre $\frac{r}{a_0} =$	0	2·55	7·66	12·77	16	19·2	24	32	$44\frac{1}{2}$
Velocity of mean square in kilometres per sec. $v =$	5·37	5·37	5·37	5·37	5·37	4·85	4·12	2·93	0
Mean free path, $\frac{L}{a_0} =$	·00028	·00038	·00157	·00511	·00900	·0122	·0199	·0552	∞
L kilom. $=$	41,600	57.000	233,000	760,000	1,340,000	1,810,000	2,960,000	8,210,000	∞
Mean free time, in days $T =$	·097	·133	·545	1·78	3·13	4·70	9·02	35·17	∞
Criterion, $\frac{D}{L} =$	..	·0000167	·0000832	·000195	·000278	·000387	·000709	·00279	∞
Criterion, $\frac{A}{C} =$	..	·0000113	·0000295	·0000633	·0000895	·000111	·000174	·000497	∞

The incidence of the several planets in the scale of distance is roughly indicated by the names written above.

The criteria show that, if the meteorites weigh $3\frac{1}{8}$ kilogrammes, the collisions are frequent enough, even beyond the orbit of Neptune, to allow the kinetic theory of gases to be applicable for such problems as are in contemplation. For, when $r/a = 32$, the two criteria (with decimal point shifted one place to the right) are ·028 and ·005, both small fractions. But, if the meteorites weigh $3\frac{1}{8}$ tonnes, the criteria cease to be very small, about $r/a = 24$. If they weigh 3125 tonnes, the applicability will cease somewhat beyond where the asteroids now are.

I conclude, then, from this discussion that we are justified in applying hydrodynamical treatment to a swarm of meteorites from which the solar system originated, even in the earliest stages of the history of the swarm.

This discussion has, of course, no bearing on the fundamental hypothesis that meteorites *can* glance from one another on impact with a virtually high degree of elasticity; nor does it do anything to justify the assumption that a swarm will consist approximately of a quasi-isothermal nucleus with a quasi-adiabatic layer over it. This latter assumption I have been led to by the considerations to which we now pass.

§ 11. *On the Diffusion of Kinetic Energy and on the Viscosity.*

In order to discuss these questions, it will be well to begin with a simple case of fluid motion.

Consider two-dimensional motion, in which there are a number of streams of equal breadth moving parallel to y with velocity V, and, interpolated between them, let there be strata of quiescent fluid; suppose then that we wish to find the motion at any time after this initial state. Let the boundaries of the streams V be from $x = ml$ to $\frac{1}{2}(2m+1)l$. Then, if u be the velocity at x, parallel to y at time t, and ν the kinetic modulus of viscosity, the equation of motion is

$$\frac{du}{dt} = \nu\frac{d^2u}{dx^2}.$$

The solution of this being of the form $e^{-p^2t}\cos px$, the complete solution satisfying the initial condition is—

$$u = \tfrac{1}{2}V + \frac{2V}{\pi}\left[e^{-\pi^2\nu t/l^2}\cos\frac{\pi x}{l} - \tfrac{1}{3}e^{-9\pi^2\nu t/l^2}\cos\frac{3\pi x}{l} + \tfrac{1}{5}e^{-25\pi^2\nu t/l^2}\cos\frac{5\pi x}{l} - \dots\right].$$

Now, if we refer time to a period τ, where $\tau = l^2/\pi^2\nu$, then after a time $\theta\tau$, which is greater than τ, the solution is sensibly

$$u = \tfrac{1}{2}V\left[1 + \frac{4}{\pi e^\theta}\cos\frac{\pi x}{l}\right].$$

It is clear that the maximum of u occurs when $x = 0$, and the minimum when $x = l$, and that they are

$$\tfrac{1}{2}V\left[1 \pm \frac{4}{\pi e^\theta}\right].$$

Hence, the difference between the maximum and minimum is $4V/\pi e^\theta$. Therefore, the ratio of the greatest difference of velocities after time $\theta\tau$ to the initial difference of velocities is $4/\pi e^\theta$. When θ is 1, 2, 3, this ratio assumes the values 1/2·135, 1/5·804, 1/15·73 respectively. Thus, after three times the interval τ, the difference of velocities is small. The time τ may be therefore taken as a convenient measure of viscosity.

In our problem the streams must be taken of a width comparable with the linear dimensions of the solar system. I therefore take l, the width of the streams, as equal to a_0, the Earth's distance from the Sun, and we have

$$\tau = \frac{a_0^2}{\pi^2\nu}.$$

Now, according to the kinetic theory of gases, the kinetic modulus of viscosity is $1/\pi$ into the mean free path multiplied by the mean velocity. Hence,

$$\nu = \frac{1}{\pi} L \left(v \sqrt{\frac{8}{3\pi}} \right).^{*}$$

Hence, we have

$$\tau = \left(\frac{a_0}{L}\right)\left(\frac{a_0}{v}\right) \sqrt{\frac{3}{8\pi}}. \qquad (42)$$

If we apply this formula to the solution which has been already found in Table IV., we obtain the following results :—

$\frac{}{a_0}$	0,	2·55,	7·66,	12·77,	16,	19·2,	24,	32,
τ years	1082,	792,	193,	59·2,	33·7,	27·5,	19·8,	61·7.

These results are applicable to meteorites weighing $3\frac{1}{8}$ grammes in a swarm extending to $44\frac{1}{2}$ a_0. If the meteorites weigh $3\frac{1}{8}$ kilogrammes, the values of τ would be one-tenth of the tabulated values. If the streams were ten times as broad, the periods would be a hundred times as long.

Now the periods τ in the above table, even if multiplied by a thousand, must be considered as short in the history of a stellar system. It thus appears that the quasi-viscosity must be such that a swarm of meteors must, if revolving, move nearly without relative motion of its parts, at least in the early stages of its evolution.

But let us consider the values of τ at different epochs in the history of the same system. If a be the radius of the isothermal sphere the formulæ (9) and (10) show that L/a_0 varies as a^3, whilst v/a_0 varies as $a^{-\frac{1}{2}}$. Hence τ varies inversely as $a^{\frac{5}{2}}$. Thus, as the swarm contracts, the periods τ increase rapidly.

Thus, later in the history, the viscosity will probably fall off so much that equalisation of angular velocity may be no longer attained, and we should then have the central portion rotating more rapidly than the outside, with a gradual transition from one angular velocity to the other.

The modulus ν gives, besides the viscosity, the rate of equalisation of the kinetic energy of agitation ; this corresponds in a true gas with the conduction of heat. The conclusion at which we thus arrive appears to justify the assumption that the whole of the central part of the swarm is endued with uniform kinetic energy of agitation, and that the mass of the quasi-isothermal nucleus is the greatest possible. With regard to the assumption that the nucleus is coated with a layer in adiabatic or convective equilibrium, it may be remarked that the velocity of agitation must decrease when we get to the outskirts of the swarm, and convective equilibrium will probably satisfy the conditions of the case better than any other. Further considerations will be adduced on this point in the Summary.

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[*] MEYER. 'Kinetische Theorie der Gase.' p. 321. The $1/\pi$ is derived from a numerical quadrature which gives the value ·318, and it is apparently only accidentally equal to $1/\pi$. The $v\sqrt{(8/3\pi)}$ is the mean velocity denoted Ω by MEYER.

§ 12. *On the Rate of Loss of Kinetic Energy through Imperfect Elasticity, and on the Heat Generated.*

In a collision between two meteorites the loss of energy is probably proportional to their relative kinetic energy before impact. Therefore, the amount of heat generated by a single meteorite per unit time is proportional to the kinetic energy (say h) and to the frequency of collision. By (10) the frequency, or reciprocal of T, varies as vws^2/m; but $m^{\frac{1}{2}}v$ is equal to $(2h)^{\frac{1}{2}}$, and s^2m^{-1} varies as $m^{-\frac{1}{3}}$. Hence, the frequency of collision varies as $h^{\frac{1}{2}}wm^{-\frac{5}{6}}$, and the amount of heat generated by a single meteorite per unit time varies as $h^{\frac{3}{2}}wm^{-\frac{5}{6}}$. But, if p be the quasi-hydrostatic pressure, p varies as hwm^{-1}, and, therefore, the heat generated by a single meteorite varies as $h^{\frac{1}{2}}pm^{\frac{1}{6}}$.

Then, to find the total heat generated per unit time and volume, we have to multiply this by the number of meteorites per unit volume, that is to say, by wm^{-1}, which is equal to $3ph^{-1}$.

Thus the amount of heat generated per unit time and volume is proportional to $p^2m^{\frac{1}{6}}h^{-\frac{1}{2}}$. With meteorites of uniform size, and with uniform kinetic energy of agitation, this becomes simply the square of the hydrostatic pressure.

The mean temperature of the gases volatilised by collisions must depend on a variety of considerations, but it would seem as if the temperature would follow, more or less closely, the variations of heat generated per unit time and volume.

§ 13. *On the Fringe of a Swarm of Meteorites.*

The law of distribution of meteorites found above depends on the frequency of collisions. But at some distance from the centre collisions must have become so rare that the statistical method is inapplicable. There must then be a sort of fringe to the swarm, which I attempt to represent by supposing that beyond a certain radius a (not the same as the former a) collisions never occur, and each meteorite describes an orbit under gravity.

Now, at any point gravity depends on the mass of all the matter lying inside a sphere whose radius is equal to the distance of that point from the centre of the swarm. Hence, the value of gravity depends on the law of density of distribution of the meteorites, which is the thing which we are seeking to discover.

We suppose, then, that from every point of a sphere of radius a a fountain of meteorites is shot up, at all inclinations to the vertical, and with velocities grouped about a mean velocity, according to the exponential law appropriate to the case. As many meteorites are supposed to fall back on to the surface as leave it, and this inward cannonade against the boundary of the sphere exactly balances the quasi-gaseous pressure on the inside of the sphere. Thus, the ideal surface may be annihilated. Since the falling half of the orbit of a meteorite is the facsimile of the rising half, we need only trace the body from projection to apocentre, and then double the distribution of density which is deduced on the hypothesis that all the meteorites are rising. Again, since every element of the sphere shoots out a similar fountain, and since collisions are precluded by hypothesis, we need only consider the velocity along the radius vector. As far as concerns the distribution of density, it is the same as if each element shot up a vertical fountain; but, of course, in determining the vertical velocity, we must pay attention to the inclination to the vertical at which the meteorite was shot out.

The mass of the matter inside the sphere, whose attraction affords the principal part of the force under which the meteorites move, is say M, and, for the sake of simplicity of notation, we shall take $2\mu M/a$ as being unit square of velocity.

Now, let $\frac{1}{2}\phi(r)$ be the potential at the point whose radius is r, and suppose that a meteorite is shot out from a point on the sphere with a velocity u, and at an inclination ϵ to the vertical; then, if r, θ be the radius vector and longitude of the meteorite at the time t, the equations of conservation of moment of momentum, and of energy are—

$$r^2 \frac{d\theta}{dt} = ua \sin \epsilon,$$

$$\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2 - \phi(r) = u^2 - \phi(a).$$

If we write $f(r) = \phi(a) - \phi(r)$, and eliminate $d\theta/dt$, we get

$$r^2 \frac{dr}{dt} = r\{r^2(u^2 - f(r)) - u^2a^2 \sin^2 \epsilon\}^{\frac{1}{2}}.$$

Now, we are to regard dr/dt as the vertical velocity in a fountain squirting up from a point on the sphere. Then, since $f(a) = 0$, it follows that at the foot of the fountain $r^2\, dr/dt$ is equal to $a^2u \cos \epsilon$. If, therefore, δ be the density at the height r, and δ_0 at the foot, the equation of continuity is

$$\delta\, r^2 \frac{dr}{dt} = \delta_0\, a^2u \cos \epsilon.$$

Therefore,

$$\frac{\delta}{\delta_0} = \frac{a^2u \cos \epsilon}{r\{r^2(u^2 - f(r)) - u^2a^2 \sin^2 \epsilon\}^{\frac{1}{2}}}.$$

But now let us suppose that the meteorites are not only shot out at inclination ϵ, but at all possible inclinations from 0° and 90°. It is then clear that this expression must be multiplied by $\sin \epsilon\, d\epsilon$, and integrated. Hence, if δ now denotes the integral density,

$$\delta = C\int_0^{\epsilon_0} \frac{a^2u \cos \epsilon \sin \epsilon\, d\epsilon}{r\{r^2(u^2 - f(r)) - u^2a^2 \sin^2 \epsilon\}^{\frac{1}{2}}},$$

where C is a constant which it will be unnecessary to determine, and where the limit ϵ_0 will be the subject of future consideration.

Effecting the integration, we have

$$\delta = -\frac{C}{ur}\{r^2(u^2-f(r)) - u^2a^2\sin^2\epsilon\}^{\frac{1}{2}}, \text{ between limits,}$$

$$= \frac{C}{ur}\left[\{r^2(u^2-f(r))\}^{\frac{1}{2}} - \{r^2(u^2-f(r)) - u^2a^2\sin^2\epsilon_0\}^{\frac{1}{2}}\right].$$

It is obvious that, if u^2 is greater than $r^2f(r)/(r^2-a^2)$, the square root involved in dr/dt does not vanish for any value of ϵ; and, hence, we must simply take $\epsilon_0 = 90°$. If, on the other hand, u^2 is less than this critical value, ϵ_0 is that value of ϵ which makes dr/dt vanish.

Thus, our formula divides into three, viz. :—

1st. u^2 greater than $\dfrac{f(r)}{1-a^2/r^2}$;

$$\delta = \frac{C}{u}\left[(u^2-f(r))^{\frac{1}{2}} - \left(1-\frac{a^2}{r^2}\right)^{\frac{1}{2}}\left(u^2-\frac{f(r)}{1-a^2/r^2}\right)^{\frac{1}{2}}\right].$$

2nd. u^2 less than $\dfrac{f(r)}{1-a^2/r^2}$;

$$\delta = \frac{C}{u}(u^2-f(r))^{\frac{1}{2}}.$$

3rd. u^2 less than $f(r)$; $\delta = 0$.

The physical meaning of this division is as follows: If we take a station near the surface of the sphere, meteorites shot out at all inclinations, even horizontally, reach the height of our station; and, when they are shot out horizontally, $\epsilon = 90°$. If we go, however, to a higher region, there is a certain inclination which just brings the meteorites at apocentre, where $dr/dt = 0$, to our height; but those shot out more nearly horizontally fail to reach us. Still higher, not even a meteorite shot up vertically can reach us, and the density vanishes.

These results only correspond to a single velocity u; but, if v^2 be the mean square of the velocity, the number of meteorites whose velocities range between u and $u+du$ is proportional to $u^2e^{-3u^2/2v^2}du$.* Hence, we have to multiply δ by this expression, and integrate from $u = \infty$ to $u = 0$.

Now, the first term of the first form for δ is the same as the second form; and in the third form δ is zero; hence, this first term when multiplied by the exponential has to be integrated from $u^2 = \infty$ to $f(r)$. The second term of the first form of δ has to be multiplied by the exponential, and integrated from $u^2 = \infty$ to $f(r)/(1-a^2/r^2)$.

Now, for the first term put

$$\frac{3}{2v^2}(u^2-f(r)) = x^2;$$

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* OSKAR MEYER, 'Die Kinetische Theorie der Gase,' 1877, pp. 271-2.

therefore,

$$u\left(u^2 - f(r)\right)^{\frac{1}{2}} du = \left(\frac{2v^2}{3}\right)^{\frac{3}{2}} x^2\, dx,$$

and the limits of x are ∞ to 0.

Hence, the first term is

$$C\left(\frac{2v^2}{3}\right)^{\frac{3}{2}} e^{-3f(r)/2r^2} \int_0 x^2 e^{-x^2}\, dx$$

Again, for the second term put

$$\frac{3}{2v^2}\left(u^2 - \frac{f(r)}{1 - a^2/r^2}\right) = x^2,$$

and similarly introduce it into the second term, and we have

$$- C\left(\frac{2v^2}{3}\right)^{\frac{3}{2}}\left(1 - \frac{a^2}{r^2}\right)^{\frac{1}{2}} e^{3f(r)/2v^2(1 - a^2/r^2)} \int_0^\infty x^2 e^{-x^2}\, dx.$$

From these expressions we may omit the constant factors; and, if w be the density at height r, whilst w_0 is the density at the sphere,

$$\frac{w}{w_0} = e^{-3f(r)/2v^2} - \left(1 - \frac{a^2}{r^2}\right)^{\frac{1}{2}} e^{-3f(r)/2v^2(1 - a^2/r^2)}.$$

In this formula unit square of velocity is $2\mu M/a$; but we have elsewhere written $v^2 = \beta^2\mu M/a$; hence, if the special unit of velocity be given up, we may write β^2 in place of $2v^2$, and the result becomes

$$\frac{w}{w_0} = e^{-3f(r)/\beta^2} - \left(1 - \frac{a^2}{r^2}\right)^{\frac{1}{2}} e^{-3f(r)/\beta^2(1 - a^2/r^2)}. \quad . \quad . \quad . \quad . \quad . \quad (43)$$

It is interesting to observe the connection between this law of density and that which would have held if the gaseous law (due to collisions) had obtained. In that case, since $\frac{1}{2}\phi(r)$ is the potential, we should have had

$$\frac{1}{w}\frac{dp}{dr} - \frac{1}{2}\frac{d}{dr}\phi(r) = 0.$$

Now, $p = \frac{1}{3}v^2 w$, and, therefore,

$$\log w - \frac{3}{2v^2}\phi(r) = \text{const.}$$

$$= \log w_0 - \frac{3}{2v^2}\phi(a).$$

Thus,

$$\log\frac{w}{w_0} = -\frac{3}{2v^2}\left[\phi(a) - \phi(r)\right] = -\frac{3}{2v^2} f(r),$$

or

$$\frac{w}{w_0} = e^{-3f(r)/2v^2} = e^{-3f(r)/\beta^2}.$$

The first term of our result, then, is exactly that resulting from the gaseous law, and the second subtractive term represents the action of the diminished velocity with which the meteorites move in the higher regions, when they are liberated from the equalising effects of continual impacts.

By previous definition, $\mu M f(r)/a$ is the excess of the potential at radius a above its value at radius r; hence,

$$\frac{\mu M}{a} f(r) = \int_a^r \frac{4\pi\mu}{r^2}\int_0^r wr^2\,dr\,.\,dr.$$

Now, since $f(r)$ is only required for values of r greater than a, we may put w equal to its mean value ρ, between the limits 0 and a. Thus,

$$\int_0^r wr^2\,dr = \int_a^r wr^2\,dr + \tfrac{1}{3}\rho a^3.$$

Hence,

$$f(r) = \frac{4\pi a}{M}\left[\int_a^r \frac{1}{r^2}\int_a^r wr^2\,dr + \tfrac{1}{3}\rho a^3 \int_a^r \frac{dr}{r^2}\right] = \left(1 - \frac{a}{r}\right) + 3\int_1^{r/a} \frac{1}{z^2}\int_1^z \frac{w}{\rho}\,z^2\,dz.$$

If this form for $f(r)$ were substituted in (43), we should obtain a very complicated differential equation for w. We may, however, find two values of $f(r)$ within which the truth must lie.

First, if we neglect the attraction of all the matter lying outside of radius a, the second term vanishes, and we have,

$$f(r) = 1 - \frac{a}{r};$$

and the law of density is

$$\frac{w}{w_0} = e^{-3(1-a/r)\beta^2} - \left(1 - \frac{a^2}{r^2}\right)^{\frac{1}{2}} e^{3(1+a/r)\beta^2}. \quad . \quad . \quad . \quad . \quad . \qquad (44)$$

Secondly, we may suppose the density to go on diminishing according to the inverse square of the distance. We have seen in the preceding solution and tables that this is roughly the law of diminution for a long way outside the isothermal nucleus. According to this assumption, $w = w_0\,a^2/r^2$. Hence, in the second term of $f(r)$ we put $w = w_0\,a^2/r^2 = w_0/z^2$.

Hence,

$$\int_1^z \frac{w}{\rho} z^2\,dz = \frac{w_0}{\rho}\int_1^z dz = \frac{w_0}{\rho}(z-1),$$

and

$$f(r) = 1 - \frac{a}{r} + \frac{w_0}{\frac{1}{3}\rho}\int_1^{r/a} \frac{z-1}{z^2}\,dz = \left(1 - \frac{w_0}{\frac{1}{3}\rho}\right)\left(1 - \frac{a}{r}\right) + \frac{w_0}{\frac{1}{3}\rho}\log\frac{r}{a}. \quad . \qquad (45)$$

The substitution of this value in (43) gives the law of density.

In order to see the kind of results to which these formulæ lead, let us suppose that, when we have reached radius 2 in the adiabatic layer, collisions have become so rare as to be negligible. Then the symbols in the formulæ of this section have numerical values; and, in order to distinguish them, let them be accented, so that, for example, we write a', β'^2, ρ', &c., in (43), (44), and (45), in place of a, β^2, ρ.

Now Table III. shows that, when $a' = 2a$, $M' = 1{\cdot}95\,M = 2M$ nearly. Hence, $M'/a' = M/a$ nearly. But, at radius $2a$ in Table III., $v^2/v_0^2 = {\cdot}298 = {\cdot}3$, and this v^2 is what we now write v'^2 or $\beta'^2\mu M'/a'$, whilst $v_0^2 = \beta^2\mu M/a$.

But $\beta^2 = \frac{6}{5}$ very nearly; hence, $\beta'^2/\beta^2 = {\cdot}3$, or $\beta'^2 = {\cdot}36$.

Thus, $3/\beta'^2 = 8{\cdot}333$.

Then, substituting $2a$ for a', and noticing that in Table III., $w/\frac{1}{3}\rho = {\cdot}163$, when $r = 2a$, the first law of density (44) becomes

$$\frac{w}{\frac{1}{3}\rho} = {\cdot}163\left[e^{-\frac{25}{3}(1-2a/r)} - \left(1 - 4\frac{a^2}{r^2}\right)^{\frac{1}{2}} e^{-\frac{25}{3}(1+2a/r)}\right]. \quad . \quad . \quad . \quad (46)$$

Again, since $M' = 2M$, and $a' = 2a$, $\rho' = \frac{1}{4}\rho$, $\frac{w_0'}{\frac{1}{3}\rho'} = 4 \times \frac{w_0'}{\frac{1}{3}\rho} = 4 \times {\cdot}163$ by Table III., and $\frac{w_0'}{\frac{1}{3}\rho'} = {\cdot}65 = \frac{2}{3}$ nearly.

Thus, according to the second assumption, we have by (45)

$$3f(r) = \left(1 - \frac{2a}{r}\right) + \log\left(\frac{r}{2a}\right)^2, \text{ and, since } \frac{1}{\beta'^2} = 2{\cdot}78,$$

$$\left.\begin{aligned} \frac{3f(r)}{\beta'^2} &= 2{\cdot}78\left(1 - \frac{2a}{r}\right) + 2{\cdot}78\log\left(\frac{r}{2a}\right)^2, \\ \frac{3f'(r)}{\beta'^2(1 - 4a^2/r^2)} &= \frac{2{\cdot}78}{1 + 2a/r} + \frac{2{\cdot}78}{1 - 4a^2/r^2}\log\left(\frac{r}{2a}\right)^2; \end{aligned}\right\}$$

and the law of density is

$$\frac{w}{\frac{1}{3}\rho} = {\cdot}163\left[e^{-3f(r)/\beta'^2} - \left(1 - \frac{4a^2}{r^2}\right)^{\frac{1}{2}} e^{-3f'(r)/\beta'^2(1-4a^2/r^2)}\right]. \quad . \quad . \quad . \quad (47)$$

The values computed from these alternative formulæ (46) and (47) will be comparable with those in Table III.

In Table III. we have the value of $w/\frac{1}{3}\rho$ computed at distances $r/a = 2{\cdot}208$, $2{\cdot}463$, $2{\cdot}786$. The following short table gives the result extracted from Table III. for comparison with the values computed from (46) and (47):—

	$r/a =$	$2{\cdot}0$,	$2{\cdot}208$,	$2{\cdot}463$,	$2{\cdot}786$.
Convective equilib.,	$\frac{w}{\frac{1}{3}\rho} =$	·163,	·092,	·033,	0.
First hypoth. (46),	$\frac{w}{\frac{1}{3}\rho} =$	·163,	·074,	·033,	·015.
Second hypoth. (47),	$\frac{w}{\frac{1}{3}\rho} =$	·163,	·071,	·029,	·011.

It appears, therefore, that the results from the two hypotheses differ but little for some distance outside the region of collisions, and either line may be taken as near enough to the correct result. We see then that the effect of annulling collisions and allowing each body to describe an orbit is that the density at first falls off more rapidly than if the medium were in convective equilibrium, and that further away the density falls off less rapidly. At more remote distances the density would be found to tend to vary as the inverse square of this distance. Thus, the formulæ would make the mass of the system infinite. In other words, the existence of meteorites with nearly parabolic and hyperbolic orbits necessitates an infinite number, if the loss to the system is constantly made good by the supply.

The subject of this section is considered further, from a physical point of view, in the Summary at the end.

§ 14. *On the Kinetic Theory where the Meteorites are of all sizes.*

In an actual swarm of meteorites all sizes occur, for, even if this were not the case initially, inequality of size would soon arise through fractures. Hence, it becomes of interest to examine the kinetic theory on the hypothesis that the colliding bodies are of all possible sizes, grouped about some mean value according to some law of frequency.

If there be two sets of elastic spheres in such numbers that there are respectively A and B in unit volume, and if the mean squares of the velocities of the two are α^2 and β^2 respectively, and if a and b are the radii of the spheres of the two sets, then it is proved that the number of collisions between them per unit time and volume is

$$2AB(a+b)^2\left[\tfrac{2}{3}\pi(\alpha^2+\beta^2)\right]^{\frac{1}{2}}\,*$$

We shall now change the notation, and for a and b write s_1 and s_2, and for α and β write u_1 and u_2.

Then, if δ be the density of the spheres, their masses are $\frac{4}{3}\pi\delta s_1^3$ and $\frac{4}{3}\pi\delta s_2^3$.

The condition for the permanence of condition is that the spheres of all masses shall have the same mean kinetic energy. Hence, we refer the mass to a mean sphere of radius ς, and the velocity to a square of velocity V^2.

Then

$$s_1^3u_1^2 = s_2^3u_2^2 = \varsigma^3V^2.$$

Thus, our formula may be written

$$2AB(s_1+s_2)^2\left[\left(\frac{\varsigma}{s_1}\right)^3+\left(\frac{\varsigma}{s_2}\right)^3\right]^{\frac{1}{2}}\left(\tfrac{2}{3}\pi V^2\right)^{\frac{1}{2}}.$$

---

* 'The Kinetic Theory of Gases,' by H. W. WATSON, p. 11.

But now suppose that there are spheres of all possible sizes, and that in unit volume the number whose radius lies between s and $s + ds$ is

$$\frac{4n}{\sigma^3\sqrt{\pi}}\, s^2 e^{-s^2/\sigma^2}\, ds.^*$$

Since the integral of this from ∞ to 0 is n, it follows that n is the number of spheres of all sizes in unit volume.

If ρ be the total mass in unit volume, or the density of distribution,

$$\begin{aligned}\rho &= \frac{4n}{\sqrt{\pi}} \cdot \int_0^\infty \tfrac{4}{3}\pi\delta s^3 \,.\, \frac{s^2}{\sigma^3}\, e^{-s^2/\sigma^2}\, ds \\ &= \frac{4n}{\sqrt{\pi}} \cdot \tfrac{4}{3}\pi\delta\sigma^3 \int_0^\infty x^5 e^{-x^2}\, dx \\ &= \frac{4n}{\sqrt{\pi}} \cdot \tfrac{4}{3}\pi\delta\sigma^3 .\end{aligned}$$

If m be the mean mass, $m = \rho/n$; but $m = \frac{4}{3}\pi\delta\varsigma^3$; hence,

$$\varsigma^3 = \frac{4}{\sqrt{\pi}}\sigma^3,$$

and

$$\left(\frac{\varsigma}{s_1}\right)^3 + \left(\frac{\varsigma}{s_2}\right)^3 = \frac{4}{\sqrt{\pi}}\left[\left(\frac{\sigma}{s_1}\right)^3 + \left(\frac{\sigma}{s_2}\right)^3\right].$$

If the A spheres of radii s_1 are those whose radii lie between s_1 and $s_1 + ds_1$, and the B spheres of radii s_2 are those whose radii lie between s_2 and $s_2 + ds_2$,

$$\begin{aligned}A &= \frac{4n}{\sqrt{\pi}}\left(\frac{s_1}{\sigma}\right)^2 e^{-s_1^2/\sigma^2}\frac{ds_1}{\sigma}, \\ B &= \frac{4n}{\sqrt{\pi}}\left(\frac{s_2}{\sigma}\right)^2 e^{-s_2^2/\sigma^2}\frac{ds_2}{\sigma}.\end{aligned}$$

Hence, the formula for collisions between the A's and B's is

$$\frac{64n^2}{\pi^{\frac{1}{2}}} \cdot \left(\tfrac{2}{3}\pi V^2\right)^{\frac{1}{2}} . \left(s_1 + s_2\right)^2 \left[\left(\frac{\sigma}{s_1}\right)^3 + \left(\frac{\sigma}{s_2}\right)^3\right]^{\frac{1}{2}} \frac{s_1{}^2 s_2{}^2}{\sigma^4}\, e^{-(s_1^2 + s_2^2)/\sigma^2}\, \frac{ds_1}{\sigma}\frac{ds_2}{\sigma},$$

or, if we write $x = s_1/\sigma$, $y = s_2/\sigma$, it is

$$\frac{64n^2}{\pi^{\frac{1}{2}}}\left(\tfrac{2}{3}\pi V^2\right)^{\frac{1}{2}} \sigma^2 (x + y)^2 (x^3 + y^3)^{\frac{1}{2}} (xy)^{\frac{1}{2}} e^{-x^2-y^2}\, dx\, dy. \quad . \quad . \quad . \qquad (48)$$

---

[*] If the spheres are grouped about a mean mass, instead of about a mean radius, according to a law of this kind, the subsequent integrals become very troublesome. Any law of the kind suffices for the discussion. If, however, I had foreseen the investigation of § 16, I should not have taken this law of frequency.

But

$$\sigma^2 = \frac{\pi^{\frac{1}{2}}}{2^{\frac{1}{2}}}\varsigma^2, \quad \text{and} \quad \frac{64}{\pi^{\frac{5}{6}}}\left(\tfrac{2}{3}\pi\right)^{\frac{1}{4}}\frac{\pi^{\frac{1}{2}}}{2^{\frac{1}{2}}} = \frac{32}{\pi^{\frac{1}{12}}}\cdot\frac{2^{\frac{3}{4}}}{3^{\frac{1}{4}}}.$$

Hence, the number of collisions per unit time and volume between spheres whose radii range between s_1 and $s_1 + ds_1$, and others with radii between s_2 and $s_2 + ds_2$, is

$$\frac{32}{\pi^{\frac{1}{12}}}\frac{2^{\frac{3}{4}}}{3^{\frac{1}{4}}}\cdot V\varsigma^2 n^2 . (x+y)^2 (x^3+y^3)^{\frac{1}{2}} (xy)^{\frac{1}{2}} e^{-x^2-y^2}\, dx\, dy.$$

The number of collisions of a single sphere per unit time is $1/n$ of this, and, since $n = \rho/m$, we have for the collisions of a single sphere the factor $\dfrac{V\varsigma^2}{m/\rho}$ instead of $V\varsigma^2 n^2$.

Then the total number of collisions of all kinds in unit time, or the reciprocal of the mean free time, is the double integral of this from ∞ to 0.

For the purpose of carrying out the integration, we may conveniently, as an algebraic artifice, change from the rectangular axes x, y to the polar coordinates r, θ. Thus,

$$\int_0^\infty\int_0^\infty (x+y)^2 (x^3+y^3)^{\frac{1}{2}} (xy)^{\frac{1}{2}} e^{-x^2-y^2}\, dx\, dy$$
$$= \int_0^\infty r^{\frac{11}{2}} e^{-r^2}\, dr \int_0^{\frac{1}{2}\pi} (\sin\theta + \cos\theta)^{\frac{5}{2}} (1 - \sin\theta\cos\theta)^{\frac{1}{2}} (\sin\theta\cos\theta)^{\frac{1}{2}}\, d\theta.$$

Now, if we put $r = z^2$,

$$\int_0^\infty r^{\frac{11}{2}} e^{-r^2}\, dr = 2\int_0^\infty z^{12} e^{-z^4}\, dz = 2\cdot\frac{9.5.1}{4.4.4}\int_0^\infty e^{-z^4}\, dz.$$

For the transformation of the second integral, put

$$z = \cos\theta - \sin\theta,$$

and we find

$$\int_0^{\frac{1}{2}\pi} (\sin\theta + \cos\theta)^{\frac{5}{2}} (1 - \sin\theta\cos\theta)^{\frac{1}{2}} (\sin\theta\cos\theta)^{\frac{1}{2}}\, d\theta = \int_{-1}^{+1} \tfrac{1}{2}(2 - z^2)^{\frac{3}{4}} (1 - z^4)^{\frac{1}{2}}\, dz$$
$$= \int_0^{+1} (2 - z^2)^{\frac{3}{4}} (1 - z^4)^{\frac{1}{2}}\, dz.$$

Hence, the whole integral is

$$\tfrac{45}{32}\int_0^\infty e^{-z^4}\, dz \int_0^1 (2 - z^2)^{\frac{3}{4}} (1 - z^4)^{\frac{1}{2}}\, dz,$$

and the mean frequency of collision of a single ball per unit time is

$$\tfrac{15}{4}\cdot\frac{3^{\frac{3}{4}}.2^{\frac{3}{4}}}{\pi^{\frac{1}{12}}}\frac{V(2\varsigma)^2}{m/\rho}\int_0^\infty e^{-z^4}\, dz \int_0^1 (2 - z^2)^{\frac{3}{4}} (1 - z^4)^{\frac{1}{2}}\, dz.$$

The second of these two integrals cannot, I think, be evaluated algebraically, but its value is easily found by quadratures. I find, then,

$$\int_0^1 (2-z^2)^{\frac{3}{4}}(1-z^4)^{\frac{1}{4}}\,dz = 1{\cdot}2999.$$

The former of the two integrals may be evaluated as follows:—

Let

$$I = \int_0^\infty e^{-x^4}\,dx,$$

then,

$$\begin{aligned} 4I^2 &= 4\int_0^\infty\int_0^\infty e^{-x^4-y^4}\,dx\,dy \\ &= 4\int_0^\infty\int_0^{\frac{1}{2}\pi} e^{-r^4(1-\frac{1}{2}\sin^2 2\theta)}\,r\,dr\,d\theta \\ &= \int_0^\infty\int_0^\pi e^{-z^2(1-\frac{1}{2}\sin^2\phi)}\,dz\,d\phi \\ &= 2\int_0^{\frac{1}{2}\pi}\int_0^\infty \frac{e^{-t^2}}{(1-\frac{1}{2}\sin^2\phi)}\,dt\,d\phi \\ &= \pi^{\frac{1}{2}}\int_0^{\frac{1}{2}\pi}\frac{d\phi}{(1-\frac{1}{2}\sin^2\phi)^{\frac{1}{2}}} \\ &= \pi^{\frac{1}{2}}F(45^\circ), \end{aligned}$$

where F is the complete elliptic integral with modulus sin 45°.

Hence,

$$I = \tfrac{1}{2}\pi^{\frac{1}{4}}F^{\frac{1}{2}}.^*$$

We thus have the frequency of collision given by

$$\tfrac{15}{8}\cdot\frac{3^{\frac{1}{4}}.2^{\frac{1}{4}}}{\pi^{\frac{1}{4}}}\cdot F^{\frac{1}{2}}\cdot 1{\cdot}2999\cdot\frac{V(2\varsigma)^2}{m/\rho}.$$

Now, LEGENDRE's Tables give

$$\log F = {\cdot}2681272,$$

with which value we easily find for T the mean free time, or $1/T$ the frequency,

$$\frac{1}{T} = 5{\cdot}3318\,\frac{V(2\varsigma)^2}{m/\rho} = \tfrac{16}{3}\,\frac{V(2\varsigma)^2}{m/\rho}\ \text{nearly.} \qquad \ldots\ldots \quad (49)$$

If $1/T_0$ be the frequency of collision when the spheres are all of the same size and mass ς and m, and are agitated with mean square of velocity V^2, we have, by the ordinary theory,

$$\frac{1}{T_0} = 4\sqrt{\frac{\pi}{3}}\cdot\frac{V(2\varsigma)^2}{m/\rho} = 4{\cdot}0935\,\frac{V(2\varsigma)^2}{m/\rho}. \qquad \ldots\ldots \quad (50)$$

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[*] I owe this to Mr. FORSYTH, and the result verifies an evaluation by quadratures which I had made.

It follows, therefore, that in our case collisions are more frequent than if the balls were all of the same size in about the proportion of 4 to 3.

In order to find the mean free path, we require to find the mean velocity.

If u^2 be the mean square of the velocity for any size s, the proportion of all the spheres of that size which move with velocities lying between v and $v + dv$ is

$$\frac{4}{\sqrt{\pi}}\, y^2 e^{-y^2}\, dy,$$

where $y^2 = 3v^2/2u^2$.

But the number of spheres of size between s and $s + ds$, in unit volume, is

$$\frac{4n}{\sqrt{\pi}}\, x^2 e^{-x^2}\, dx,$$

where $x = s/\sigma$.

Hence, the mean velocity U is given by

$$U = \frac{16}{\pi}\int_0^\infty\int_0^\infty v x^2 y^2 e^{-x^2-y^2}\, dx\, dy.$$

Now,

$$v = \sqrt{\tfrac{2}{3}}\,.\, uy, \text{ and } s^3u^2 = \mathfrak{s}^3 V^2, \text{ or } x^3u^2 = \left(\frac{\mathfrak{s}}{\sigma}\right)^3 V^2 = \frac{4}{\sqrt{\pi}}\, V^2,$$

so that

$$u = \frac{2}{\pi^{\frac{1}{4}}}\, x^{-\frac{3}{2}} V, \text{ and } v = \frac{2\sqrt{2}}{\pi^{\frac{1}{4}}\sqrt{3}}\, x^{-\frac{3}{2}} y V.$$

Therefore,

$$U = \frac{32\sqrt{2}}{\pi^{\frac{1}{4}}\sqrt{3}}\, V\int_0^\infty\int_0^\infty x^{\frac{1}{2}} y^3 e^{-x^2-y^2}\, dx\, dy.$$

But

$$\int_0^\infty y^3 e^{-y^2}\, dy = \tfrac{1}{2}, \text{ and } \int_0^\infty x^{\frac{1}{2}} e^{-x^2}\, dx = 2\int_0^\infty z^2 e^{-z^4}\, dz\,;$$

therefore,

$$U = \frac{32\sqrt{2}}{\pi^{\frac{1}{4}}\sqrt{3}}\, V\int_0^\infty z^2 e^{-z^4}\, dz.$$

This integral may be evaluated as follows :—

Let

$$J = \int_0^\infty x^2 e^{-x^4}\, dx,$$

$$\begin{aligned}
4J^2 &= 4\int_0^\infty\int_0^\infty x^2 y^2 e^{-x^4-y^4}\, dx\, dy \\
&= \int_0^\infty\int_0^{\frac{1}{2}\pi} r^4 \sin^2 2\theta\; e^{-r^4(1-\frac{1}{2}\sin^2 2\theta)}\, r\, dr\, d\theta \\
&= \tfrac{1}{4}\int_0^\infty\int_0^{\pi} z^2 \sin^2\phi\; e^{-z^2(1-\frac{1}{2}\sin^2\phi)}\, dz\, d\phi \\
&= \tfrac{1}{2}\int_0^\infty\int_0^{\frac{1}{2}\pi} \frac{\sin^2\phi}{(1-\frac{1}{2}\sin^2\phi)^{\frac{3}{2}}}\, t^2 e^{-t^2}\, dt\, d\phi \\
&= \tfrac{1}{4}\pi^{\frac{1}{2}}\int_0^{\frac{1}{2}\pi} \frac{\frac{1}{2}\sin^2\phi}{(1-\frac{1}{2}\sin^2\phi)^{\frac{3}{2}}}\, d\phi \\
&= \tfrac{1}{4}\pi^{\frac{1}{2}}\left[\int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1-\frac{1}{2}\sin^2\phi)^{\frac{3}{2}}} - F\right].
\end{aligned}$$

Now,

$$\int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1 - k^2 \sin^2 \phi)^{\frac{3}{2}}} = \frac{E}{k'^2},$$

and in the present case $k^2 = k'^2 = \frac{1}{2}$.

Hence,

$$J = \tfrac{1}{4}\pi^{\frac{1}{2}} [2E - F]^{\frac{1}{2}}\text{*},$$

where E and F are the complete elliptic integrals with modulus sin 45°.

In LEGENDRE'S Tables, we find

$$E = 1·350644, \quad F = 1·854075, \quad \text{and } 2E - F = ·847213.$$

Then,

$$\frac{U}{V} = \frac{8}{\pi} \sqrt{\tfrac{2}{3}} \sqrt{(2E - F)} = 1·91377.$$

The mean free path

$$L = UT = 1·9138\ VT = \frac{1·9138}{5·3318} \frac{m/\rho}{(2\varsigma)^2},$$

and thus

$$L = \frac{1}{2·786} \frac{m/\rho}{(2\varsigma)^2}. \quad \ldots\ldots\ldots \quad (51)$$

If the spheres had all been of the same size, we should have had

$$L_0 = \frac{m/\rho}{\pi (2\varsigma)^2 \sqrt{2}} = \frac{1}{4·44} \frac{m/\rho}{(2\varsigma)^2}. \quad \ldots\ldots \quad (52)$$

Hence, finally from (49) to (52), if there be a number of spherical meteorites, of uniform density, of all sizes with radii grouped about a mean radius according to the law of error, and if S be the *diameter* of the meteorite of mean mass m, and ρ be the density of the distribution of meteorites in space, and $\frac{1}{2} m V^2$ their mean kinetic energy of agitation, then the mean free path L, mean free time T, and mean velocity U are given by

$$\left.\begin{aligned} L &= \frac{1}{2·786} \frac{m/\rho}{S^2} = \frac{5}{14} \frac{m/\rho}{S^2} \text{ nearly,} \\ T &= \frac{1}{5·332} \frac{m/\rho}{VS^2} = \frac{3}{16} \frac{m/\rho}{VS^2} \text{ nearly,} \\ U &= 1·9138\ V = 2V \text{ nearly.} \end{aligned}\right\} \quad \ldots\ldots \quad (53)$$

Also the mean free path is about $\frac{7}{11}$ths, and the mean free time about $\frac{3}{4}$ of that which would have held if the meteorites had all been of the same size m and had had the same mean kinetic energy $\frac{1}{2} m V^2$.

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* I owe this to Mr. FORSYTH.

§ 15. *On the Variation of Mean Frequency of Collision and Mean Free Path for the several sizes of balls.*

Each size of ball has its own mean frequency of collision and mean free path, and it is well to trace how the total means evaluated in the last section are made up.

We have already seen in (48) that (substituting for σ its value in terms of ς) the number of collisions per unit time and volume between balls of sizes s to $s + ds$ and balls of sizes s' to $s' + ds'$ is

$$\frac{64n^2}{\pi^{\frac{1}{2}}}\left(\tfrac{2}{3}\pi V^2\right)^{\frac{1}{2}}.\frac{\pi^{\frac{1}{2}}}{2^{\frac{1}{2}}}\varsigma^2(x+y)^2(x^3+y^3)^{\frac{1}{2}}(xy)^{\frac{1}{2}}e^{-x^2-y^2}\,dx\,dy,$$

where $x = s/\sigma$, $y = s'/\sigma$.

But the number of balls of size s to $s + ds$ in unit volume is

$$\frac{4n}{\sqrt{\pi}}x^2e^{-x^2}\,dx.$$

Hence, the mean frequency of collision for a ball of size s with all others is

$$\frac{\pi^{\frac{1}{2}}}{4n}\cdot\frac{64n^2}{\pi^{\frac{1}{2}}}\left(\tfrac{2}{3}\pi V^2\right)^{\frac{1}{2}}\frac{\pi^{\frac{1}{2}}}{2^{\frac{1}{2}}}\varsigma^2\int_0^\infty(x+y)^2(x^3+y^3)^{\frac{1}{2}}x^{-\frac{3}{2}}y^{\frac{1}{2}}e^{-y^2}\,dy.$$

Now,

$$x^{-\frac{3}{2}} = \left(\frac{\sigma}{s}\right)^{\frac{3}{2}} = \tfrac{1}{2}\pi^{\frac{3}{4}}.\left(\frac{\varsigma}{s}\right)^{\frac{3}{2}}.$$

Therefore, if we write $1/\tau$ for the frequency of collision of a ball of size s with all others, we have

$$\frac{1}{\tau} = \frac{2^{\frac{1}{3}}.\pi^{\frac{1}{4}}}{3^{\frac{1}{2}}}\cdot\frac{V(2\varsigma)^2}{m/\rho}\left(\frac{\varsigma}{s}\right)^{\frac{3}{2}}\int_0^\infty(x+y)^2(x^3+y^3)^{\frac{1}{2}}y^{\frac{1}{2}}e^{-y^2}\,dy.$$

Now, the mean frequency for all sizes is given by

$$\frac{1}{T} = 5\cdot3318\cdot\frac{V(2\varsigma)^2}{m/\rho}\cdot$$

Hence,

$$\frac{T}{\tau} = \frac{1}{5\cdot3318}\cdot\frac{2^{\frac{1}{3}}.\pi^{\frac{1}{4}}}{3^{\frac{1}{2}}}\cdot\left(\frac{\varsigma}{s}\right)^{\frac{3}{2}}\int_0^\infty(x+y)^2(x^3+y^3)^{\frac{1}{2}}y^{\frac{1}{2}}e^{-y^2}\,dy.$$

$$= \cdot1780\cdot\left(\frac{\varsigma}{s}\right)^{\frac{3}{2}}\int_0^\infty(x+y)^2(x^3+y^3)^{\frac{1}{2}}y^{\frac{1}{2}}e^{-y^2}\,dy. \quad . \quad . \quad . \quad . \quad . \quad (54)$$

The integral involved here cannot in general be determined algebraically; but, if x be very small, or very great, we can find an approximate value for it.

If x be very small, the integral becomes

$$\int_0^\infty y^4e^{-y^2}\,dy = \tfrac{3}{8}\sqrt{\pi}, \qquad \text{and} \qquad \frac{T}{\tau} = \cdot118\left(\frac{\varsigma}{s}\right)^{\frac{3}{2}}.$$

If x be very large, the integral becomes

$$x^{\frac{7}{2}}\int_0^\infty y^{\frac{1}{2}}e^{-y^2}\,dy = 2x^{\frac{7}{2}}\int_0^\infty z^2 e^{-z^4}\,dz.$$

Now,

$$x^{\frac{7}{2}} = \left(\frac{s}{\sigma}\right)^{\frac{7}{2}} = \frac{2^{\frac{7}{4}}}{\pi^{\frac{7}{12}}}\left(\frac{s}{\varsigma}\right)^{\frac{7}{2}}, \qquad \text{and} \qquad \int_0^\infty z^2 e^{-z^4}\,dz = \tfrac{1}{4}\pi^{\frac{1}{4}}(2E-F)^{\frac{1}{2}}.$$

Therefore, the integral becomes

$$\frac{2^{\frac{3}{4}}}{\pi^{\frac{1}{3}}}(2E-F)^{\frac{1}{2}}\left(\frac{s}{\varsigma}\right)^{\frac{7}{2}},$$

and with the known values of E and F this gives us

$$\frac{T}{\tau} = \cdot282\left(\frac{s}{\varsigma}\right)^2.$$

For intermediate values of s recourse must be had to quadratures for evaluating the integral. I have therefore determined, by a rough numerical process, sufficient values of the integral to render possible the drawing of a curve for the values of T/τ for all values of s. The following table gives the results for the integral $\int_0^\infty (x+y)^2(x^3+y^3)^{\frac{1}{2}}y^{\frac{1}{2}}e^{-y^2}\,dy$, which may be denoted by K:—

	K
$s = \frac{1}{2}\varsigma$	1·71 ,
$s = \frac{3}{4}\varsigma$	2·90 ,
$s = \varsigma$	4·94 ,
$s = \frac{3}{2}\varsigma$	12·97 ,
$s = 2\varsigma$	28·75 .

If these values be introduced in the formula

$$\frac{T}{\tau} = \cdot1780\,K\left(\frac{\varsigma}{s}\right)^{\frac{3}{2}},$$

we obtain

	T/τ
$s = \frac{1}{2}\varsigma$	·86 ,
$s = \frac{3}{4}\varsigma$	·80 ,
$s = \varsigma$	·88 ,
$s = \frac{3}{2}\varsigma$	1·26 ,
$s = 2\varsigma$	1·81 .

These values are used for forming the curve, entitled "frequency of collision," in fig. 1 below, and they are supplemented by the values found above for T/τ, in the case where s/ς is either very small or very large.

The frequency becomes infinite when the balls are infinitely small, because of the infinite velocity with which they move, and again infinite for infinitely large balls, because of their infinite size. But it must be remembered that there are infinitely few balls of these two limiting sizes.

We have now to consider the mean free path, say λ, for the several sizes.

If u^2 be the mean square of velocity for the size s, the mean velocity for that size is $u\sqrt{(8/3\pi)}$, by the ordinary kinetic theory.

From the constancy of mean kinetic energy for all sizes, we have

$$s^3 u^2 = \varsigma^3 V^2,$$

so that the mean velocity for size s is

$$V(\varsigma/s)^{\frac{3}{2}}\sqrt{(8/3\pi)}.$$

But, if U be the mean velocity, and L the mean free path, and T the mean free time for all sizes together, we have

$$V = \frac{U}{1{\cdot}9138} = \frac{1}{1{\cdot}9138}\frac{L}{T}.$$

Therefore, the mean velocity for size s is

$$\frac{\sqrt{(8/3\pi)}}{1{\cdot}9138}\left(\frac{\varsigma}{s}\right)^{\frac{3}{2}}\frac{L}{T} = {\cdot}4815\left(\frac{\varsigma}{s}\right)^{\frac{3}{2}}\frac{L}{T}\cdot$$

But the mean velocity for size s is λ/τ; hence,

$$\frac{\lambda}{\mathrm{L}} = {\cdot}4815\left(\frac{\varsigma}{s}\right)^{\frac{3}{2}}\frac{\tau}{\mathrm{T}} = \frac{4815}{1780}\cdot\frac{1}{\mathrm{K}}$$
$$= \frac{2{\cdot}705}{\mathrm{K}}\cdot$$

When s is very small, we find $\lambda/\mathrm{L} = 4$, and, when s is very large, $\lambda/\mathrm{L} = 1{\cdot}7\,(\varsigma/s)^{\frac{7}{2}}$. Thus, for small values of s, the mean free path reaches a constant limit 4, and for large values it becomes infinitely small.

The intermediate values, sufficient for drawing a curve, are given in the following short table :—

	λ/L.
$s = \frac{1}{2}\varsigma$	1·58.
$s = \frac{3}{4}\varsigma$	·93.
$s = \varsigma$	·55.
$s = \frac{3}{2}\varsigma$	·21.
$s = 2\varsigma$	·09.

result, that the mean mass in the centre of the swarm becomes infinite. The existence of very large meteorites in sufficient numbers to give statistical constancy in a volume which is not a considerable fraction of the volume of the whole swarm is physically improbable. We shall, therefore, treat the case best by absolutely excluding very large masses. When such masses occur, they must not be treated statistically; this is a question which I hope to consider in a future paper. Had I foreseen this conclusion when the investigations of the last two sections were carried out, a different law of frequency of mass would have been assumed. But the results of those sections are amply sufficient to indicate the conclusions which would have been reached with another law of frequency, and, therefore, it does not seem worth while to recompute the results by means of a fresh series of laborious quadratures.

Any law of frequency would suffice for our purpose which excludes masses greater than a certain limit and rises to a maximum for a certain mean mass. For the present, I do not specify that law precisely, but merely assume that at some radius, which may conveniently be taken as that of the isothermal sphere, where $r = a$, the number of meteorites whose masses lie between x and $x + \delta x$ is $f(x)\,\delta x$; it is also assumed that x may range from M * to zero.

The meteorites whose masses range from x to $x + \delta x$ may be deemed to constitute a gas. Suppose that at radius r the number of its molecules per unit volume is δn, its density δw, its pressure δp, and let the same symbols, with suffix 0, denote the same things at radius a. Since all the partial gases are in the permanent state, they all have the same mean kinetic energy of agitation, equal to $\frac{1}{2}h$, suppose. Throughout the isothermal sphere, this h is constant, and equal, say, to h_0, but varies with the radius in the adiabatic layer over it. It follows, therefore, that the mean square of the velocity of the particular partial gas x to $x + \delta x$ is equal to h/x, and the relation between δp and δw is

$$\delta p = \tfrac{1}{3}\frac{h}{x}\,\delta w.$$

Let $-\chi$ be the excess of the gravitation potential of the *whole* swarm at radius r above its value at radius a.

Then, since each partial gas behaves as though it existed by itself, the equation of hydrostatic equilibrium of the partial gas x to $x + \delta x$ is

$$\frac{1}{\delta w}\frac{d\,\delta p}{dr} + \frac{d\chi}{dr} = 0.$$

The investigation must now divide into two, according as whether we are considering the isothermal sphere or the adiabatic layer.

The Isothermal Sphere.

Here we have h a constant and equal to h_0, and δp varies as δw, so that

---

* This M is not to be confused with M, the mass of the isothermal sphere.

$$\tfrac{1}{3}\frac{h_0}{x}\log\frac{\delta w}{\delta w_0} = -\chi,$$

or

$$\frac{\delta w}{\delta w_0} = e^{-3\chi x/h_0}.$$

Now it is obvious that $\delta n/\delta n_0 = \delta w/\delta w_0$; and, therefore,

$$\delta n = e^{-3\chi x/h_0}\,\delta n_0.$$

But, by the definition of $f(x)$,

$$\delta n_0 = f(x)\,\delta x\,;$$

hence,

$$\delta n = e^{-3\chi x/h_0} f(x)\,\delta x. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (55)$$

This is the law of frequency of mass x to $x + \delta x$ at radius r.

Now, if m, m_0 be the mean masses at radii r and a respectively,

$$m = \frac{\int_0^{\mathrm{M}} x\, e^{-3\chi x/h_0} f(x)\, dx}{\int_0^{\mathrm{M}} e^{-3\chi x/h_0} f(x)\, dx}\,; \quad . \; . \; . \; . \; . \; . \; . \quad (56)$$

and, if we put $\chi = 0$, we obtain m_0 from the same formula.

It is also clear that, if w be the total density of the swarm at radius r,

$$w = \int x\, dn = \int_0^{\mathrm{M}} x\, e^{-3\chi x/h_0} f(x)\, dx. \quad . \; . \; . \; . \; . \; . \quad (57)$$

By the definition of χ, and in consequence of the supposed spherical arrangement of matter, we have

$$\chi = \int_a^r \frac{1}{r^2}\left(\int_o^r 4\pi\mu w r^2 dr\right) dr.$$

If this value were substituted in (57), we should obtain a very complicated differential equation to determine w, the solution of which is hopelessly difficult. We may, however, assume without much error that the w in the integral expressing χ is the density of meteorites, all of which are of the same size m', and which are agitated with mean kinetic energy $\tfrac{1}{2}h_0$. If this density be written w, we then clearly have

$$\chi = -\frac{h_0}{3m'}\log\frac{\mathrm{w}}{\mathrm{w}_0}.$$

The values of w and w_0 may be extracted from Table III. of solutions in § 6.

Then we have

$$-\frac{3\chi x}{h_0} = \frac{x}{m'} \log \frac{\text{w}}{\text{w}_0} = qx, \text{ suppose,}$$

where q is rigorously equal to $-3\chi/h_0$; but for computing the approximate value $(1/m') \log (\text{w}/\text{w}_0)$ is to be employed.

In order to proceed to the evaluation of the mean mass at various distances, we must assume some form for $f(x)$.

I assume, then, that

$$f(x) = \frac{6n_0}{\text{M}^3} x (\text{M} - x).$$

It is easy to show that

$$\int_0^{\text{M}} f(x)\, dx = n_0, \quad \text{and} \quad \frac{1}{n_0} \int_0^{\text{M}} x f(x)\, dx = \tfrac{1}{2}\text{M}.$$

Hence, the mean mass $m_0 = \frac{1}{2}\text{M}$, and the maximum frequency is for masses equal to m_0.

Then, by (56), we have for the mean mass at radius r

$$m = \frac{\int_0^{\text{M}} x^2 (\text{M} - x)\, e^{qx} dx}{\int_0^{\text{M}} x (\text{M} - x)\, e^{qx} dx}.$$

But

$$\left.\begin{aligned} \int_0^{\text{M}} x^2 (\text{M} - x)\, e^{qx} dx &= \frac{1}{q^4} [e^{\text{M}q} (\text{M}^2 q^2 - 4\text{M}q + 6) - 2(\text{M}q + 3)], \\ \int_0^{\text{M}} x (\text{M} - x)\, e^{qx} dx &= \frac{1}{q^3} [e^{\text{M}q} (\text{M}q - 2) + (\text{M}q + 2)]. \end{aligned}\right\} \quad . \quad . \quad (58)$$

It may be remarked that, if Mq be treated as small, we have the first of these integrals equal to $\frac{1}{12}\text{M}^4 (1 + \frac{3}{5}\text{M}q)$, and the second equal to $\frac{1}{6}\text{M}^3 (1 + \frac{1}{2}\text{M}q)$, and the ratio of the first to the second is $\frac{1}{2}\text{M} (1 + \frac{1}{10}\text{M}q)$.

In order to evaluate m, we proceed to introduce the approximate value for q.

Now,

$$q = \frac{1}{m'} \log \frac{\text{w}}{\text{w}_0}, \quad \text{and} \quad e^{\text{M}q} = \left(\frac{\text{w}}{\text{w}_0}\right)^{\text{M}/m'};$$

then, writing for brevity,

$$P = \log \left(\frac{\text{w}}{\text{w}_0}\right)^{\text{M}/m'},$$

we have

$$\frac{m}{\frac{1}{2}M} = \frac{2}{P} \cdot \frac{\left(\frac{\text{w}}{\text{w}_0}\right)^{\text{M}/m'} (P^2 - 4P + 6) - 2(P + 3)}{\left(\frac{\text{w}}{\text{w}_0}\right)^{\text{M}/m'} (P - 2) + (P + 2)}. \quad . \quad . \quad . \quad . \quad . \quad (59)$$

Also, if P be small, the approximate result is

$$\frac{m}{\frac{1}{2}\mathrm{M}} = 1 + \tfrac{1}{10}P.$$

Before proceeding to give numerical values for the fall of mean mass as we proceed outwards from the centre of the isothermal sphere, we must consider

The Adiabatic Layer.

In this case we assume, as before, that the ratio of the two specific heats is $1\frac{2}{3}$, and we therefore have for the relationship between δp and δw at radius r,

$$\frac{\delta p}{\delta p_0} = \left(\frac{\delta w}{\delta w_0}\right)^{\frac{5}{3}}.$$

Hence,

$$\frac{1}{\delta w}\frac{d\,\delta p}{dr} = \tfrac{5}{6}\left[\frac{3\,\delta p_0}{\delta w_0}\right]\frac{d}{dr}\left(\frac{\delta w}{\delta w_0}\right)^{\frac{2}{3}}.$$

But, since δp_0, δw_0 apply to the radius a where $h = h_0$, a constant,

$$\frac{3\,\delta p_0}{\delta w_0} = \frac{h_0}{x}.$$

Thus, in the adiabatic layer the equation of hydrostatic equilibrium is

$$\tfrac{5}{6}\frac{h_0}{x}\frac{d}{dr}\left(\frac{\delta w}{\delta w_0}\right)^{\frac{2}{3}} + \frac{d\chi}{dr} = 0,$$

whence,

$$\chi = \tfrac{5}{6}\frac{h_0}{x}\left(1 - \left(\frac{\delta w}{\delta w_0}\right)^{\frac{2}{3}}\right), \quad \ldots\ldots\ldots \quad (60)$$

or

$$\delta w = \delta w_0\left[1 - \frac{6\chi x}{5h_0}\right]^{\frac{3}{2}}.$$

The investigation now follows a line parallel to that taken before.

We have $\delta n/\delta n_0 = \delta w/\delta w_0$, and $\delta n_0 = f(x)\,dx$, so that

$$\delta n = \left(1 - \frac{6\chi x}{5h_0}\right)^{\frac{3}{2}} f(x)\,\delta x.$$

This is the law of frequency of masses lying between x and $x + \delta x$ at radius r.

As δn can never be negative, we see that there can be no mass greater than $\frac{5}{6}h_0/\chi$; and, if M be the greatest positive value of the expression $f(x)$, there can be no mass greater than the smaller of $\frac{5}{6}h_0/\chi$ or M.

Thus, if m be the mean mass at radius r,

$$m = \frac{\int_0^\alpha x\left(1 - \frac{6\chi x}{5h_0}\right)^{\frac{3}{2}} f(x)\,dx}{\int_0^\alpha \left(1 - \frac{6\chi x}{5h_0}\right)^{\frac{3}{2}} f(x)\,dx}, \quad \ldots \ldots \ldots \quad (61)$$

where α is the smaller of $\frac{5}{6}h_0/\chi$ and M.

If we put $\chi = 0$ in (61), we obtain m_0, the mean mass at radius a.

It is clear also that, if w be the total density of the swarm at radius r,

$$w = \int x\,dn = \int_0^\alpha x\left(1 - \frac{6\chi x}{5h_0}\right)^{\frac{3}{2}} f(x)\,dx. \quad \ldots \ldots \quad (62)$$

By definition of χ, and in consequence of the supposed spherical arrangement of matter, we have

$$\chi = \int_a^r \frac{1}{r^2}\left(\int_0^r 4\pi\mu w r^2\,dr\right) dr.$$

If this value were substituted in (62), we might obtain a complicated differential equation for w. It is clear, however, that an adequate approximation may be obtained by assuming that the w in the integral expressing χ is the density of meteorites, all of which are of the same size m', arranged in a layer in convective equilibrium, and with kinetic energy of agitation at the limit $r = a$ equal to $\frac{1}{2}h_0$.

If this density be written w, and if v^2 be the mean square of velocity of agitation at radius r, we have, by (60), and in consequence of the relationship $(\mathrm{w}/\mathrm{w}_0)^{\frac{2}{3}} = (\mathrm{v}/\mathrm{v}_0)^2$,

$$\chi = \frac{5}{6}\frac{h_0}{m'}\left(1 - \left(\frac{\mathrm{w}}{\mathrm{w}_0}\right)^{\frac{2}{3}}\right) = \frac{5}{6}\mathrm{v}_0^2\left(1 - \frac{\mathrm{v}^2}{\mathrm{v}_0^2}\right),$$

and

$$\frac{6}{5}\frac{\chi x}{h_0} = \frac{x}{m'}\left(1 - \frac{\mathrm{v}^2}{\mathrm{v}_0^2}\right).$$

Let

$$\frac{1}{\beta} = \frac{1}{m'}\left(1 - \frac{\mathrm{v}^2}{\mathrm{v}_0^2}\right)$$

for brevity; then, adopting the law of frequency $f(x) = \frac{6n_0}{\mathrm{M}^3}x(\mathrm{M} - x)$, as before, we have for the mean mass at radius r

$$m = \frac{\int_0^\alpha x^2(\mathrm{M} - x)\left(1 - \frac{x}{\beta}\right)^{\frac{3}{2}} dx}{\int_0^\alpha x(\mathrm{M} - x)\left(1 - \frac{x}{\beta}\right)^{\frac{3}{2}} dx}, \quad \ldots \ldots \ldots \quad (63)$$

where α is equal to the smaller of M and β.

The solution now becomes different according as M or β is the smaller.

First, suppose M is the smaller. Then the limits of integration are M and 0.

If we put

$$z = 1 - \frac{x}{\beta}.$$

$$x^n \left(1 - \frac{x}{\beta}\right)^{\frac{3}{2}} dx = - \beta^{n+1} z^{\frac{3}{2}} \left(1 - nz + \frac{n \, . \, n-1}{1 \, . \, 2} z^2 - \ldots \right) dz,$$

so that the numerator and denominator of m are easily integrable.

If now we write

$$Q = 1 - \frac{\mathrm{M}}{\beta},$$

$$\int_0^{\mathrm{M}} z^2 (\mathrm{M} - x) \left(1 - \frac{x}{\beta}\right)^{\frac{3}{2}} dx = 2\beta^4 \left[\tfrac{1}{5} \left(\frac{\mathrm{M}}{\beta} - 1\right)(1 - Q^{\frac{5}{2}}) - \tfrac{1}{7} \left(2\frac{\mathrm{M}}{\beta} - 3\right)(1 - Q^{\frac{7}{2}}) \right.$$
$$\left. + \tfrac{1}{9} \left(\frac{\mathrm{M}}{\beta} - 3\right)(1 - Q^{\frac{9}{2}}) + \tfrac{1}{11}(1 - Q^{\frac{11}{2}}) \right]$$
$$= 2\beta^4 \left[\frac{8}{7 \, . \, 9 \, . \, 11} - \frac{8}{5 \, . \, 7 \, . \, 9} Q + \frac{2}{5 \, . \, 7} Q^{\frac{7}{2}} - \frac{4}{7 \, . \, 9} Q^{\frac{9}{2}} + \frac{2}{9 \, . \, 11} Q^{\frac{11}{2}} \right],$$

$$\int_0^{\mathrm{M}} x (\mathrm{M} - x) \left(1 - \frac{x}{\beta}\right)^{\frac{3}{2}} dx = 2\beta^3 \left[\tfrac{1}{5} \left(\frac{\mathrm{M}}{\beta} - 1\right)(1 - Q^{\frac{5}{2}}) - \tfrac{1}{7} \left(\frac{\mathrm{M}}{\beta} - 2\right)(1 - Q^{\frac{7}{2}}) - \tfrac{1}{9}(1 - Q^{\frac{9}{2}}) \right]$$
$$= 2\beta^3 \left[\frac{2}{7 \, . \, 9} - \frac{2}{5 \, . \, 7} Q + \frac{2}{5 \, . \, 7} Q^{\frac{7}{2}} - \frac{2}{7 \, . \, 9} Q^{\frac{9}{2}} \right].$$

Then, since $\beta = (1 - Q)/M$, we have

$$\frac{m}{M} = \frac{\frac{4}{11} - \frac{4}{5} Q + \frac{9}{5} Q^{\frac{7}{2}} - 2Q^{\frac{9}{2}} + \frac{7}{11} Q^{\frac{11}{2}}}{(1 - Q)(1 - \frac{9}{5} Q + \frac{9}{5} Q^{\frac{7}{2}} - Q^{\frac{9}{2}})} . \quad \ldots \ldots \quad (64)$$

This expression has a high order of indeterminateness when $Q = 1$, but I find that when Q is nearly equal to unity

$$\frac{m}{\mathrm{M}} = \tfrac{1}{2} [1 - \tfrac{3}{10} (1 - Q^{\frac{1}{2}})] \text{ nearly.} \quad \ldots \ldots \quad (65)$$

Thus, the mean mass is $\frac{1}{2}$M where $r = a$, which we know to be correct.

Secondly, suppose that β is smaller than M. Then effecting the integrations in the same manner as before, we have

$$\int_0^{\beta} x^2 (\mathrm{M} - x) \left(1 - \frac{x}{\beta}\right)^{\frac{3}{2}} dx = 2\beta^4 \left[\tfrac{1}{5} \left(\frac{\mathrm{M}}{\beta} - 1\right) - \tfrac{1}{7} \left(\frac{2\mathrm{M}}{\beta} - 3\right) + \tfrac{1}{9} \left(\frac{\mathrm{M}}{\beta} - 3\right) + \tfrac{1}{11} \right]$$
$$= \frac{2 \, . \, 8}{5 \, . \, 7 \, . \, 9} \beta^4 \left(\frac{\mathrm{M}}{\beta} - \tfrac{6}{11}\right),$$

$$\int_0^\beta x\,(\mathrm{M} - x)\left(1 - \frac{x}{\beta}\right)^{\frac{3}{2}} dx = 2\beta^3\left[\tfrac{1}{5}\left(\frac{\mathrm{M}}{\beta} - 1\right) - \tfrac{1}{7}\left(\frac{\mathrm{M}}{\beta} - 2\right) - \tfrac{1}{9}\right]$$

$$= \frac{2\,.\,2}{5\,.\,7}\beta^3\left[\frac{\mathrm{M}}{\beta} - \tfrac{4}{9}\right].$$

Therefore,

$$m = \tfrac{4}{9}\beta\,.\,\frac{\dfrac{\mathrm{M}}{\beta} - \frac{6}{11}}{\dfrac{\mathrm{M}}{\beta} - \frac{4}{9}},$$

or

$$\frac{m}{\mathrm{M}} = \tfrac{4}{9}\,.\,\frac{\dfrac{m'}{\mathrm{M}}}{1 - \dfrac{\mathrm{v}^2}{\mathrm{v}_0^{\,2}}}\,.\,\frac{\dfrac{\mathrm{M}}{m'}\left(1 - \dfrac{\mathrm{v}^2}{\mathrm{v}_0^{\,2}}\right) - \frac{6}{11}}{\dfrac{\mathrm{M}}{m'}\left(1 - \dfrac{\mathrm{v}^2}{\mathrm{v}_0^{\,2}}\right) - \frac{4}{9}}\quad.\quad.\quad.\quad.\quad.\quad.\quad(66)$$

In order to compute from the formulæ, (59), (64), (66), it is necessary to make an assumption as to the value of m' the mass of the meteorites of uniform size whose arrangement of density is supposed to be the same as that of the heterogeneous meteorites.

We have supposed that the law of frequency of masses is known at radius a, and that the mean mass is there equal to $\frac{1}{2}M$. Now, inside of that radius the larger masses are more frequent, and outside of it the smaller masses. I suppose, then, that throughout the isothermal sphere m' lies half way between m_0 or $\frac{1}{2}\mathrm{M}$ and the maximum mass M, and in the adiabatic layer that it lies half way between m_0 or $\frac{1}{2}\mathrm{M}$ and the minimum mass 0.

Thus, inside I take $m' = \frac{3}{4}\mathrm{M}$, and outside $m' = \frac{1}{4}M$.

As we only want to consider the general nature of the sorting process, these assumptions will suffice. It may also be remarked that a large variation of m' is required to make any considerable difference in the numerical results.

We now have—

In the isothermal sphere (where $\mathrm{w}_0 = \frac{1}{3}\rho$),

$$\frac{\mathrm{M}}{m'} = \tfrac{4}{3}, \qquad \mathrm{P} = \log_e\left(\frac{\mathrm{w}}{\frac{1}{3}\rho}\right)^{\frac{3}{2}}, \qquad \tfrac{1}{2}\mathrm{M} = m_0;$$

In the adiabatic layer,

$$\frac{\mathrm{M}}{m'} = 4, \qquad Q = 1 - \frac{\mathrm{M}}{m'}\left(1 - \frac{\mathrm{v}^2}{\mathrm{v}_0^{\,2}}\right) = 4\,\frac{\mathrm{v}^2}{\mathrm{v}_0^{\,2}} - 3\,.\,.$$

Thus, our formulæ are :—

In the isothermal sphere, from (59),

$$\frac{m}{m_0} = \frac{2}{P}\,.\,\frac{\left(\dfrac{\mathrm{w}}{\frac{1}{3}\rho}\right)^{\frac{3}{2}}(P^2 - 4P + 6) - 2\,(P + 3)}{\left(\dfrac{\mathrm{w}}{\frac{1}{3}\rho}\right)^{\frac{3}{2}}(P - 2) + (P + 2)};\quad.\quad.\quad.\quad.\quad.\quad(67)$$

in the adiabatic layer,

when $\frac{\mathrm{v}^2}{\mathrm{v}_0^2} > \frac{3}{4}$, from (64),

$$\frac{m}{m_0} = \frac{\frac{4}{11} - \frac{4}{5} Q + \frac{9}{5} Q^{\frac{4}{3}} - 2 Q^{\frac{5}{3}} + \frac{7}{11} Q^{\frac{11}{3}}}{\frac{1}{2}(1 - Q)(1 - \frac{9}{5} Q + \frac{9}{5} Q^{\frac{4}{3}} - Q^{\frac{5}{3}})}, \quad \ldots\ldots \quad (68)$$

when $\frac{\mathrm{v}^2}{\mathrm{v}_0^2} < \frac{3}{4}$, from (66),

$$\frac{m}{m_0} = \frac{2}{9(1 - \mathrm{v}^2/\mathrm{v}_0^2)} \cdot \frac{\frac{19}{22} - \mathrm{v}^2/\mathrm{v}_0^2}{\frac{8}{9} - \mathrm{v}^2/\mathrm{v}_0^2}. \quad \ldots\ldots \quad (69)$$

The values of $\mathrm{w}/\frac{1}{3}\rho$ and of $\mathrm{v}^2/\mathrm{v}_0^2$ are tabulated in Table III., and from these I compute—

	isothermal.					$\frac{\mathrm{v}^2}{\mathrm{v}_0^2} > \frac{3}{4}$		$\frac{\mathrm{v}^2}{\mathrm{v}_0^2} < \frac{3}{4}$						
$\frac{r}{a} =$	0	·16	·48	·80	1·0	1·09	1·2	1·33	1·5	1·71	2·00	2·21	2·46	2·79
$\frac{m}{m_0} =$	1·41	1·38	1·22	1·11	1·0	·92	·83	·66	·49	·38	·30	·27	·24	·22

These values (together with two others in the isothermal part) are set out in fig. 2, and show the law of diminution of mean mass from centre to outside.

Fig. 2.

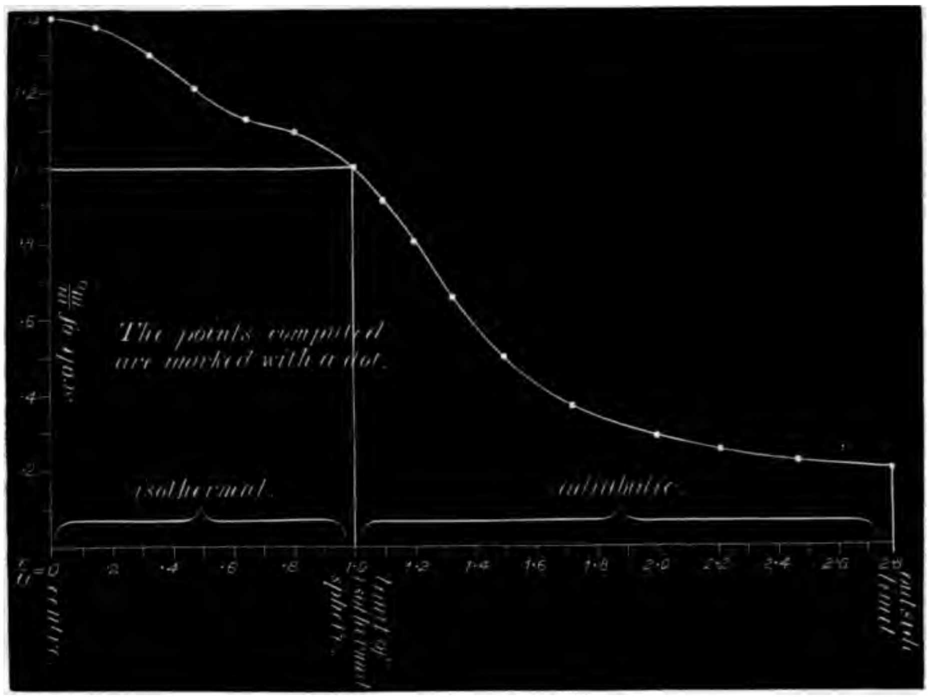

Diagram showing diminution of mean mass from centre to outside.

The evaluation of mean mass in the fringe (see § 13), where collisions are supposed to be non-existent, is not very difficult, although it involves some troublesome algebra. I do not give the investigation, merely remarking that it leads to almost exactly the same kind of law of diminution of mean mass as we have found in the adiabatic layer.

§ 17. *Summary.*

The first and second sections only involved arguments of a general character in which mathematical analysis was unnecessary. The reader who does not wish to concern himself with details may therefore be supposed to have passed from §§ 1 and 2 to this Summary.

In order to submit the theory to an adequate test, it is necessary to discuss some definite case of the aggregation of a swarm of meteorites, and it is obvious that the only system of which we possess any knowledge is our own. It is accordingly supposed that a number of meteorites have fallen together from a condition of wide dispersion, and have ultimately coalesced so as to leave the Sun and planets as their progeny. The object of this paper is to consider the mechanical condition of the system after the cessation of any considerable supply of meteorites from outside, and before the coalescence of the swarm into a star with attendant planets.

For the sake of simplicity, the meteorites are considered to be spherical, and are treated, at least in the first instance, as being of uniform size.

It is assumed provisionally that the kinetic theory of gases may be applied for the determination of the distribution of the meteorites in space. No account being taken of the rotation of the system, the meteorites will be arranged in concentric spherical layers of equal density of distribution, and the quasi-gas, whose molecules are meteorites, being compressible, the density will be greater towards the centre of the swarm.

The elasticity of a gas depends on the kinetic energy of agitation of its molecules; and, therefore, in order to determine the law of density in the swarm, we must know the distribution of kinetic energy of agitation. It is assumed that, when the swarm comes under our notice, uniformity of distribution of energy has been attained throughout a central sphere, which is surrounded by a layer of meteorites with that distribution of kinetic energy which in a gas corresponds to convective equilibrium. In other words, we have a quasi-isothermal sphere surrounded by what may be called an atmosphere in convective equilibrium, and with continuity of density and velocity of agitation at the sphere of separation. Since in a gas in convective equilibrium the law connecting pressure and density is that which holds when the gas is contained in a vessel impermeable to heat, such an arrangement of gas has been called by M. RITTER "an isothermal-adiabatic sphere," and the same term is adopted here as applicable to a swarm of meteorites. The justifiability of these assumptions will be considered later.

The first problem which presents itself, then, is the equilibrium of an isothermal sphere of gas under its own gravitation. The law of density is determined in § 4; but it will here suffice to remark that, if a given mass be enclosed in an envelope of given radius, there is a minimum temperature (or energy of agitation) at which isothermal equilibrium is possible. The minimum energy of agitation is found to be such that the mean square of velocity of the meteorites is almost exactly $\frac{6}{5}$ (viz. 1·1917) of the square of the velocity of a satellite grazing the surface of the sphere in a circular orbit.

As indicated above, it is supposed that in the meteor-swarm the rigid envelope bounding the isothermal sphere is replaced by a layer of meteorites in convective equilibrium. The law of density in the adiabatic layer is determined in § 5, and it appears that, when the isothermal sphere has minimum temperature, the mass of the adiabatic atmosphere is a minimum relatively to that of the isothermal sphere. Numerical calculation shows, in fact, that the isothermal sphere cannot amount in mass to more than 46 per cent. of the mass of the whole isothermal-adiabatic sphere, and that the limit of the adiabatic atmosphere is at a distance equal to 2·786 times the radius of the isothermal sphere.* A table of various quantities in such a system, at various distances from the centre, is given in Table III., § 6.

It is next proved, in § 7, that the total energy, existing in the form of energy of agitation in an isothermal-adiabatic sphere, is exactly one-half of the potential energy lost in the concentration of the matter from a condition of infinite dispersion. This result is brought about by a continual transfer of energy from a molar to a molecular form, for a portion of the kinetic energy of a meteorite is constantly being transferred into the form of thermal energy in the volatilised gases generated on collision. The thermal energy is then lost by radiation.

It is impossible as yet to sum up all the considerations which go to justify the assumption of the isothermal-adiabatic arrangement; but it is clear that uniformity of kinetic energy of agitation in the isothermal sphere must be principally brought about by a process of diffusion. It is, therefore, interesting to consider what amount of inequality in the kinetic energy would have to be smoothed away.

The arrangement of density in the isothermal-adiabatic sphere being given, it is easy to compute what the kinetic energy would be at any part of the swarm, if each meteorite fell from infinity to the neighbourhood where we find it, and there retained all the velocity due to such fall. The variation of the square of this velocity gives an indication of the amount of inequality of kinetic energy which has to be degraded by conversion into heat and redistributed by diffusion in the attainment of uniformity. This may be called "the theoretical value of the kinetic energy"; it is tabulated in Table III., on the line called "square of velocity of satellite." It rises from zero at the centre of the sphere to a maximum, which is attained nearly half way through the adiabatic layer, and then falls again. If the radius of the isothermal sphere be unity, then from $\frac{1}{2}$ to 2 the variations of this theoretical value of the kinetic energy are small. Since this "theoretical value of the kinetic energy" is zero at the centre, there must have been diffusion of energy from without inwards, and considerations of the same kind show that when a planet consolidates there must be a cooling of the middle strata both outwards and inwards.

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* These results had been previously discovered by M. RITTER.

We must now consider the nature of the criterion which determines whether the hydrodynamical treatment of a swarm of meteorites is permissible.

The hydrodynamical treatment of an ideal plenum of gas leads to the same result as the kinetic theory with regard to any phenomenon involving purely a mass, when that mass is a large multiple of the mass of a molecule; to any phenomenon involving purely a length, when the cube of that length contains a large number of molecules; and to any phenomenon involving purely a time, when that time is a large multiple of the mean interval between collisions. Again, any velocity to be justly deduced from hydrodynamical principles must be expressible as the edge of a cube containing many molecules passed over in a time containing many collisions of a single molecule; and a similar statement must hold of any other function of mass, length, and time.

Beyond these limits, we must go back to the kinetic theory itself, and in using it care must be taken that enough molecules are considered at once to impart statistical constancy to their properties.

There are limits, then, to the hydrodynamical treatment of gases, and the like must hold of the parallel treatment of meteorites.

The principal question involved in the nebular hypothesis seems to be the stability of a rotating mass of gas; but, unfortunately, this has remained up to now an untouched field of mathematical research. We can only judge of probable results from the investigations which have been made concerning the stability of a rotating mass of liquid. Now, it appears that the instability of a rotating mass of liquid first enters through the graver modes of gravitational oscillation. In the case of a rotating spheroid of revolution the gravest mode of oscillation is an elliptic deformation, and its period does not differ much from that of a satellite which revolves round the spheroid so as to graze its surface. Hence, assuming for the moment that a kinetic theory of liquids had been formulated, we should not be justified in applying the hydrodynamical method to this discussion of stability unless the periodic time of such a satellite were a large multiple of the analogue of the mean free time of a molecule of liquid.*

Carrying, then, this conclusion on to the kinetic theory of meteorites, it seems probable that hydrodynamical treatment must be inapplicable for the discussion of such a theory as the meteoric-nebular hypothesis, unless a similar relation holds good.

These considerations, although of a very general character, will afford a criterion of the applicability of hydrodynamics to the discussion of the mechanical conditions of a swarm of meteorites in the kind of problem suggested by the nebular hypothesis.

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* If the molecules of liquid describe orbits about one another, the analogue would probably be the mean periodic time of one molecule about another.

In § 9 two criteria, suggested by this line of thought, are found. They measure, roughly speaking, the degree of curvature of the average path pursued by a meteorite between two collisions. These two criteria, denoted D/L and A/C, will afford a measure of the applicability of hydrodynamics in the sense above indicated.

After these preliminary investigations, we have to consider what kind of meeting of two meteorites will amount to an "encounter" within the meaning of the kinetic theory. Is it possible, in fact, that two meteorites can considerably bend their paths under the influence of gravitation when they pass near one another? This question is answered in § 8, where a formula is found for the deflection of the path of each of a pair of meteorites, when, moving with their mean relative velocity, they graze past one another without striking. It appears from the formula that, unless they have the dimensions of small planets, the mutual gravitational influence is practically insensible. Hence, nothing short of absolute impact is to be considered an encounter in the kinetic theory; and what is called the radius of "the sphere of action" is simply the distance between the centres of a pair when they graze, and is, therefore, the sum of their radii, or, if of uniform size, the diameter of one of them.

The next point to consider is the mass and size which must be attributed to the meteorites.

The few samples which have been found on the earth prove that no great error can be committed if the average density of a meteorite be taken as a little less than that of iron, and I accordingly suppose their density to be six times that of water.

Undoubtedly, in a swarm of meteorites all sizes exist (a supposition considered hereafter); for, even if originally of one uniform size, they would, by subsequent fracture, be rendered diverse. But in the first consideration of the problem they have been treated as of uniform size, and, as actual average sizes are nearly unknown, results are given in the numerical table for meteorites weighing $3\frac{1}{8}$ grammes. By merely shifting the decimal point one, two, or three places to the right the results become applicable to meteorites weighing $3\frac{1}{8}$ kilogrammes, $3\frac{1}{8}$ tonnes, 3125 tonnes, and so on.

It is known that meteorites are actually of irregular shapes, but certainly no material error can be incurred when we treat them as being spheres.

The object of all these investigations is to apply the formulæ to a concrete example. The mass of the system is therefore taken as equal to that of the Sun, and the limit of the swarm at any arbitrary distance from the present Sun's centre. The theory is, of course, most severely tested the wider the dispersion of the swarm; and, accordingly, in the numerical example the outside limit of the Solar swarm is taken at $44\frac{1}{2}$ times the Earth's distance from the Sun, or further beyond the planet Neptune than Saturn is from the Sun. This assumption makes the limit of the isothermal sphere at distance 16, about half way between Saturn and Uranus.

The results, applicable to meteorites of $3\frac{1}{8}$ grammes, are exhibited in Table IV., § 10.

The velocity of mean square in the isothermal sphere is $\sqrt{(6/5)}$ of the linear velocity of a planet at distance 16, revolving about a central body with a mass equal to 46 per cent. of that of the Sun, viz., $5\frac{1}{3}$ kilometres per second; and in the adiabatic layer it diminishes down to zero at distance $44\frac{1}{2}$. This velocity is independent of the size of the meteorites.

The mean free path between collisions ranges from 42,000 kilometres at the centre, to 1,300,000 kilometres at radius 16, and to infinity at radius $44\frac{1}{2}$. The mean interval between collisions ranges from a tenth of a day at the centre, to three days at radius 16, and to infinity at radius $44\frac{1}{2}$. The criterion D/L ranges from $\frac{1}{60,000}$ at the distance of the asteroids, to $\frac{1}{3600}$ at radius 16, and to infinity at radius $44\frac{1}{2}$. The criterion A/C is somewhat smaller than D/L. All these quantities are ten times as great for meteorites of $3\frac{1}{8}$ kilogrammes, and a hundred times as great for meteorites of $3\frac{1}{8}$ tonnes.

From a consideration of the table it appears that, with meteorites of $3\frac{1}{8}$ kilogrammes, the collisions are sufficiently frequent, even beyond the orbit of Neptune, to allow the kinetic theory to be applicable in the sense explained. But, if the meteorites weigh $3\frac{1}{8}$ tonnes, the criteria cease to be very small about distance 24; and, if they weigh 3125 tonnes, they cease to be very small at about the orbit of Jupiter.

It may be concluded, then, that, as far as frequency of collision is concerned, the hydrodynamical treatment of a swarm of meteorites is justifiable.

Although these numerical results are necessarily affected by the conjectural values of the mass and density of the meteorites, yet it was impossible to arrive at any conclusion whatever as to the validity of the theory without numerical values, and such a discussion as the above was therefore necessary. If the particular values used are not such as to commend themselves to the judgment of the reader, it is easy to substitute others in the formulæ, and so submit the theory to another test.

I now pass on to consider some results of this view of a swarm of meteorites, and to consider the justifiability of the assumption of an isothermal-adiabatic arrangement of density.

With regard to the uniformity of distribution of kinetic energy in the isothermal sphere, it is important to ask whether or not sufficient time can have elapsed in the history of the system to allow of the equalisation by diffusion.

In § 11 the rate of diffusion of the kinetic energy of agitation is considered, and it is shown that, in the case of our numerical example, primitive inequalities of distribution would, in a few thousand years, be sensibly equalised over a distance some ten times as great as our distance from the Sun. This result, then, goes to show that we are justified in assuming an isothermal sphere as the centre of the swarm. As, however, the swarm contracts, the rate of diffusion diminishes as the inverse $\frac{5}{2}$ power of its linear dimensions, whilst the rate of generation of inequalities of distribution of kinetic energy, through the imperfect elasticity of the meteorites, increases. Hence, in a late stage of the swarm inequalities of kinetic energy would be set up; thus, there would be a tendency to the production of convective currents, and the whole swarm would probably settle down to the condition of convective equilibrium throughout.

It may be conjectured, then, that the best hypothesis in the early stages of the swarm is the isothermal adiabatic arrangement, and later an adiabatic sphere. It has not seemed to me worth while to discuss the latter hypothesis in detail at present.

The investigation of § 11 also gives the coefficient of viscosity of the quasi-gas, and shows that it is so great that the meteor-swarm must, if rotating, revolve nearly without relative motion of its parts, other than the motion of agitation. But, as the viscosity diminishes when the swarm contracts, this would probably not be true in the later stages of its history, and the central portion would probably rotate more rapidly than the outside. It forms, however, no part of the scope of this paper to consider the rotation of the system.

In § 12 the rate of loss of kinetic energy through imperfect elasticity is considered, and it appears that the rate estimated per unit time and volume must vary directly as the square of the quasi-pressure and inversely as the mean velocity of agitation. Since the kinetic energy lost is taken up in volatilising solid matter, it follows that the heat generated must follow the same law. The mean temperature of the gases generated in any part of the swarm depends on a great variety of circumstances, but it seems probable that its variation would be according to some law of the same kind. Thus, if the spectroscope enables us to form an idea of the temperature in various parts of a nebula, we shall at the same time obtain some idea of the distribution of density.

It has been assumed that the outer portion of the swarm is in convective equilibrium, and therefore there is a definite limit beyond which it cannot extend. Now, a medium can only be said to be in convective equilibrium when it obeys the laws of gases, and the applicability of those laws depends on the frequency of collisions. But at the boundary of the adiabatic layer the velocity of agitation vanishes, and collisions become infinitely rare. These two propositions are mutually destructive of one another, and it is impossible to push the conception of convective equilibrium to its logical conclusion. There must, in fact, be some degree of rarity of density, and of collisions, at which the statistical treatment of the medium breaks down.

I have sought to obtain some representation of the state of things by supposing that collisions never occur beyond a certain distance from the centre of the swarm. Then, from every point of the surface of the sphere, which limits the regions of collisions, a fountain of meteorites is shot out, in all azimuths and inclinations to the vertical, and with velocities grouped about a mean according to the law of error. These meteorites ascend to various heights without collision, and, in falling back on to the limiting sphere, cannonade its surface, so as to counterbalance the hydrostatic pressure at the limiting sphere.

The distribution of meteorites, thus shot out, is investigated in § 13, and it is found that near the limiting sphere the decrease in density is somewhat more rapid than the decrease corresponding to convective equilibrium. But at more remote distances the decrease is less rapid, and the density ultimately tends to vary inversely as the square of the distance from the centre.

It is clear, then, that, according to this hypothesis, the mass of the system is infinite in a mathematical sense, for the existence of meteorites with nearly parabolic and hyperbolic orbits necessitates an infinite number, if the loss of the system shall be made good by the supply.*

But, if we consider the subject from a physical point of view, this conclusion appears unobjectionable. The ejection of molecules with exceptionally high velocities from the surface of a liquid is called evaporation, and the absorption of others is called condensation. The general history of a swarm, as stated in § 2, may then be put in different words, for we may say that at first a swarm gains by condensation, that condensation and evaporation balance, and, finally, that evaporation gains the day.

If the hypothesis of convective equilibrium be pushed to its logical conclusion, we reach a definite limit to the swarm, whereas, if collisions be entirely annulled, the density goes on decreasing inversely as the square of the distance. The truth must clearly lie between these two hypotheses. It is thus certain that even the very small amount of evaporation shown by the formulæ derived from the hypothesis of no collisions must be in excess of the truth; and it may be that there are enough waifs and strays in space, ejected from other systems, to make up for the loss. Whether or not the compensation is perfect, a swarm of meteorites would pursue its evolution without being sensibly affected by a slow evaporation.

Up to this point the meteorites have been considered as of uniform size, but it is well to examine the more truthful hypothesis, that they are of all sizes, grouped about a mean according to a law of error.

It appears, from the investigation in § 14, that the larger stones move slower, the smaller ones faster; and the law is that the mean kinetic energy is the same for all sizes.

It is proved that the mean path between collisions is shorter in the proportion of 7 to 11, and the mean frequency greater in the proportion of 4 to 3, than if the meteorites were of uniform mass, equal to their mean. Hence, the previous numerical results for uniform size are applicable to non-uniform meteorites of mean mass about a third greater than the uniform mass; for example, the results for uniform meteorites of $3\frac{1}{8}$ tonnes apply to non-uniform ones of mean mass, a little over 4 tonnes.

The means here spoken of refer to all sizes grouped together, but there are a separate mean free path and a mean frequency appropriate to each size. These are investigated

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[* It must also be borne in mind that the very high velocities, which occur occasionally in a medium with perfectly elastic molecules, must happen with great rarity amongst meteorites. An impact of such violence that it *ought* to generate a hyperbolic velocity will probably merely cause fracture.—Added Nov. 23, 1888.]

in § 15, and their various values are illustrated in fig. 1. The horizontal scale in that figure gives the ratio of the radius of each size to the radius of the meteorite of mean mass. The vertical scales are the ratio of the mean free path of any size to that of all sizes together, and the ratio of the mean frequency for any size to that of all sizes together. The figure shows that collisions become infinitely frequent for the infinitely small ones, because of their infinite velocity; and again infinitely frequent for the infinitely large ones, because of their infinite size. There is a minimum frequency of collision for a certain size, a little less in radius than the mean, and considerably less in mass than the mean mass.

For infinitely small meteorites, the mean free path reaches a finite limit, equal to about four times the grand mean free path; but this could not be shown in the figure without a considerable extension of it upwards. For infinitely large ones, the mean free path becomes infinitely short. It must be borne in mind that there are infinitely few of the infinitely large and small meteorites.

Variety of size does not, then, so far, materially affect the results.

But a difference arises when we come to consider the different parts of the swarm. The larger meteorites, moving with smaller velocities, form a quasi-gas of less elasticity than do the smaller ones. Hence, the larger meteorites are more condensed towards the centre than are the smaller ones, or the large ones have a tendency to sink down, whilst the small ones have a tendency to rise. Accordingly, the various kinds are to some extent sorted according to size.

In § 16, an investigation is made of the mean mass of the meteorites at various distances from the centre, both inside and outside of the isothermal sphere, and fig. 2 is drawn to illustrate the law of diminution of mean mass.

It is also clear that the loss of the system through evaporation must fall more heavily on the small meteorites than on the large ones.

After the foregoing summary, it will be well to briefly recapitulate the principal conclusions which seem to be legitimately deducible from the whole investigation; and, in this recapitulation, qualifications must necessarily be omitted, or stated with great brevity.

When two meteorites are in collision, they are virtually highly elastic, although ordinary elasticity must be nearly inoperative.

A swarm of meteorites is analogous with a gas, and the laws governing gases may be applied to the discussion of its mechanical properties. This is true of the swarm from which the Solar system was formed, when it extended beyond the orbit of the planet Neptune.

When the swarm was very widely dispersed, the arrangement of density and of velocity of agitation of the meteorites was that of an isothermal-adiabatic sphere. Later in its history, when the swarm had contracted, it was probably throughout in convective equilibrium.

The actual mean velocity of the meteorites is determinable in a swarm of given mass, when expanded to a given extent.

The total energy of agitation in an isothermal-adiabatic sphere is half the potential energy lost in the concentration from a condition of infinite dispersion.

The half of the potential energy lost, which does not reappear as kinetic energy of agitation, is expended in volatilising solid matter and heating the gases produced on the impact of meteorites. The heat so generated is gradually lost by radiation.

The amount of heat generated per unit time and volume varies as the square of the quasi-hydrostatic pressure, and inversely as the mean velocity of agitation. The temperature of the gases volatilised probably varies by some law of the same nature.

The path of the meteorites is approximately straight, except when abruptly deflected by a collision with another. This ceases to be true at the outskirts of the swarm, where the collisions have become rare. The meteorites here describe orbits, under gravity, which are approximately elliptic, parabolic, and hyperbolic.

In this fringe to the swarm the distribution of density ceases to be that of a gas under gravity, and, as we recede from the centre, the density at first decreases more rapidly, and afterwards less rapidly, than if the medium were a gas.

Throughout all stages of the history of a swarm there is a sort of evaporation, by which the swarm very slowly loses in mass, but this loss is more or less counterbalanced by condensation. In the early stages, the gain by condensation outbalances the loss by evaporation; they then equilibrate; and, finally, the evaporation may be greater than the condensation.

Throughout the swarm the meteorites are partially sorted, according to size. As we recede from the centre, the number of small ones preponderates more and more and, thus, the mean mass continually diminishes with increasing distance. The loss to the system by evaporation falls principally on the smaller meteorites.

A meteor-swarm is subject to gaseous viscosity, which is greater the more widely diffused is the swarm. In consequence of this, a widely extended swarm, if in rotation, will revolve like a rigid body, without relative movement of its parts. Later in its history, the viscosity will, probably, not suffice to secure uniformity of rotation, and the central portion will revolve more rapidly than the outside.

[The kinetic theory of meteorites may be held to present a fair approximation to the truth in the earlier stages of the evolution of the system. But ultimately the majority of the meteors must have been absorbed by the central Sun and its attendant planets, and amongst the meteors which remain free the relative motion of agitation must have been largely diminished. These free meteorites—the dust and refuse of the system—probably move in clouds, but with so little remaining motion of agitation that (except, perhaps, near the perihelion of very eccentric orbits) it would scarcely be permissible to treat the cloud as in any respects possessing the mechanical properties of a gas.*]

The value of this whole investigation will appear very different to different minds. To some it will stand condemned, as altogether too speculative; others may think that it is better to risk error on the chance of winning truth. To me, at least, it appears that the line of thought flows in a true channel; that it may help to give a meaning to the observations of the spectroscopist; and that many interesting problems, here barely alluded to, may, perhaps, be solved with sufficient completeness to throw light on the evolution of nebulæ and of planetary systems.

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* Added Nov. 23, 1888.

II. *A Class of Functional Invariants.*

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The investigations herein contained are indirectly connected with some results in an earlier memoir.* In that memoir functions called quotient-derivatives are obtained in the form of certain combinations of differential coefficients of a quantity y dependent on a single independent variable x; and they are there shown to possess the property of invariance for isolated homographic transformations of the dependent and the independent variables. It is evident, however, from their form that they do not constitute the complete aggregate of irreducible invariants for the case of a single independent variable; and the deduction of this aggregate and an investigation of the relation in which they stand to a particular class of reciprocants were made in a subsequent paper.† The present memoir is a continuation of the theory of functional invariants, the invariants herein considered being constituted by combinations of the differential coefficients of a function of more than one independent variable which are such that, when the independent variables are transformed, each combination is reproduced save as to a factor depending on the transformations to which the variables are subjected. The transformations, in the case of which any detailed results are given, are of the general homographic type; and the investigations are limited to invariantive derivatives of a function of two independent variables only, a limitation introduced partly for the sake of conciseness. The characteristic properties, such as the symmetry of the invariants and the forms of the simultaneous linear partial differential equations satisfied by them, can in the case of more than two independent variables be inferred from the properties actually given; but many of the deductions made are necessarily proper to functions of only two independent variables.

In the matter of notation it is convenient here to state that the independent variables are denoted by x and y, and the dependent variable by z. The general differential coefficient $\partial^{m+n}z/\partial x^m\,\partial y^n$ is represented by $z_{m,n}$; but frequently the following modifications for the notation of particular coefficients are made, viz.:

p, q replace z_{10}, z_{01}:
r, s, t replace z_{20}, z_{11}, z_{02}:
a, b, c, d . . . z_{30}, z_{21}, z_{12}, z_{03}:
e, f, g, h, i . . . z_{40}, z_{31}, z_{22}, z_{13}, z_{04}:

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* "Invariants, Covariants, and Quotient-Derivatives associated with Linear Differential Equations," 'Phil. Trans.,' A, 1888, pp. 377–489.

† "Homographic Invariants and Quotient-Derivatives," 'Mess. of Math.,' vol. 17 (1888), pp. 154–192.

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respectively. The transformed independent variables are denoted by X and Y; and quantities bearing to them the same relation as the foregoing bear to x and y are denoted by $Z_{m,n}$, P, Q, ... And in three different instances it has been necessary, for the sake of uniformity of notation for similar successions of quantities, to use different symbols for the same quantity occurring in different successions; these are $u_4 = A_0$ (§§ 3, 12), $u_8 = -A_1$ (§§ 12, 13), $u_{13} = A_2$ (§§ 16, 17).

The general results of the memoir may be stated as follows :—

Every invariant is explicitly free from the variables themselves, viz., the dependent and the two [m] independent variables; it is homogeneous in the differential coefficients of the dependent variable; it is of uniform grade in differentiations with regard to each of the dependent variables, and it is either symmetric or skew symmetric with regard to such differentiations.

It satisfies six [$m^2 + m$] linear partial differential equations, all of the first order, of which four [m^2] are characteristic equations and determine the form of the invariant, and the remaining two [m] are index equations and are identically satisfied when the form is known and the index is derived by inspection from the form.

Every invariant involves the two [m] differential coefficients of the first order.

The following results relative to irreducible invariants derived from a single dependent variable z are given :—The invariants can be ranged in sets, each set being proper[*] to a particular rank. There is no invariant proper to the rank 1; there is one proper to the rank 2; there are three invariants proper to the rank 3; and, for a value of n greater than 3, there are $n + 1$ invariants proper to the rank n, which can be chosen so as to be linear in the differential coefficients of order n. Every invariant can be expressed in terms of these irreducible invariants; and the expression involves invariants of rank no higher than the order of the highest differential coefficient which occurs in that invariant.

In the case of irreducible invariants, involving differential coefficients of two dependent variables, it is shown that there is a single one proper to the rank 1, and that there are four proper to the rank 2.

Some eductive operators are given; and in one case the educts are discussed so as to select those of the invariants thus obtained which are evidently reducible. Some general results analogous to reversor operations are derived.

Finally, it is shown how the theory of binary forms can be partly connected with the theory of functional invariants; for functional invariants are expressible in terms of the simultaneous concomitants of a certain set of quantities, viewed as binary quantics of successive orders in q and $-p$ as variables.

[Note added December 5, 1888.[†]—The invariants in the present memoir are distinct in character from the differential invariants of M. HALPHEN and the ternary reciprocants of Mr. ELLIOTT.

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[*] An invariant is said to be *proper* to the rank n when the highest differential coefficient of z occurring in it is of order n.

[†] This addition is due to a desire which has been expressed that some indication should be given of the difference between the functions considered in the present memoir and invariantive functions of

The earliest record of M. HALPHEN's investigations is his well-known thesis,* wherein he considers the invariance of a differential equation $f(x, y, y', y'', \ldots) = 0$, when the single independent variable x and the single dependent variable y are (p. 20, *loc. cit.*) subjected to the transformation

$$\frac{X}{ax + by + c} = \frac{Y}{a'x + b'y + c'} = \frac{1}{a''x + b''y + c''}.$$

The only reference in the thesis to the case of three variables is (p. 60) in the concluding paragraph, where it is said that the theory can be extended to the case of one dependent variable z and two dependent variables x and y, the transformation suggested, but not explicitly stated, being

$$\frac{X}{\alpha x + \beta y + \gamma z + \delta} = \frac{Y}{\alpha' x + \beta' y + \gamma' z + \delta'} = \frac{Z}{\alpha'' x + \beta'' y + \gamma'' z + \delta''} = \frac{1}{\alpha''' x + \beta''' y + \gamma''' z + \delta'''}.$$

M. HALPHEN, again,† considers differential invariants, in which the last transformation is effected on functions of the three variables; but in this investigation y and z are taken to be two dependent variables of the single independent variable x.

Mr. ELLIOTT's theory ‡ of ternary reciprocants is closely connected with the concluding paragraph of M. HALPHEN's thesis; the functions are invariantive for interchanges of z, x, y, where z is a variable dependent on x and y; and the pure reciprocants are invariantive for the above-suggested transformations.

The theory in this memoir deals almost entirely with the case of three variables, z, x, y, where z is a dependent variable, and x and y are independent variables. The transformations, through which the invariance is maintained, refer to the independent variables only; they are—

$$\frac{x}{\alpha_1 + \beta_1 X + \gamma_1 Y} = \frac{y}{\alpha_2 + \beta_2 X + \gamma_2 Y} = \frac{1}{\alpha_3 + \beta_3 X + \gamma_3 Y}.$$

The dependent variable is left untransformed; it does not enter into the equations of transformation.

It follows, from the difference between the transformation in the theory here

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other classes, such as the differential invariants of M. HALPHEN and the ternary reciprocants of Mr. ELLIOTT.

* 'Sur les Invariants Différentiels,' Paris, 1878.

† "Sur les Invariants Différentiels des Courbes gauches," 'Journ. de l'École Polytechnique,' vol. 28, 1880, pp. 1–102.

‡ "On Ternary and *n*-ary Reciprocants," 'London Math. Soc. Proc.' vol. 17 (1886), pp. 171–196; "On the Linear Partial Differential Equations satisfied by pure Ternary Reciprocants," *ibid.*, vol. 18 (1887), pp. 142–164; "On pure Ternary Reciprocants and Functions allied to them," *ibid.*, vol. 19 (1888), pp. 6–23.

exposed in which the dependent variable does not enter into the equations of transformation, and the transformations above indicated in which the occurrence of the dependent variable in the equations of transformation is essential, that different results will be obtained. Two examples will suffice. First, a comparison of the characteristic equations of ELLIOTT's reciprocants and of those characteristic of the present functional invariants may be made from the forms expressed in the notations of this memoir :—

Annihilators of ELLIOTT's reciprocants.	Annihilators of Invariants in this memoir.
$\Omega_1 = r\frac{\partial}{\partial s} + 2s\frac{\partial}{\partial t} + a\frac{\partial}{\partial b} + 2b\frac{\partial}{\partial c} + 3c\frac{\partial}{\partial d} + \dots$	$\Delta_4 = p\frac{\partial}{\partial q} + r\frac{\partial}{\partial s} + 2s\frac{\partial}{\partial t}$ $+ a\frac{\partial}{\partial b} + 2b\frac{\partial}{\partial c} + 3c\frac{\partial}{\partial d} + \dots$
$\Omega_2 = t\frac{\partial}{\partial s} + 2s\frac{\partial}{\partial r} + d\frac{\partial}{\partial c} + 2c\frac{\partial}{\partial b} + 3c\frac{\partial}{\partial a} + \dots$	$\Delta_3 = q\frac{\partial}{\partial p} + t\frac{\partial}{\partial s} + 2s\frac{\partial}{\partial r}$ $+ d\frac{\partial}{\partial c} + 2c\frac{\partial}{\partial b} + 3b\frac{\partial}{\partial a} + \dots$
$-\mathrm{E}_1 = -\mu + 3r\frac{\partial}{\partial r} + 2s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}$ $+ 4a\frac{\partial}{\partial a} + 3b\frac{\partial}{\partial b} + 2c\frac{\partial}{\partial c} + d\frac{\partial}{\partial d} + \dots$	$\Omega_0 = -3\lambda + 2p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q}$ $+ 4r\frac{\partial}{\partial r} + 3s\frac{\partial}{\partial s} + 2t\frac{\partial}{\partial t}$ $+ 6a\frac{\partial}{\partial a} + 5b\frac{\partial}{\partial b} + 4c\frac{\partial}{\partial c} + 3d\frac{\partial}{\partial d} + \dots$
$-\mathrm{E}_2 = -\mu + r\frac{\partial}{\partial r} + 2s\frac{\partial}{\partial s} + 3t\frac{\partial}{\partial t}$ $+ a\frac{\partial}{\partial a} + 2b\frac{\partial}{\partial b} + 3c\frac{\partial}{\partial c} + 4d\frac{\partial}{\partial d} + \dots$	$\Omega_1 = -3\lambda + p\frac{\partial}{\partial p} + 2q\frac{\partial}{\partial q}$ $+ 2r\frac{\partial}{\partial r} + 3s\frac{\partial}{\partial s} + 4t\frac{\partial}{\partial t}$ $+ 3a\frac{\partial}{\partial a} + 4b\frac{\partial}{\partial b} + 5c\frac{\partial}{\partial c} + 6d\frac{\partial}{\partial d} + \dots$
$\mathrm{V}_1 = 3r^2\frac{\partial}{\partial a} + 3rs\frac{\partial}{\partial b} + (rt + 2s^2)\frac{\partial}{\partial c}$ $+ 3st\frac{\partial}{\partial d} + \dots$	$\Delta_1 = 1\left(q\frac{\partial}{\partial s} + 2p\frac{\partial}{\partial r}\right)$ $+ 2\left(t\frac{\partial}{\partial c} + 2s\frac{\partial}{\partial b} + 3r\frac{\partial}{\partial a}\right) + \dots$
$\mathrm{V}_2 = 3rs\frac{\partial}{\partial a} + (rt + 2s^2)\frac{\partial}{\partial b} + 3st\frac{\partial}{\partial c}$ $+ 3t^2\frac{\partial}{\partial d} + \dots$	$\Delta_2 = 1\left(p\frac{\partial}{\partial s} + 2q\frac{\partial}{\partial t}\right)$ $+ 2\left(r\frac{\partial}{\partial b} + 2s\frac{\partial}{\partial c} + 3t\frac{\partial}{\partial d}\right) + \dots$

Second, as an inference from the equations $\Omega_1 = 0$, $\Omega_3 = 0$, in ELLIOTT's theory, it follows that all pure reciprocants are invariants of the binary quantics $(r, s, t \between \xi, \eta)^2$, $(a, b, c, d \between \xi, \eta)^3, \ldots$ —but all invariants are not reciprocants—and that there are no covariants among these reciprocants. From the equations $\Delta_4 = 0$, $\Delta_3 = 0$, in the present theory it follows that all the functional invariants are algebraical covariants of the binary quantics $(r, s, t \between q, -p)^2$, $(a, b, c, d \between q, -p)^3, \ldots$ —but not all algebraical covariants are functional invariants; and, from the other equations, that no algebraical invariants of these quantics are functional invariants. In particular, $rt - s^2$ is a reciprocant, but not a functional invariant; $q^2r - 2pqs + p^2t$ is a functional invariant, but not a reciprocant.]

Isolated Transformations.

1. We may briefly consider functions which are *invariantive for merely isolated changes of the independent variables*, that is, for changes which are effected by one relation between x and X only, and one relation between y and Y only. For such transformations we have

$$\mathrm{P} = p\frac{dx}{d\mathrm{X}}, \quad \mathrm{Q} = q\frac{dy}{d\mathrm{Y}}, \quad \mathrm{S} = s\frac{dx}{d\mathrm{X}}\frac{dy}{d\mathrm{Y}},$$

so that $s \div pq$ is an absolute invariant. Again,

$$\frac{\partial}{\partial \mathrm{X}} = \frac{dx}{d\mathrm{X}}\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial \mathrm{Y}} = \frac{dy}{d\mathrm{Y}}\frac{\partial}{\partial y},$$

so that $1/p\ \partial/\partial x$ and $1/q\ \partial/\partial y$ are absolute invariantive operators, which, when applied to absolute invariants, will produce absolute invariants. We therefore have the series

$$\frac{1}{p}\frac{\partial}{\partial x}\left(\frac{s}{pq}\right), \quad \frac{1}{q}\frac{\partial}{\partial y}\left(\frac{s}{pq}\right);$$

$$\left(\frac{1}{p}\frac{\partial}{\partial x}\right)^2\frac{s}{pq}, \quad \left(\frac{1}{p}\frac{\partial}{\partial x}\frac{1}{q}\frac{\partial}{\partial y}\right)\frac{s}{pq}, \quad \left(\frac{1}{q}\frac{\partial}{\partial y}\frac{1}{p}\frac{\partial}{\partial x}\right)\frac{s}{pq}, \quad \left(\frac{1}{q}\frac{\partial}{\partial y}\right)^2\frac{s}{pq},$$

and so on. The operators $1/p\ \partial/\partial x$ and $1/q\ \partial/\partial y$ may be applied, any number of times in any order, to the absolute invariant s/pq (or any other invariant which is absolute), and the result will be an absolute invariant.

These invariants possess their property for any general isolated transformations of x and y; but, if special isolated transformations are effected on x and on y, *e.g.*, the homographic transformations of the form

$$x = \frac{a\mathrm{X} + b}{c\mathrm{X} + d}, \quad y = \frac{a'\mathrm{Y} + b'}{c'\mathrm{Y} + d'},$$

additional invariants will be introduced. For instance, we then have

$$\mathrm{A}(z) = \frac{z_{30}z_{10} - \frac{3}{2}z_{20}^2}{z_{10}^4}, \qquad \mathrm{B}(z) = \frac{z_{03}z_{01} - \frac{3}{2}z_{02}^2}{z_{01}^4},$$

both absolute invariants; in the former the variation of y, and in the latter the variation of x, do not come into consideration. From these we can derive a series of educts by the application of combinations of the absolute invariantive operators $1/p\ \partial/\partial x$ and $1/q\ \partial/\partial y$. When any such educt is invariantive, say

$$\mathrm{A}_m = \left(\frac{1}{p}\frac{\partial}{\partial x}\right)^m \mathrm{A}(z),$$

we may obtain from it other invariants by taking as the function, the differential coefficients of which are to enter, not z, but any educt which is an absolute invariant. All such invariants, however, thus obtained are expressible in terms of the educts obtained from $\mathrm{A}(z)$ and $\mathrm{B}(z)$ by repeated application of $1/p\ \partial/\partial x$ and $1/q\ \partial/\partial y$ in all possible combinations. Thus, it is easy to verify that if I be any absolute invariant, and I_1, I_2, I_3 its first, second, and third educts due to successive operations on I by $1/p\ \partial/\partial x$, the equation

$$\mathrm{A}(\mathrm{I}) = \frac{\mathrm{I}_1\mathrm{I}_3 - \frac{3}{2}\mathrm{I}_2^2}{\mathrm{I}_1^4} - \frac{\mathrm{A}(z)}{\mathrm{I}_1^2}$$

is satisfied; and the law is general.

2. Nor is it necessary to consider in any detail functions of the differential coefficients of z, which are *invariantive for isolated transformation of the dependent variable;* that is, for a transformation which connects z with a new variable ζ, without regard to the dependent variables. Such a transformation can be effected by means of an equation,

$$\phi(z, \zeta) = 0;$$

and we then have

$$z_{01} = \zeta_{01}\frac{dz}{d\zeta}, \qquad z_{10} = \zeta_{10}\frac{dz}{d\zeta}.$$

Since $dz/d\zeta$ is determinate from the transforming equation, it follows that

$$\frac{z_{01}}{z_{10}}$$

is an absolute invariant for the transformations at present under consideration. Moreover, in the present case $\partial/\partial x$ and $\partial/\partial y$ are absolute invariantive operators; and therefore

$$\frac{\partial^{m+n}}{\partial x^m\,\partial y^n}\left(\frac{z_{01}}{z_{10}}\right)$$

will, for all values of m and n, be an absolute invariant. Thus, taking in succession $m = 1$ and $n = 0$, and $m = 0$ and $n = 1$, we have

$$\frac{z_{10}z_{11} - z_{01}z_{20}}{z_{10}^2} \quad \text{and} \quad \frac{z_{10}z_{02} - z_{01}z_{11}}{z_{10}^2},$$

the former of which, multiplied by z_{01}/z_{10} and subtracted from the latter, gives

$$\frac{z_{01}^2 z_{20} - 2z_{01}z_{10}z_{11} + z_{10}^2 z_{02}}{z_{10}^3}$$

as an absolute invariant, or $z_{10}^2 z_{20} - 2z_{01} z_{10} z_{11} + z_{10}^2 z_{02}$ as a relative invariant, for the present transformation.

General Transformation.

3. We now proceed to the consideration of functions which are invariantive for the general simultaneous homographic transformation of the independent variables represented by

$$\frac{x}{\alpha_1 + \beta_1 X + \gamma_1 Y} = \frac{y}{\alpha_2 + \beta_2 X + \gamma_2 Y} = \frac{1}{\alpha_3 + \beta_3 X + \gamma_3 Y}.$$

As it will be convenient to have some one invariant at least, a relative invariant for these transformations can be obtained as follows. An integral relation given by

$$z = \frac{a + bx + cy}{a' + b'x + c'y} = \frac{u}{v}$$

reproduces itself in form when the independent variables are subjected to the above transformation; and the differential equation which is the equivalent of this integral relation will, therefore, also reproduce itself, and so will furnish an invariant.

Now, both u and v satisfy the three equations

$$\frac{\partial^2}{\partial x^2} = 0, \quad \frac{\partial^2}{\partial x\, \partial y} = 0, \quad \frac{\partial^2}{\partial y^2} = 0;$$

and therefore, substituting vz as the value of u in these, we have

$$\begin{aligned} 0 &= z_{20}v + 2z_{10}v_{10}, \\ 0 &= z_{11}v + z_{10}v_{01} + z_{01}v_{10}, \\ 0 &= z_{02}v + 2z_{01}v_{01}. \end{aligned}$$

The elimination of v, v_{10}, v_{01} between these leads to the result

$$0 = \begin{vmatrix} z_{20}, & 2z_{10}, & 0 \\ z_{11}, & z_{01}, & z_{10} \\ z_{02}, & 0, & 2z_{01} \end{vmatrix} = 2\left(z_{01}^2 z_{20} - 2z_{10}z_{01}z_{11} + z_{10}^2 z_{02}\right);$$

and, therefore,

$$A_0 = z_{01}^2 z_{20} - 2z_{10}z_{01}z_{11} + z_{10}^2 z_{02} = (z_{20}, z_{11}, z_{02} \between z_{01}, -z_{10})^2$$

is an invariant.

The integral of a partial differential equation of the second order, which is most general so far as concerns the number of arbitrary constants, contains five such independent arbitrary constants; and, therefore, a general integral of

$$A_0 = 0$$

is

$$z = \frac{a + bx + cy}{a' + b'x + c'y}.$$

It has already appeared that A_0 is an invariant for arbitrary change of z; and therefore, an immediate corollary is that

$$z = \phi\left(\frac{a + bx + cy}{a' + b'x + c'y}\right),$$

where ϕ is arbitrary, is a general integral of the equation $A_0 = 0$.

4. As an invariant is self-reproductive after transformations have been effected, save as to a factor, it is necessary to obtain the form of this factor. For this purpose it will be sufficient to consider a simple case.

Let z_1 and z_2 be two functions, and suppose the transformations of the variables to be any whatever, say of the form

$$x = \phi(X, Y), \qquad y = \psi(X, Y).$$

Then we have

$$P_1 = p_1\frac{\partial x}{\partial X} + q_1\frac{\partial y}{\partial X}, \qquad P_2 = p_2\frac{\partial x}{\partial X} + q_2\frac{\partial y}{\partial X},$$

$$Q_1 = p_1\frac{\partial x}{\partial Y} + q_1\frac{\partial y}{\partial Y}, \qquad Q_2 = p_2\frac{\partial x}{\partial Y} + q_2\frac{\partial y}{\partial Y};$$

and therefore

$$\begin{vmatrix} P_1, & Q_1 \\ P_2, & Q_2 \end{vmatrix} = \begin{vmatrix} p_1, & q_1 \\ p_2, & q_2 \end{vmatrix} \frac{\partial(x, y)}{\partial(X, Y)}.$$

Hence, in the present case, the factor is $\partial(x, y)/\partial(X, Y) = J$; and, by the analogy of all invariants, the factor for any one will be some power of J.

The invariants at present under consideration may, therefore, be defined as follows:—

A function ϕ of the partial differential coefficients of z with regard to x and to y is called an invariant if, when the independent variables are changed to X *and* Y *and the same function* Φ *of the new variables is formed, the equation*

$$\Phi = J^m \phi$$

is satisfied, where

$$J = \frac{\partial(x, y)}{\partial(X, Y)}.$$

5. The following properties of *irreducible* invariants are easily obtained :—

(i.) An invariant does not contain the dependent variable, nor either of the independent variables.

(ii.) An invariant is homogeneous in the differential coefficients.

(iii.) An invariant is of uniform grade,* equal to its index m, in differentiation with regard to x; and of uniform grade, also equal to its index m, in differentiation with regard to y.

(iv.) An invariant is either symmetric or skew symmetric in differentiation with regard to the independent variables.

All these properties hold of A_0, the index of which is easily seen to be 2; it is a symmetric invariant, that is, it is unchanged if x and y be interchanged.

The index of a symmetric invariant is an even integer; the index of a skew symmetric invariant is an odd integer.

These properties hold for functions which are invariants for any general transformation, and not merely for the homographic transformations to be adopted; but the forms of possible functions, as well as the value of J, will be determined by the character of the transformation. And, in particular, for the homographic transformation it is easy to prove that

$$J = \begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2, & \beta_3 \\ \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix} (\alpha_3 + \beta_3 X + \gamma_3 Y)^{-3}.$$

6. The method adopted for the determination of the forms of invariants will be to obtain the partial differential equations satisfied by them; these equations can be obtained, as in a similar case,† by using the principle of complete infinitesimal variation. For this purpose it will be necessary to have the formulæ expressing the relations between differential coefficients of z when the variables are transformed. This relation is given in the following proposition, the transformations being supposed any whatever. The special application to the homographic transformation will afterwards be made.

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* The grade of a term is the sum of the orders of differentiation with regard to one variable of the factors; thus, the x-grade of A_0 is 2; the y-grade is 2.

† "Homographic Invariants and Quotient-Derivatives," 'Mess. of Math.,' vol. 17 (1888), pp. 154-192.

Let $z = \theta(x, y)$, and let the variables be transformed by the equations

$$x = \phi(\mathrm{X}, \mathrm{Y}), \qquad y = \psi(\mathrm{X}, \mathrm{Y}).$$

Let

$$\Phi = \phi(\mathrm{X} + \rho,\ \mathrm{Y} + \sigma) - x,$$
$$\Psi = \psi(\mathrm{X} + \rho,\ \mathrm{Y} + \sigma) - y,$$

so that Φ and Ψ vanish with ρ and σ. Then, by the generalised form of TAYLOR'S Theorem,

$$\frac{1}{m!\,n!}\frac{\partial^{m+n}z}{\partial\mathrm{X}^m\,\partial\mathrm{Y}^n}$$

is the coefficient of $\rho^m\sigma^n$ in the expansion in ascending powers of

$$\theta\{\phi(\mathrm{X} + \rho,\ \mathrm{Y} + \sigma),\ \psi(\mathrm{X} + \rho,\ \mathrm{Y} + \sigma)\}$$
$$= \theta(x + \Phi,\ y + \Psi),$$

where ρ and σ occur only in Φ and Ψ. Now,

$$\theta(x + \Phi,\ y + \Psi) = \sum_{m'=0}\sum_{n'=0}\frac{1}{m'!\,n'!}\frac{\partial^{m'+n'}z}{\partial x^{m'}\,\partial y^{n'}}\Phi^{m'}\Psi^{n'};$$

and therefore

$$\frac{1}{m!\,n!}\frac{\partial^{m+n}z}{\partial\mathrm{X}^m\,\partial\mathrm{Y}^n} = \sum_{m'=0}\sum_{n'=0}\frac{1}{m'!\,n'!}\frac{\partial^{m'+n'}z}{\partial x^{m'}\,\partial y^{n'}}\mathrm{C}_{m,n}(\Phi^{m'}\Psi^{n'}),$$

where $\mathrm{C}_{m,n}(\Phi^{m'}\Psi^{n'})$ denotes the coefficient of $\rho^m\sigma^n$ in the expansion of $\Phi^{m'}\Psi^{n'}$ in ascending powers of ρ and σ. When m' and n' both vanish, or when $m' + n' > m + n$, the coefficient $\mathrm{C}_{m,n}(\Phi^{m'}\Psi^{n'})$ is zero.

The form of the corresponding theorem for the case of any number of independent variables is evident.

Homographic Transformation: Characteristic Equations.

7. When we consider the general homographic transformation, we may take α_1 and α_2 to be zero, for the invariants do not explicitly contain x and y, but only differential coefficients with regard to them, and so they may be modified by the subtraction of the respective constants α_1/α_3, α_2/α_3; and then the general forms are equivalent to

$$\frac{x}{\mathrm{X} + \alpha\mathrm{Y}} = \frac{y}{\mathrm{Y} + \beta\mathrm{X}} = \frac{1}{\alpha_3 + \beta_3\mathrm{X} + \gamma_3\mathrm{Y}}.$$

In order to apply the method of infinitesimal variation, it is sufficient to make the factor J nearly equal to unity, or, what is the same thing, to make x nearly equal to

X and y nearly equal to Y. Hence, we take α_3 to be unity, and β_3 and γ_3 small, say, $-\epsilon$ and $-\theta$ respectively; and α and β are to be considered small, quantities of the first order being retained. Thus, we have

$$\begin{aligned} \mathrm{J} &= \begin{vmatrix} 0, & 0, & 1 \\ 1, & \beta, & -\epsilon \\ \alpha, & 1, & -\theta \end{vmatrix} (1 - \epsilon\mathrm{X} - \theta\mathrm{Y})^{-3}, \\ &= 1 + 3\epsilon\mathrm{X} + 3\theta\mathrm{Y}, \end{aligned}$$

so far as quantities of the first order. Also

$$\begin{aligned} x &= \frac{\mathrm{X} + \alpha\mathrm{Y}}{1 - \epsilon\mathrm{X} - \theta\mathrm{Y}} = \mathrm{X} + \alpha\mathrm{Y} + \epsilon\mathrm{X}^2 + \theta\mathrm{XY} &&= \phi(\mathrm{X}, \mathrm{Y}), \\ y &= \frac{\mathrm{Y} + \beta\mathrm{X}}{1 - \epsilon\mathrm{X} - \theta\mathrm{Y}} = \beta\mathrm{X} + \mathrm{Y} \qquad + \epsilon\mathrm{XY} + \theta\mathrm{Y}^2 &&= \psi(\mathrm{X}, \mathrm{Y}), \end{aligned}$$

to the same order; and therefore

$$\begin{aligned} \Phi &= \phi(\mathrm{X} + \rho,\ \mathrm{Y} + \sigma) - \phi(\mathrm{X}, \mathrm{Y}) \\ &= \rho + \alpha\sigma + \epsilon(\rho^2 + 2\mathrm{X}\rho) + \theta(\rho\sigma + \rho\mathrm{Y} + \sigma\mathrm{X}), \\ \Psi &= \psi(\mathrm{X} + \rho,\ \mathrm{Y} + \sigma) - \psi(\mathrm{X}, \mathrm{Y}) \\ &= \sigma + \beta\rho + \epsilon(\rho\sigma + \rho\mathrm{Y} + \sigma\mathrm{X}) + \theta(\sigma^2 + 2\mathrm{Y}\sigma). \end{aligned}$$

Hence, to the first order inclusive, we have

$$\begin{aligned} \Phi^{m'}\Psi^{n'} = \rho^{m'}\sigma^{n'} &+ m'\rho^{m'-1}\sigma^{n'}\{\alpha\sigma + \epsilon(\rho^2 + 2\mathrm{X}\rho) + \theta(\rho\sigma + \rho\mathrm{Y} + \sigma\mathrm{X})\} \\ &+ n'\rho^{m'}\sigma^{n'-1}\{\beta\rho + \epsilon(\rho\sigma + \rho\mathrm{Y} + \sigma\mathrm{X}) + \theta(\sigma^2 + 2\mathrm{Y}\sigma)\}, \end{aligned}$$

and therefore

$$\begin{gathered} \mathrm{C}_{m', n'} = 1 + m'(2\epsilon\mathrm{X} + \theta\mathrm{Y}) + n'(\epsilon\mathrm{X} + 2\theta\mathrm{Y}), \\ \mathrm{C}_{m'-1, n'+1} = m'(\alpha + \theta\mathrm{X}), \quad \mathrm{C}_{m'+1, n'-1} = n'(\beta + \epsilon\mathrm{Y}), \\ \mathrm{C}_{m'+1, n'} = (m' + n')\,\epsilon, \qquad \mathrm{C}_{m', n'+1} = (m' + n')\,\theta. \end{gathered}$$

All other coefficients are negligible, being of a higher order of small quantities or zero (non-occurring); and these give all the combinations of values of m and n for $\mathrm{C}_{m, n}(\Phi^{m'}\Psi^{n'})$. Therefore, for all values of m and n, we have

$$\begin{aligned} \frac{\partial^{m+n} z}{\partial\mathrm{X}^m\,\partial\mathrm{Y}^n} = \frac{\partial^{m+n} z}{\partial x^m\,\partial y^n}&\{1 + m(2\epsilon\mathrm{X} + \theta\mathrm{Y}) + n(\epsilon\mathrm{X} + 2\theta\mathrm{Y})\} \\ &+ \frac{\partial^{m+n} z}{\partial x^{m+1}\,\partial y^{n-1}}\,n(\alpha + \theta\mathrm{X}) + \frac{\partial^{m+n} z}{\partial x^{m-1}\,\partial y^{n+1}}\,n(\beta + \epsilon\mathrm{Y}) \\ &+ \frac{\partial^{m+n-1} z}{\partial x^{m-1}\,\partial y^n}\,m(m + n - 1)\,\epsilon + \frac{\partial^{m+n-1} z}{\partial x^m\,\partial y^{n-1}}\,n(m + n - 1)\,\theta. \end{aligned}$$

8. If, then, we have an invariant f of index λ, such that

$$F(\ldots, Z_{m,n}, \ldots) = J^{\lambda} f(\ldots, z_{m,n}, \ldots),$$

and we substitute for J and for all differential coefficients $Z_{m,n}$, and then expand, retaining all small quantities of the first order, we have the following equations derived from a comparison of corresponding terms.

From the terms which are multiplied by ϵX

$$\Sigma\Sigma (2m + n) z_{m,n} \frac{\partial f}{\partial z_{m,n}} = 3\lambda f \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(i.)};$$

from the terms in θY

$$\Sigma\Sigma (2n + m) z_{m,n} \frac{\partial f}{\partial z_{m,n}} = 3\lambda f \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(ii.)};$$

from the terms in ϵ

$$\Delta_1 f = \Sigma\Sigma m (m + n - 1) z_{m-1,n} \frac{\partial f}{\partial z_{m,n}} = 0 \quad . \; . \; . \; . \; . \quad \text{(iii.)};$$

from the terms in θ

$$\Delta_2 f = \Sigma\Sigma n (m + n - 1) z_{m,n-1} \frac{\partial f}{\partial z_{m,n}} = 0 \quad . \; . \; . \; . \; . \quad \text{(iv.)};$$

from the terms in $\beta + \epsilon Y$

$$\Delta_3 f = \Sigma\Sigma m z_{m-1,n+1} \frac{\partial f}{\partial z_{m,n}} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(v.)};$$

and from the terms in $\alpha + \theta X$

$$\Delta_4 f = \Sigma\Sigma n z_{m+1,n-1} \frac{\partial f}{\partial z_{m,n}} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(vi.)}.$$

Equations (iii.)–(vi.) determine the form of the function f; when the form is obtained, the index is derivable by inspection, and equations (i.) and (ii.) are then identically satisfied.

9. Before considering these equations, characteristic of the invariants, one remark should be made. If the quantities ϵ and θ are absolutely zero so that the transformations are

$$x = X + \alpha Y, \qquad y = \beta X + Y,$$

that is, transformations to which a binary form is subject, the terms which, in what precedes, give rise to equations (i.)–(iv.) do not exist, and, therefore, these equations do not exist; but there are terms in β and α, and, therefore, equations (v.) and (vi.) survive, being in fact the partial differential equations determining those covariants which can be expressed in terms of partial differential coefficients of the form with regard to the variables.

Invariants in the Second Order.

10. First, let us consider invariants which involve no partial differential coefficients of order higher than the second. The differential equations to be satisfied are, in the non-subscript notation,

$$\text{(iii.)}\ q\frac{\partial f}{\partial s} + 2p\frac{\partial f}{\partial r} = 0,$$

$$\text{(iv.)}\ p\frac{\partial f}{\partial s} + 2q\frac{\partial f}{\partial t} = 0,$$

$$\text{(v.)}\ q\frac{\partial f}{\partial p} + 2s\frac{\partial f}{\partial r} + t\frac{\partial f}{\partial s} = 0,$$

$$\text{(vi.)}\ p\frac{\partial f}{\partial q} + 2s\frac{\partial f}{\partial t} + r\frac{\partial f}{\partial s} = 0,$$

so far as concerns the form of the function. From these equations we have

$$\frac{\frac{\partial f}{\partial r}}{q^2} = \frac{\frac{\partial f}{\partial s}}{-2pq} = \frac{\frac{\partial f}{\partial t}}{p^2} = \frac{\frac{\partial f}{\partial p}}{2(pt-qs)} = \frac{\frac{\partial f}{\partial q}}{2(qr-sp)}.$$

When Θ is taken to be the common value of these fractions, it follows that

$$\begin{aligned} df &= \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial s}ds + \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial p}dp + \frac{\partial f}{\partial q}dq \\ &= \Theta\, d(q^2r - 2pqs + p^2t). \end{aligned}$$

Now, df is a perfect differential, and therefore Θ is some function of $q^2r - 2pqs + p^2t$; hence, f also is some function of $q^2r - 2pqs + p^2t$, and therefore the *only irreducible invariant which contains differential coefficients of order not higher than the second is*

$$q^2r - 2pqs + p^2t.$$

This is the function A_0, already (§ 3) considered; the integral equation corresponding to the vanishing of this invariant is known.

Invariants in the Third Order.

11. When we come to consider invariants which involve differential coefficients of higher order, the method just used is no longer available, because the four differential equations are not sufficient to determine the ratios of the differential coefficients which

enter. If the determination of the invariants be made from the point of view that they are simultaneous solutions of the four equations, one method of proceeding will be to adopt JACOBI's process.

Any function f, which satisfies (iii.)–(vi.), must also satisfy each of the equations

$$[\Delta_r, \Delta_s] = 0$$

for $r, s = 1, 2, 3, 4$. Forming these, it is easy to show that

$$\begin{aligned}[\Delta_4, \Delta_3] &= \Sigma\,\Sigma\left\{\frac{\partial(\Delta_4 f)}{\partial\left(\dfrac{\partial f}{\partial z_{m,n}}\right)}\frac{\partial(\Delta_3 f)}{\partial z_{m,n}} - \frac{\partial(\Delta_4 f)}{\partial z_{m,n}}\frac{\partial(\Delta_3 f)}{\partial\left(\dfrac{\partial f}{\partial z_{m,n}}\right)}\right\}\\ &= \Sigma\Sigma\,(m-n)\,z_{m,n}\frac{\partial f}{\partial z_{m,n}}\end{aligned}$$

after substitution and collection. Since this does not vanish in virtue of any one of the given equations, we must have a new equation

$$\Delta_5 f = \Sigma\Sigma\,(m-n)\,z_{m,n}\frac{\partial f}{\partial z_{m,n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{vii.}),$$

to be associated with the rest. But this is the only additional equation; for

$$\begin{aligned}&[\Delta_4, \Delta_2] = 0; \quad [\Delta_4, \Delta_1] = -\Delta_2 f = 0; \quad [\Delta_3, \Delta_2] = -\Delta_1 f = 0;\\ &[\Delta_3, \Delta_1] = 0; \quad [\Delta_2, \Delta_1] = 0; \quad [\Delta_5, \Delta_4] = 2\Delta_4 f = 0;\\ &[\Delta_5, \Delta_3] = -2\Delta_3 f = 0; \quad [\Delta_5, \Delta_2] = 0; \quad [\Delta_5, \Delta_1] = 0.\end{aligned}$$

It is easy to verify that $f = A_0$ satisfies (vii.). Every invariant will be a simultaneous solution of (iii.)–(vii.).

It may be noticed that the equation (vii.) can be otherwise obtained; it arises by equating the left-hand sides of equations (i.) and (ii.) to one another, for, on re-arrangement of this, we have

$$\Sigma\Sigma\,(m-n)\,z_{m,n}\frac{\partial f}{\partial z_{m,n}} = 0.$$

Hence, equation (vii.) may be considered as replacing either (i.) or (ii.); and any function, which is a simultaneous solution of (iii.)–(vi.) and for the same value of λ satisfies (i.) and (ii.), will also satisfy (vii.).

One deduction as to the character of the invariants can at once be made from the form of the equations.

Every Irreducible Invariant must involve z_{01} *and* z_{10}.

For every irreducible invariant f satisfies the five equations; if then it be independent of z_{01}, we have $\partial f/\partial z_{01} = 0$. Since there is no term in it which involves z_{01}, and since there is a single term involving z_{01} in $\Delta_3 f = 0$, viz., $z_{01}\,(\partial f/\partial z_{10})$, we must therefore have $\partial f/\partial z_{10}$, *i.e.*, the function must be independent of z_{10}. From $\Delta_1 f = 0$, it then follows that $\partial f/\partial z_{20}$ and $\partial f/\partial z_{11}$ both vanish; from $\Delta_2 f = 0$, it then follows that $\partial f/\partial z_{11}$ and $\partial f/\partial z_{02}$ both vanish, and, therefore, that f involves no differential coefficients of the second order. Proceeding in this way to the successive orders, it appears that f involves no differential coefficients whatever; so that it cannot be an invariant, other than a constant or z.

12. Proceeding now to the consideration of invariants which involve differential coefficients of the third order as the highest, and denoting them for convenience by a, b, c, d ($= z_{30}$, z_{12}, z_{21}, z_{30}, respectively), we have, as the subsidiary equations necessary for the construction of the general solution of $\Delta_3 f = 0$, the set

$$\frac{dp}{q} = \frac{dq}{0} = \frac{dr}{2s} = \frac{ds}{t} = \frac{dt}{0} = \frac{da}{3b} = \frac{db}{2c} = \frac{dc}{d} = \frac{dd}{0}.$$

To deduce that general solution, eight independent integrals of the subsidiary set must be obtained; bearing in mind the character of the invariants (§ 11) ultimately to be arrived at, we take these integrals in the form

$$\begin{aligned}
u_1 &= q, \\
u_2 &= t, \\
u_3 &= qs - pt, \\
u_4 &= q^2 r - 2pqs + p^2 t, \\
u_5 &= d, \\
u_6 &= pd - qc, \\
u_7 &= p^2 d - 2pqc + q^2 b, \\
u_8 &= p^3 d - 3p^2 qc + 3pq^2 b - q^3 a.
\end{aligned}$$

Any solution of the equation $\Delta_3 f = 0$ can be expressed as a functional combination of $u_1, u_2, \ldots, u_8$; thus

$$rt - s^2 = \frac{u_2 u_4 - u_3^2}{u_1^2},$$

$$bd - c^2 = \frac{u_5 u_7 - u_6^2}{u_1^2},$$

$$(ad - bc)^2 - 4\,(ac - b^2)\,(bd - c^2) = \frac{(u_5 u_8 - u_6 u_7)^2 - 4\,(u_5 u_7 - u_6^2)\,(u_6 u_8 - u_7^2)}{u_1^6},$$

and so on.

In order to obtain the most general solution which simultaneously satisfies $\Delta_3 f = 0$ and $\Delta_1 f = 0$, it will be sufficient to obtain the irreducible functional combinations of $u_1, u_2, \ldots, u_8$, which satisfy $\Delta_1 f = 0$. Now,

$$\Delta_1 u_1 = 0, \quad \Delta_1 u_2 = 0, \quad \Delta_1 u_4 = 0, \quad \Delta_1 u_5 = 0\,;$$

and

$$\begin{aligned} \Delta_1 u_3 &= u_1^2, \\ \Delta_1 u_6 &= -\ 2u_1 u_2, \\ \Delta_1 u_7 &= 4u_1 u_3, \\ \Delta_1 u_8 &= -\ 6u_1 u_4\,; \end{aligned}$$

so that

$$\begin{aligned} \Delta_1 (u_1 u_6 + 2u_2 u_3) &= 0, \\ \Delta_1 (u_1 u_7 - 2u_3^2) &= 0, \\ \Delta_1 (u_1 u_8 + 6u_3 u_4) &= 0. \end{aligned}$$

Hence, the most general simultaneous solution of $\Delta_3 f = 0$, and $\Delta_1 f = 0$, can be expressed as a functional combination of

$$\begin{aligned} &u_1,\ u_2,\ u_4,\ u_5, \\ &v_6 = u_1 u_6 + 2u_2 u_3, \\ &v_7 = u_1 u_7 - 2u_3^2, \\ &v_8 = u_1 u_8 + 6u_3 u_4. \end{aligned}$$

In order to obtain the most general simultaneous solution of $\Delta_3 f = 0$, $\Delta_1 f = 0$, $\Delta_2 f = 0$, it will be sufficient to obtain the irreducible functional combinations of $u_1, u_2, u_4, u_5, v_6, v_7, v_8$ which satisfy $\Delta_2 f = 0$. Now, it is easy to show that

$$\Delta_2 u_1 = 0, \quad \Delta_2 u_4 = 0, \quad \Delta_2 v_6 = 0, \quad \Delta_2 v_8 = 0\,;$$

and

$$\begin{aligned} \Delta_2 u_2 &= 2u_1, \\ \Delta_2 u_5 &= 6u_2, \\ \Delta_2 v_7 &= 2u_1 u_4\,; \end{aligned}$$

so that

$$\begin{aligned} \Delta_2 (u_1 u_5 - \tfrac{2}{3} u_2^2) &= 0, \\ \Delta_2 (v_7 - u_2 u_4) &= 0. \end{aligned}$$

Hence, the most general simultaneous solution of $\Delta_3 f = 0$, $\Delta_1 f = 0$, $\Delta_2 f = 0$ can be expressed as a functional combination of

$$\begin{aligned} &u_1, \quad u_4, \quad v_6, \quad v_8, \\ &v_5 = u_1 u_5 - \tfrac{3}{2} u_2^2, \\ &w_7 = v_7 - u_2 u_4 = u_1 u_7 - u_2 u_4 - 2u_3^2. \end{aligned}$$

In order to obtain the most general simultaneous solution of $\Delta_3 f = 0$, $\Delta_1 f = 0$, $\Delta_2 f = 0$, $\Delta_4 f = 0$, it will be sufficient to obtain the irreducible functional combinations of u_1, u_4, v_5, v_6, w_7, v_8 which satisfy $\Delta_4 f = 0$. Now it is easy enough to show that

$$\Delta_4 u_4 = 0\,;$$

and that

$$\begin{aligned} \Delta_4 u_1 &= p, \\ u_1 \Delta_4 v_5 &= 4pv_5 - 3v_6, \\ u_1 \Delta_4 v_6 &= 3pv_6 - 2w_7, \\ u_1 \Delta_4 w_7 &= 2pw_7 - v_8, \\ u_1 \Delta_4 v_8 &= pv_8 + 6u_4^2. \end{aligned}$$

If, then, we write

$$\frac{v_8}{u_1} = \mathrm{P}_8, \quad \frac{w_7}{u_1^2} = \mathrm{P}_7, \quad \frac{v_6}{u_1^3} = \mathrm{P}_6, \quad \frac{v_5}{u_1^4} = \mathrm{P}_5,$$

these equations become

$$\begin{aligned} u_1^2 \Delta_4 \mathrm{P}_8 &= 6u_4^2, \\ u_1^2 \Delta_4 \mathrm{P}_7 &= -\mathrm{P}_8, \\ u_1^2 \Delta_4 \mathrm{P}_6 &= -2\mathrm{P}_7, \\ u_1^2 \Delta_4 \mathrm{P}_5 &= -3\mathrm{P}_6\,; \end{aligned}$$

and therefore, bearing in mind that $\Delta_4 u_4 = 0$, we have

$$\begin{aligned} &\Delta_4 (\mathrm{P}_8^2 + 12u_4^2 \mathrm{P}_7) = 0, \\ &\Delta_4 (\mathrm{P}_8^3 + 18u_4^2 \mathrm{P}_7 \mathrm{P}_8 + 54u_4^4 \mathrm{P}_6) = 0, \\ &\Delta_4 (\mathrm{P}_8 \mathrm{P}_6 - \mathrm{P}_7^2 + 2u_4^2 \mathrm{P}_5) = 0. \end{aligned}$$

Hence the most general simultaneous solution of $\Delta_1 f = 0$, $\Delta_2 f = 0$, $\Delta_3 f = 0$, $\Delta_4 f = 0$ can be expressed as a functional combination of u_4, and

$$\begin{aligned} \mathrm{Q}_7 &= \mathrm{P}_8^2 + 12u_4^2 \mathrm{P}_7, \\ \mathrm{Q}_6 &= \mathrm{P}_8^3 + 18u_4^2 \mathrm{P}_7 \mathrm{P}_8 + 54u_4^4 \mathrm{P}_6, \\ \mathrm{Q}_5 &= \mathrm{P}_8 \mathrm{P}_6 - \mathrm{P}_7^2 + 2u_4^2 \mathrm{P}_5. \end{aligned}$$

13. Before considering the question as to whether these functions satisfy (i.) and (ii.), and, therefore, also (vii.), it is desirable to modify their expressions.

We have already had the quantity u_4; it is the same as A_0, so that we write

$$u_4 = \mathrm{A}_0\,;$$

and it will be convenient to write

$$-u_8 = \mathrm{A}_1 = (z_{30}, z_{21}, z_{12}, z_{03})(z_{01}, -z_{10})^3.$$

Then we have

$$Q_7 = \frac{1}{u_1^2}(v_8^2 + 12u_4^2 w_7)$$
$$= \frac{1}{u_1^2}\{u_1^2 u_8^2 + 12u_1 u_4 (u_3 u_8 + u_4 u_7) + 12u_4^2 (u_3^2 - u_2 u_4)\}.$$

But

$$u_3 u_8 + u_4 u_7 = (qs - pt)(p^3 d - 3p^2 qc + 3pq^2 b - q^3 a)$$
$$+ \{q(qr - ps) - p(qs - pt)\}(p^2 d - 2pqc + q^2 b)$$
$$= q(qr - ps)(p^2 d - 2pqc + q^2 b) + q(qs - pt)(-p^2 c + 2pqb - q^2 a)$$
$$= \tfrac{1}{6} q \left(\frac{\partial A_0}{\partial p}\frac{\partial A_1}{\partial q} - \frac{\partial A_0}{\partial q}\frac{\partial A_1}{\partial p}\right) = \tfrac{1}{6} u_1 J_{01},$$

where J_{01} denotes the Jacobian of A_0, A_1 with regard to u_{10} and u_{01}. Similarly,

$$u_3^2 - u_2 u_4 = u_1^2 (s^2 - rt) = -u_1^2 H_0;$$

so that

$$Q_7 = A_1^2 + 2A_0 J_{01} + 12A_0^2 H_0,$$

H_0 being the discriminant of A_0.

For the modification of the expression of Q_6 we have, on substituting for the quantities P in terms of the quantities u,

$$Q_6 = u_8^3 + 18\frac{u_4 u_8}{u_1}(u_3 u_8 + u_4 u_7) + 18\frac{u_4^2}{u_1^2}(4u_3^2 u_8 - u_2 u_4 u_8 + 6u_3 u_4 u_7 + 3u_4^2 u_6).$$

The modification of the second term has already been given; for the third we have

$$4u_3^2 u_8 - 4u_2 u_4 u_8 = 4u_1^2 A_1 H_0;$$
$$u_2 u_8 + 2u_3 u_7 + u_4 u_6 = u_{12}\{r(pd - qc) - 2s(pc - qb) + t(pb - qa)\};$$
$$= \tfrac{1}{12} u_1^2 \left\{\frac{\partial^2 A_0}{\partial q^2}\frac{\partial^2 A_1}{\partial p^2} - 2\frac{\partial^2 A_0}{\partial p\,\partial q}\frac{\partial^2 A_1}{\partial p\,\partial q} + \frac{\partial^2 A_0}{\partial p^2}\frac{\partial^2 A_1}{\partial q^2}\right\};$$
$$= -\tfrac{1}{12} u_1^2 H_{01},$$

where H_{01} denotes the simultaneous Hessian of A_0 and A_1 with regard to u_{01} and u_{10}. Hence

$$Q_6 = -A_1^3 - 3A_0 A_1 J_{01} + 72 A_0^2 A_1 H_0 - \tfrac{9}{2} A_0^3 H_{01}.$$

For Q_5 we have, after substitution for P_5, P_6, P_7, P_8, the form

$$Q_5 = \frac{1}{u_1^2}(u_6 u_8 - u_7^2)$$
$$+ \frac{2}{u_1^3}(u_2 u_3 u_8 + u_4^2 u_5 + 3u_3 u_4 u_6 + u_2 u_4 u_7 + 2u_3^2 u_7)$$
$$- \frac{1}{u_1^4}(4u_2^2 u_4^2 - 8u_2 u_4 u_3^2 + 4u_3^4).$$

Now, for the first set of terms

$$u_6 u_8 - u_7^2 = q^2 \{q^2 (ac - b^2) - pq (ad - bc) + p^2 (bd - c^2)\}$$
$$= u_1^2 \mathrm{H}_1$$

where H_1 is the Hessian of A_1 considered as a ground-form in q and $-p$; and the third set of terms is

$$-\frac{4}{u_1^4} (u_2 u_4 - u_3^2)^2 = -4\mathrm{H}_0^2.$$

For the middle set of terms it is easily found, by the results already proved, that the terms within the bracket can be expressed in the form

$$u_1 [q \{r^2 d - 3rsc + (2s^2 + rt) b - sta\} + p \{t^2 a - 3tsb + (2s^2 + rt) c - rsd\}]$$
$$= u_1^3 \mathrm{L}_2,$$

say; so that we have

$$\mathrm{Q}_3 = \mathrm{H}_1 + 2\mathrm{L}_2 - 4\mathrm{H}_0^2.$$

14. Considering now the question as to whether each of the functions thus obtained will satisfy (i.) and (ii.), for one and the same numerical value of the index λ properly associated with it, we may proceed as follows. Writing the equations in the form

$$\Omega_1 f = 3\lambda f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{i.})$$

$$\Omega_2 f = 3\lambda f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{ii.})$$

we have the following result :—

$f =$	$\Omega_1 f =$	$\Omega_2 f =$	$\lambda =$
A_1	$6\mathrm{A}_1$	$6\mathrm{A}_1$	2
A_2	$9\mathrm{A}_2$	$9\mathrm{A}_2$	3
[illegible]	[illegible]	[illegible]	4
[illegible]	[illegible]	[illegible]	4
[illegible]	[illegible]	[illegible]	3
[illegible]	[illegible]	[illegible]	2
[illegible]	[illegible]	[illegible]	4

By means of these results we at once find

$$\begin{aligned}\Omega_0 A_0 &= 6A_0 = \Omega_1 A_0 ;\\ \Omega_0 Q_7 &= 18Q_7 = \Omega_1 Q_7 ;\\ \Omega_0 Q_6 &= 27Q_6 = \Omega_1 Q_6 ;\\ \Omega_0 Q_5 &= 12Q_5 = \Omega_1 Q_5 .\end{aligned}$$

Hence A_0, Q_5, Q_6, Q_7 satisfy all the necessary equations, and they are therefore invariants; their respective indices are 2, 4, 9, 6 ; and, therefore, *every invariant which involves differential coefficients of z of order not higher than* 3 *can be expressed as an algebraical function of* A_0, Q_5, Q_6, Q_7, *where* (changing the sign of Q_6 from § 13)

$$\begin{aligned}Q_5 &= H_1 + 2L_2 - 4H_0^2,\\ Q_6 &= A_1^3 + 3A_0A_1J_{01} - 72A_0^2A_1H_0 + \tfrac{9}{2}A_0^3H_{01},\\ Q_7 &= A_1^2 + 2A_0J_{01} + 12A_0^2H_0,\end{aligned}$$

and the quantities A_0, A_1, J_{01}, H_0, H_1, H_{01}, L_2 *are given by the equations*

$$\begin{aligned}A_0 &= (z_{20}, z_{11}, z_{02} \between z_{01}, -z_{10})^2 ;\\ A_1 &= (z_{30}, z_{21}, z_{12}, z_{03} \between z_{01}, -z_{10})^3 ;\\ J_{01} &= \frac{\partial A_0}{\partial z_{10}}\frac{\partial A_1}{\partial z_{01}} - \frac{\partial A_0}{\partial z_{01}}\frac{\partial A_1}{\partial z_{10}} ;\\ H_0 &= z_{20}z_{02} - z_{11}^2 ;\\ H_{01} &= 12\,\{z_{01}(z_{20}z_{12} - 2z_{11}z_{21} + z_{02}z_{30}) - z_{10}(z_{20}z_{03} - 2z_{11}z_{12} + z_{02}z_{21})\}\\ &= \frac{\partial^2 A_0}{\partial z_{10}^2}\frac{\partial^2 A_1}{\partial z_{01}^2} - 2\,\frac{\partial^2 A_0}{\partial z_{10}\,\partial z_{01}}\,\frac{\partial^2 A_1}{\partial z_{10}\,\partial z_{01}} + \frac{\partial^2 A_0}{\partial z_{01}^2}\frac{\partial^2 A_1}{\partial z_{10}^2} ;\\ H_1 &= (z_{30}z_{12} - z_{21}^2,\ z_{30}z_{03} - z_{21}z_{12},\ z_{21}z_{03} - z_{12}^2 \between z_{01}, -z_{10})^2 ;\\ L_2 &= z_{01}\,\{z_{20}^2z_{03} - 3z_{20}z_{11}z_{12} + (2z_{11}^2 + z_{20}z_{02})\,z_{21} - z_{11}z_{02}z_{30}\}\\ &\qquad - z_{10}\,\{z_{20}z_{11}z_{03} - (2z_{11}^2 + z_{20}z_{02})\,z_{12} + 3z_{11}z_{02}z_{21} - z_{02}^2z_{30}\}\\ &= \tfrac{1}{24}\left(\frac{\partial A_0}{\partial z_{10}}\frac{\partial H_{01}}{\partial z_{01}} - \frac{\partial A_0}{\partial z_{01}}\frac{\partial H_{01}}{\partial z_{10}}\right).\end{aligned}$$

The invariant A_0 is that which was obtained before, and it may be called the irreducible invariant of the second order ; the invariants Q_5, Q_6, Q_7 may be called the irreducible invariants of the third order.

15. It may be remarked that the quantities additional to A_1 and A_0 which are necessary for the expression of Q_5, Q_6, Q_7 all belong to the simultaneous concomitant

system of A_0 and A_1 regarded as binary ground-forms in z_{01}, $-z_{10}$ as variables.[*] To this we shall return (§ 34).

A Special Series of Invariants.

16. There is a succession of invariants of consecutive orders, comparatively simple in form, which can be derived by using the remark made in § 9. A set of invariants of the form suggested by the covariants of a binary quantic, which involve only differential coefficients of the quantic with respect to the variables, is derivable by considering the functions in z analogous to HERMITE's "associated covariants" which may be taken to be

$$A_{m-2} = (z_{m,0},\quad z_{m-1,1},\quad z_{m-2,2}, \ldots\ldots, \quad z_{2,m-2},\quad z_{1,m-1},\quad z_{0,m})(z_{01}, -z_{10})^m$$

for values 2, 3, 4, . . . of m.

It is easy to see that each of these functions satisfies the equations (v.) and (vi.), viz., $\Delta_3 f = 0$ and $\Delta_4 f = 0$; these, in fact, are the equations which suggest the functions.

But, when we consider the operators Δ_1 and Δ_2 which do not arise in connexion with covariants of binary forms, we have

$$\begin{aligned}
&\Delta_1 A_{m-2} \\
&= (m-1)\left(z_{0,m-1} - \frac{\partial}{\partial z_{1,m-1}} + 2z_{1,m-2}\frac{\partial}{\partial z_{2,m-2}} + 3z_{2,m-3}\frac{\partial}{\partial z_{3,m-3}} + \ldots + mz_{m-1,0}\frac{\partial}{\partial z_{m,0}}\right)A_{m-2} \\
&= (m-1)\left\{mz_{m-1,0}\,z_{01}{}^m - (m-1)\,z_{m-2,1}\,mz_{01}{}^{m-1}z_{10} + \ldots\right\} \\
&= m\,(m-1)\,z_{10}A_{m-3}\,;
\end{aligned}$$

and, similarly,

$$\Delta_2 A_{m-2} = -m\,(m-1)\,z_{10}A_{m-3}.$$

Again, in regard to the operators which occur in (i.) and (ii.), it is easy to show that

$$\Omega_0 A_{m-2} = 3mA_{m-2},$$
$$\Omega_1 A_{m-2} = 3mA_{m-2}.$$

If, then, we can obtain combinations of A_0, A_1, A_2, . . . which are homogeneous and of uniform grade, such as to satisfy $\Delta_1 f = 0$ and $\Delta_2 f = 0$, these combinations will be invariants; and it follows from the effect of the linear operators Δ_1 and Δ_2 on the quantities A that any combination of the A's which satisfies $\Delta_1 f = 0$ will also satisfy $\Delta_2 f = 0$.

---

[*] SALMON, 'Higher Algebra' (3rd edition), § 198; CLEBSCH, 'Theorie der binären Formen,' § 59; GORDAN, 'Vorlesungen über Invariantentheorie,' vol. 2, § 31. The quantities A_1, A_0, J_{01}, L_2, H_0, H_{01}, H_1 are, save as to numerical factors, respectively the same as SALMON's symbols u, v, $(1, 1)$, L_2, Δ, L_1, $(2, 0)$; as CLEBSCH's symbols ϕ, f, ϑ, q, D, p, Δ; as GORDAN's symbols ϕ, f, ϑ, q, A_{ff}, p, Δ.

Combinations of this kind, which are of uniform grade and are homogeneous, are A_0, $A_0A_2 - k_1A_1^2$, $A_0^2A_3 - l_1A_0A_1A_2 + l_2A_1^3$, $A_0A_4 - m_1A_1A_3 + m_2A_2^2$, and so on. When these are substituted in $\Delta_1 f = 0$ and the coefficients $k, l, m, \ldots$ are determined so that the equation is satisfied, we find the following set of invariants :—

$$
\begin{aligned}
U_0 &= A_0, \\
U_2 &= A_0A_2 - A_1^2, \\
U_3 &= A_0^2A_3 - \tfrac{10}{3}A_0A_1A_2 + \tfrac{20}{9}A_1^3, \\
U_4 &= A_0A_4 - 5A_1A_3 + \tfrac{25}{6}A_2^2, \\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

These combinations suggest an analogy with the coefficients of the principal irreducible covariants of a quantic. If we change the symbols by the relation

$$A_{m-2} = m\,!\,(m-1)\,C_{m-2},$$

then, except as to numerical factors, the functions are

$$
\begin{aligned}
&C_0, \\
&C_0C_2 - C_1^2, \\
&C_0^2C_3 - 3C_0C_1C_2 + 2C_1^3, \\
&C_0C_4 - 4C_1C_3 + 3C_2^2, \\
&\ldots\ldots\ldots\ldots
\end{aligned}
$$

that is, they follow the same law of formation as the leading terms of the covariants referred to; and they can therefore be expressed in terms of the quantities C and can thence be deduced in terms of the quantities A.

All these functions satisfy the equation

$$C_0\frac{\partial U}{\partial C_1} + 2C_1\frac{\partial U}{\partial C_2} + 3C_2\frac{\partial U}{\partial C_3} + \ldots \quad = 0,$$

that is, they satisfy the equation

$$\sum_{n=1}(n+2)(n+1)\,A_{n-1}\frac{\partial U}{\partial A_n} = 0,$$

by means of which the numerical coefficients in U can be directly determined.

It is evident from the form of U_{m-2} that the highest order of differential coefficient which enters is the *m*th, that all the differential coefficients of the *m*th order enter linearly and into only one set of terms, and that the remaining terms all involve coefficients of lower order of differentiation.

Invariants in the Fourth Order.

17. To obtain the irreducible invariants which involve no differential coefficient of order higher than four, we may proceed as in § 11 by forming the irreducible functions which satisfy the differential equations; among these functions the invariants already obtained will occur.

For convenience, let the differential coefficients of the fourth order be denoted by e, f, g, h, i ($= z_{40}, z_{31}, z_{22}, z_{13}, z_{04}$ respectively). Then, beginning as before with $\Delta_3 f = 0$, the subsidiary equations additional to those already (§ 12) considered are

$$\left[\frac{dp}{q} = \frac{dq}{0}\right] = \frac{de}{4f} = \frac{df}{3g} = \frac{dg}{2h} = \frac{dh}{i} = \frac{di}{0};$$

of which the irreducible independent integrals are

$$\begin{aligned}
u_9 &= i, \\
u_{10} &= pi - qh, \\
u_{11} &= p^2 i - 2pqh + q^2 g, \\
u_{12} &= p^3 i - 3p^2 qh + 3pq^2 g - q^3 f, \\
u_{13} &= p^4 i - 4p^3 qh + 6p^2 q^2 g - 4pq^3 f + q^4 e;
\end{aligned}$$

and any solution of $\Delta_3 f = 0$ is expressible as a function of $u_1, u_2, \ldots, u_{13}$.

The remainder of the analysis is very similar to that which has been used for the earlier question, and so it is not here reproduced; the following are the results:—

(i) The functional combinations of the thirteen quantities u which satisfy $\Delta_1 f = 0$ (and which are, therefore, the irreducible simultaneous solutions of $\Delta_3 f = 0 = \Delta_1 f$) are

$$\begin{aligned}
&u_1, u_2, u_4, u_5, u_9; \\
v_6 &= u_1 u_6 + 2u_2 u_3, \\
v_7 &= u_1 u_7 - 2u_3^2, \\
v_8 &= u_1 u_8 + 6u_3 u_4; \\
v_{10} &= u_1 u_{10} + 3u_3 u_5, \\
v_{11} &= u_1 u_{11} + 6u_3 u_6 + 3u_2 u_7, \\
v_{12} &= u_1^2 u_{12} + 9u_1 u_3 u_7 - 12u_3^3, \\
v_{13} &= u_1^2 u_{13} + 12u_1 u_3 u_8 + 36u_3^2 u_4.
\end{aligned}$$

(ii) The functional combinations of these twelve quantities which satisfy $\Delta_2 f = 0$ (and which are therefore the irreducible simultaneous solutions of $\Delta_3 f = \Delta_1 f = \Delta_2 f = 0$) are

$$
\begin{aligned}
&u_1,\ u_4,\ v_6,\ v_8,\ v_{13}\,;\\
v_5 &= u_1 u_5 - \tfrac{3}{2} u_2^2,\\
w_7 &= v_7 - u_2 u_4\,;\\
v_9 &= u_1^2 u_9 - 6u_1 u_2 u_5 + 6u_2^3,\\
w_{10} &= u_1 v_{10} - \tfrac{9}{2} u_2 v_6,\\
w_{11} &= u_1 v_{11} - 6u_2 u_7 + \tfrac{3}{2} u_2^2 u_4,\\
w_{12} &= v_{12} - \tfrac{3}{2} u_2 v_8.
\end{aligned}
$$

(iii) When these functional combinations are substituted in turn for f in $\Delta_4 f$, the equations additional to those in § 12 can be transformed to

$$
\begin{aligned}
u_1\,\Delta_4 v_9 &= 6pv_9 - 4w_{10},\\
u_1\,\Delta_4 w_{10} &= 5pw_{10} - 3w_{11} + 3u_4 v_5,\\
u_1\,\Delta_4 w_{11} &= 4pw_{11} - 2w_{12} + 6u_4 v_6,\\
u_1\,\Delta_4 w_{12} &= 3pw_{12} - v_{13} + 9u_4 w_7,\\
u_1\,\Delta_4 v_{13} &= 2pv_{13} + 12u_4 v_8\,;
\end{aligned}
$$

and therefore, if we write

$$
\frac{v_{13}}{u_1^2} = P_{13},\quad \frac{w_{12}}{u_1^3} = P_{12},\quad \frac{w_{11}}{u_1^4} = P_{11},\quad \frac{w_{10}}{u_1^5} = P_{10},\quad \frac{v_9}{u_1^6} = P_9,
$$

these equations become

$$
\begin{aligned}
u_1^2\Delta_4 P_9 &= -4P_{10},\\
u_1^2\Delta_4 P_{10} &= -3P_{11} + 3u_4 P_5,\\
u_1^2\Delta_4 P_{11} &= -2P_{12} + 6u_4 P_6,\\
u_1^2\Delta_4 P_{12} &= -P_{13} + 9u_4 P_7,\\
u_1^2\Delta_4 P_{13} &= 12u_4 P_8.
\end{aligned}
$$

In addition to the former irreducible solutions, Q_7, Q_6, Q_5, which were obtained from equations in § 13, the following irreducible solutions can be obtained :—

$$
\begin{aligned}
Q_{13} &= u_4 P_{13} - P_8^2,\\
Q_{12} &= 18u_4^3 P_{12} + 3u_4 P_8 P_{13} + 81u_4^4 P_6 - 2P_8^3,\\
Q_{11} &= 72u_4^5 P_{11} + 24u_4^3 P_8 P_{12} + 2u_4 P_8^2 P_{13} + 144u_4^6 P_5 + 108u_4^4 P_7^2 - P_8^4,\\
Q_{10} &= 216u_4^7 P_{10} + 108u_4^5 P_8 P_{11} + 18u_4^3 P_8^2 P_{12} + u_4 P_8^3 P_{13}\\
&\qquad - (108u_4^6 P_8 P_5 + 81u_4^4 P_8^2 P_6 + 18u_4^2 P_8^3 P_7 + P_8^5),\\
Q_9 &= 1296u_4^9 P_9 + 864u_4^7 P_8 P_{10} + 216u_4^5 P_8^2 P_{11} + 24u_4^3 P_8^3 P_{12} + u_4 P_8^4 P_{13}\\
&\qquad - (216u_4^6 P_8^2 P_5 + 108u_4^4 P_8^3 P_6 + 18u_4^2 P_8^4 P_7 + \tfrac{5}{6} P_8^6).
\end{aligned}
$$

These are not necessarily the simplest forms obtainable, but every simultaneous solution of $\Delta_1 f = \Delta_2 f = \Delta_3 f = \Delta_4 f = 0$ can be expressed as a functional combination of u_4, Q_5, Q_6, Q_7, $Q_9, \ldots, Q_{13}$.

It will be seen that the new irreducible functions $Q_9, \ldots, Q_{13}$ are linear in the quantities P_9, $P_{10}, \ldots, P_{13}$, and are therefore linear in the partial differential coefficients of the fourth order. In this respect they apparently differ from Q_5, Q_6, Q_7, which are the irreducible invariants of the third rank in differentiation; but, if we take instead of Q_5 an equivalent invariant $144u_4^2Q_5 + Q_7^2$, which is

$$288u_4^6P_5 + 144u_4^4P_6P_5 + 24u_4^2P_7P_8^2 + P_5^4,$$

the law of successive formation of the invariants (the new) Q_5, Q_6, Q_7 is similar to that for the functions Q_9, $Q_{10}, \ldots, Q_{13}$.

18. But, before it can be asserted that Q_9, $Q_{10} \ldots Q_{13}$ are invariants, it must be shown that they severally for a common value of λ satisfy the equations (i'.) and (ii'.). Now the following results are easily obtained:—

$f =$	$\Omega_0 f =$	$\Omega_1 f =$	$\lambda =$
P_5	0	0	0
P_6	$3P_6$	$3P_6$	1
P_7	$6P_7$	$6P_7$	2
P_8	$9P_8$	$9P_8$	3
P_9	0	0	0
P_{10}	$3P_{10}$	$3P_{10}$	1
P_{11}	$6P_{11}$	$6P_{11}$	2
P_{12}	$9P_{12}$	$9P_{12}$	3
P_{13}	$12P_{13}$	$12P_{13}$	4

From this Table it at once follows that

$$\begin{aligned}\Omega_0 Q_{13} &= 18 Q_{13} = \Omega_1 Q_{13},\\ \Omega_0 Q_{12} &= 27 Q_{12} = \Omega_1 Q_{12},\\ \Omega_0 Q_{11} &= 36 Q_{11} = \Omega_1 Q_{11},\\ \Omega_0 Q_{10} &= 45 Q_{10} = \Omega_1 Q_{10},\\ \Omega_0 Q_{9} &= 54 Q_{9} = \Omega_1 Q_{9}.\end{aligned}$$

Hence, Q_9, Q_{10}, Q_{11}, Q_{12}, Q_{13} *are invariants of indices* 18, 15, 12, 9, 6 *respectively; and every invariant which involves no differential coefficient of order higher than the fourth can be expressed as a function of* u_4, Q_5, Q_6, Q_7, Q_9, Q_{10}, Q_{11}, Q_{12}, Q_{13}.

General Inferences.

19. And if, among the sets of irreducible invariants thus obtained, those invariants which involve the partial differential coefficients of the nth order as the highest that occur, and which are linear in those partial differential coefficients of highest order, are called irreducible invariants *proper to the rank n*, then we have the following propositions relating to the complete aggregate of invariants :—

(i) The irreducible invariants can be ranged in sets, each set being proper to a particular rank ;
(ii) There is no irreducible invariant proper to the rank unity ;
(iii) There is a single irreducible invariant ($= u_4 = A_0$) proper to the rank 2 ;
(iv) There are three irreducible invariants ($= Q_5, Q_6, Q_7$) proper to the rank 3 ;
(v) For every value of n greater than 3, there are $n + 1$ irreducible invariants proper to the rank n, and they can be so chosen as to be linear in the differential coefficients of order n ;
(vi) Every invariant can be expressed as a function of the irreducible invariants ; and, if such an invariant have differential coefficients of order r as those of highest order occurring in it, the functional equivalent involves some or all of the aggregate of irreducible invariants proper to ranks not greater than r; it involves some of the irreducible invariants proper to the rank r, but no irreducible invariant proper to a rank greater than r.

Simultaneous Invariants of Two Functions.

20. Hitherto we have considered invariants of only a single dependent variable which is a function of the two independent variables ; but we may consider a second dependent variable, say z', which is also a function of x and y. The two quantities

$$\Theta_3 u_1 = 0, \quad \Theta_3 u'_1 = 0; \qquad \Theta_2 u_4 = 0, \quad \Theta_2 u'_4 = 0; \qquad \Theta_2 u_5 = 0;$$

and

$$\begin{aligned} \Theta_2 u_2 &= \quad 2u_1, \\ \Theta_2 u'_2 &= \quad 2u'_1, \\ \Theta_2 v_3 &= -3u_5. \end{aligned}$$

Hence

$$\begin{aligned} \Theta_2 (u_1 u'_2 - u_2 u'_1) &= 0, \\ \Theta_2 (3u_2 u_5 + 2u_1 v_3) &= 0; \end{aligned}$$

and therefore, if

$$\begin{aligned} \mathrm{P} &= u_1 u'_2 - u_2 u'_1 = qt' - q't, \\ \mathrm{Q} &= 3u_2 u_5 + 2u_1 v_3, \end{aligned}$$

the irreducible combinations which satisfy $\Theta_1 \psi = 0 = \Theta_2 \psi = \Theta_3 \psi$, are u_1, u'_1, P, u_4, u'_4, u_5, Q.

It is now necessary to obtain the irreducible combinations of these seven quantities which satisfy $\Theta_4 \psi = 0$. We have

$$\Theta_4 u_1 = p, \qquad \Theta_4 u'_1 = p';$$

so that

$$u'_1 \Theta_4 u_1 - u_1 \Theta_4 u'_1 = u_5.$$

Again,

$$\Theta_4 \mathrm{P} = pt' - p't + 2qs' - 2q's.$$

Now,

$$q(pt' - p't) = p(\mathrm{P} + q't) - p'qt = p\mathrm{P} + u_5 u_2;$$

so that

$$\begin{aligned} \Theta_4 \mathrm{P} &= 3(pt' - p't) + 2v_3 \\ &= \frac{3}{q}(p\mathrm{P} + u_2 u_5) + 2v_3; \end{aligned}$$

and hence

$$u_1 \Theta_4 \mathrm{P} - 3p\mathrm{P} = \mathrm{Q}.$$

Again,

$$\tfrac{1}{2}\Theta_4 \mathrm{Q} = 3su_5 + pv_3 + q\Theta_4 v_3;$$

but

$$\Theta_4 v_3 = qr' - ps' - (q'r - p's);$$

and

$$\begin{aligned} qr' - ps' &= \frac{1}{q}(q^2 r' - 2pqs' + p^2 t') + \frac{p}{q}(qs' - pt') \\ &= \frac{1}{u_1 u_1'^2}(u_1^2 u'_4 - 2u_1' u_3 u_5 - u'_2 u_5^2) + \frac{pu'_3}{u_1}, \\ q'r - p's &= \frac{1}{u_1' u_1^2}(u_1'^2 u_4 + 2u'_1 u_3 u_5 - u'_2 u_5^2) + \frac{p'u'_3}{u'_1}; \end{aligned}$$

so that

$$\Theta_4 v_3 = \frac{p}{u_1} u_3 - \frac{p' u'_3}{u'_1} + \frac{u_1 u'_4}{u_1^2} - \frac{u'_1 u_4}{u_1^2} - \frac{2u_5}{u_1 u'_1}(u_3 + u_3') - \frac{u_5^2}{u_1^2 u'^2_1} \mathrm{P}.$$

Hence

$$pv_3 + q\Theta_4 v_3 = 2pv_3 + \frac{u_1^2}{u'^2_1} u'_4 - \frac{u'_1}{u_1} u_4 - \frac{u_5^2}{u_1 u'^2_1} \mathrm{P} - \frac{u_5}{u'_1}(2u_3 + u'_3).$$

Hence

$$\begin{aligned}\tfrac{1}{2} u_1 \Theta_4 \mathrm{Q} - p\mathrm{Q} &= u_1\left(\frac{u_1^2}{u'^2_1} u_4' - \frac{u'_1}{u_1} u_4\right) - \frac{u_5^2}{u'^2_1} \mathrm{P} - \frac{u_5 u_1}{u'_1}\left(2u_3 + u'_3 - 3q's + 3\frac{pq'}{q} t\right) \\ &= u_1\left(\frac{u_1^2}{u'^2_1} u_4' - \frac{u'_1}{u_1} u_4\right) - \frac{u_5^2}{u'^2_1} \mathrm{P} - \frac{u_5 \mathrm{Q}}{u'_1}.\end{aligned}$$

If now we write

$$\frac{u_1}{u'_1} = \mathrm{C},$$

$$\frac{\mathrm{P}}{u_1^3} = \mathrm{B},$$

$$\frac{\mathrm{Q}}{u_1^2} = \mathrm{A},$$

the three results can be put into the forms

$$\begin{aligned} u'^2_1 \Theta_4 \mathrm{C} &= u_5, \\ u'^2_1 \Theta_4 \mathrm{B} &= \mathrm{AC}^{-4}, \\ u'^2_1 \Theta_4 \mathrm{A} &= 2\mathrm{C}^2 u'_4 - 2\mathrm{C}^{-1} u_4 - 2\mathrm{BC}^2 u_5^2. \end{aligned}$$

And, further, we have

$$\Theta_4 u_4 = 0, \qquad \Theta_4 u_4' = 0, \qquad \Theta_4 u_5 = 0.$$

Since the result of operating with Θ_4 on B gives a quantity into which A enters linearly, and the result of operating with Θ_4 on A gives another quantity into which B enters linearly, we are led to assume that the irreducible solution (or solutions) of $\Theta_4 \psi = 0$ are of the form

$$\mathrm{RA} + \mathrm{SB} + \mathrm{T},$$

where R, S, T are independent of A and B. If this be a solution, we have

$$u'^2_1 \Theta_4 \mathrm{T} = \mathrm{R}(2\mathrm{C}^2 u'_4 - 2\mathrm{C}^{-1} u_4 - 2\mathrm{BC}^2 u_5^2) + \mathrm{SAC}^{-4} + \mathrm{A} u'^2_1 \Theta_4 \mathrm{R} + \mathrm{B} u'^2_1 \Theta_4 \mathrm{S};$$

and we suppose R and S so determined that

$$\begin{aligned} \mathrm{SC}^{-4} &= - u'^2_1 \Theta_4 \mathrm{R}, \\ 2\mathrm{RC}^2 u_5^2 &= u'^2_1 \Theta_4 \mathrm{S}. \end{aligned}$$

But, if $R = C^\kappa$, then

$$SC^{-4} = -nu'^2_1 C^{\kappa-1} \Theta_4 C = -nu_5 C^{\kappa-1};$$

so that

$$S = -nu_5 C^{\kappa+3};$$

and therefore

$$\begin{aligned} 2u_5^2 C^{\kappa+2} = 2RC^2 u_5^2 &= -n(n+3)\,u_5 C^{\kappa+2} u'^2_1 \Theta_4 C \\ &= -n(n+3)\,u_5^2 C^{\kappa+2}: \end{aligned}$$

whence

$$n = -1 \quad \text{or} \quad -2.$$

First, taking $n = -1$, we have

$$u'^2_1 \Theta_4 T = 2Cu'_4 - 2C^{-2} u_4,$$

and therefore

$$Tu_5 = C^2 u'_4 + 2u_4 C^{-1}.$$

Hence we may take as one irreducible solution

$$X = \frac{A}{C} + BC^2 u_5 - \frac{2u_4 + C^3 u'_4}{u_5 C}.$$

Second, taking $n = -2$, we have

$$u'^2_1 \Theta_4 T = 2u'_4 - 2C^{-3} u_4;$$

and therefore

$$Tu_5 = 2Cu'_4 + C^{-2} u_4.$$

Hence we may take as another irreducible solution

$$Y = \frac{A}{C^2} + 2BCu_5 - \frac{u_4 + 2C^3 u'_4}{u_5 C^2}.$$

And it follows from the method of derivation, and by an application of the theory of linear partial differential equations, that every simultaneous solution of the equations $\Theta_1\psi = 0 = \Theta_2\psi = \Theta_3\psi = \Theta_4\psi$ which involves no quantity of order higher than r, s, t, r', s', t' can be expressed as a functional combination of u_5; u_4, u'_4; X, Y.

23. It is now necessary to consider the index equations. We have for u_5 (= J)

$$\Phi_0 J = 3J,$$
$$\Phi_1 J = 3J;$$

so that J is an invariant of index unity. For u_4 (= A_0) we have

$$\Phi_0 A_0 = 6A_0,$$
$$\Phi_1 A_0 = 6A_0;$$

so that A_0 is an invariant of index 2; and similarly for $u'_4 (= A'_0)$,

$$\Phi_0 A'_0 = 6A'_0,$$
$$\Phi_1 A_0' = 6A'_0;$$

so that A'_0 is an invariant of index 2.

It is desirable to modify the forms of X and Y, so as to express them explicitly in terms of the quantities $p, p', \ldots$. When the values of A, B, C, u_4, u'_4, u_5 are substituted in X, and it is multiplied by $-u_5$, it takes the form

$$q^2 r' - 2pqs' + p^2 t' + 2\{qq'r - (pq' + p'q)s + pp't\},$$

which may be denoted by $\mathfrak{A}_0$; and when exactly the same operations are applied to Y, it takes the form

$$q'^2 r - 2p'q's + p'^2 t + 2\{qq'r' - (pq' + p'q)s' + pp't'\},$$

which may be denoted by $\mathfrak{A}'_0$.

It is now easy to verify that

$$\Phi_0 \mathfrak{A}_0 = 6\mathfrak{A}_0,$$
$$\Phi_1 \mathfrak{A}_0 = 6\mathfrak{A}_0;$$

so that $\mathfrak{A}_0$ is an invariant of index 2; and similarly that

$$\Phi_0 \mathfrak{A}'_0 = 6\mathfrak{A}'_0,$$
$$\Phi_1 \mathfrak{A}'_0 = 6\mathfrak{A}'_0;$$

so that $\mathfrak{A}'_0$ is an invariant of index 2.

24. The general result of the preceding investigation can be enunciated as follows:—

Every simultaneous invariant of two functions z and z′ of two independent variables, which involves no differential coefficients of order higher than the second, can be expressed in terms of the five irreducible invariants J (*of index* 1) *and* A_0, A'_0, $\mathfrak{A}_0$, $\mathfrak{A}'_0$ (*each of index* 2) *where*

$$J = pq' - p'q,$$
$$A_0 = q^2 r - 2pqs + p^2 t,$$
$$A'_0 = q'^2 r' - 2p'q's' + p'^2 t',$$
$$\mathfrak{A}_0 = q^2 r' - 2pqs' + p^2 t' + 2\{qq'r - (pq' + p'q)s + pp't\},$$
$$\mathfrak{A}'_0 = q'^2 r - 2p'q's + p'^2 t + 2\{qq'r' - (pq' + p'q)s' + pp't'\},$$

*and p, q, r, s, t; p′, q′, r′, s′, t′; have their ordinary significations as partial differential coefficients of z and of z′**.

---

* It is easy to see that the invariant A_0, formed for $z + \lambda z'$ is

$$A_0 + \lambda \mathfrak{A}_0 + \lambda^2 \mathfrak{A}'_0 + \lambda^3 A'_0.$$

This remark is practically due to Professor CAYLEY.

Theory of Eduction.

25. It has already appeared from § 4 that the operator

$$q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}$$

operating on z' produces an invariant of index unity. But for the purposes of this operation z' may be regarded merely as an unchanging quantity, and, therefore, it may be replaced by an absolute invariant (of index zero); and, when the operator acts upon an absolute invariant, there results a new invariant, of the next higher rank in the differential coefficient of the variable and of index unity.

We can, however, make the operator an absolute invariant, for the index of A_0 is 2; and, therefore,

$$A_0^{-\frac{1}{2}}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right)$$

is an absolute invariantive operator which, when it operates on an absolute invariant, generates a new absolute invariant of next higher rank.

The operator can evidently be applied any number of times in succession, so that, if I be an absolute invariant,

$$\left\{A_0^{-\frac{1}{2}}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right)\right\}^r \mathrm{I}$$

is an absolute invariant for all values of the index r.

Similarly, the result of operating upon any absolute invariant with the operator

$$q'\frac{\partial}{\partial x} - p'\frac{\partial}{\partial y}$$

is to give a relative invariant of index unity; and, if we are considering simultaneous invariants in two variables z and z', then

$$\frac{1}{\mathrm{J}}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right),$$

$$\frac{1}{\mathrm{J}}\left(q'\frac{\partial}{\partial x} - p'\frac{\partial}{\partial y}\right)$$

are absolute invariantive operators, which, when applied to absolute invariants, produce absolute invariants.

26. Thus, in the case of a single dependent variable, we have

$$\mathrm{A} = \mathrm{Q}_5\mathrm{A}_0^{-2}, \qquad \mathrm{B} = \mathrm{Q}_6\mathrm{A}_0^{-\frac{5}{2}}, \qquad \mathrm{C} = \mathrm{Q}_7\mathrm{A}_0^{-3}$$

as the three irreducible invariants proper to the rank three, and they form the complete system of irreducible absolute invariants within this rank. Hence

$$A' = A_0^{-\frac{1}{2}}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right)A,$$

$$B' = A_0^{-\frac{1}{2}}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right)B,$$

$$C' = A_0^{-\frac{1}{2}}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right)C,$$

are absolute invariants proper to the rank four. But it is not to be inferred that A, B, C, A′, B′, C′ constitute the complete system of irreducible absolute invariants within the rank four.

Again, in the operator the quantities p and q are first differential coefficients of an unchanging quantity z; they can be replaced by first differential coefficients of any absolute invariant I, and then

$$A_0^{-\frac{1}{2}}\left(\frac{\partial I}{\partial y}\frac{\partial}{\partial x} - \frac{\partial I}{\partial x}\frac{\partial}{\partial y}\right)$$

is an absolute invariantive operator, the operation of which on absolute invariants produces other absolute invariants. Hence

$$D = A_0^{-\frac{1}{2}}\left(\frac{\partial B}{\partial y}\frac{\partial}{\partial x} - \frac{\partial B}{\partial x}\frac{\partial}{\partial y}\right)C,$$

$$E = A_0^{-\frac{1}{2}}\left(\frac{\partial C}{\partial y}\frac{\partial}{\partial x} - \frac{\partial C}{\partial x}\frac{\partial}{\partial y}\right)A,$$

$$F = A_0^{-\frac{1}{2}}\left(\frac{\partial A}{\partial y}\frac{\partial}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial}{\partial y}\right)B,$$

are absolute invariants proper to the rank four, and they are of the second degree in the differential coefficients of the fourth order. Among the six quantities A′, B′, C′, D, E, F there is the relation,

$$A'D + B'E + C'F = 0;$$

so that only five of them can be independent.

27. In any higher rank n let I, J, K, . . . be the invariants, absolute and irreducible, proper to that rank; let I′ denote the absolute invariant educed from I by the operator

$$A_0^{-\frac{1}{2}}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right),$$

and I_m the absolute invariant educed from I by the operator

$$A_0^{-\frac{1}{2}}\left(\frac{\partial M}{\partial y}\frac{\partial}{\partial x} - \frac{\partial M}{\partial x}\frac{\partial}{\partial y}\right),$$

M being an absolute invariant. Then, by means of all the eductive operators associated with absolute invariants of successive ranks, we can obtain from I the set of educed invariants

$$\begin{array}{l} \mathrm{I}' ; \\ \mathrm{I}_a, \ \mathrm{I}_b, \ \mathrm{I}_c ; \\ \mathrm{I}_{a'}, \ \mathrm{I}_{b'}, \ \mathrm{I}_{c'}, \ \mathrm{I}_d, \ \mathrm{I}_e, \ \mathrm{I}_f ; \\ . \quad . \quad . \quad . \quad . \quad . \quad . ; \\ \mathrm{I}_j, \ \mathrm{I}_l, \ \ldots , \end{array}$$

all proper to the rank $n+1$; and there is a similar set from each of the other invariants J, K, . . .

This number, however, can be at once reduced; for, if I_m be any educed invariant other than I′ and I_a, we have

$$\begin{vmatrix} \mathrm{I}', & q, & p \\ \mathrm{I}_a, & \dfrac{\partial \mathrm{A}}{\partial y}, & \dfrac{\partial \mathrm{A}}{\partial x} \\ \mathrm{I}_m, & \dfrac{\partial \mathrm{M}}{\partial y}, & \dfrac{\partial \mathrm{M}}{\partial x} \end{vmatrix} = 0 ;$$

and therefore

$$\mathrm{I}_a \mathrm{M}' = \mathrm{I}'\mathrm{M}_a + \mathrm{I}_m \mathrm{A}',$$

which shows that I_m can be expressed in terms of I′ and I_a, and of invariants proper to lower ranks if M be different from J, K, . . ., and that, if M coincide with one of the invariants J, K, . . ., the invariant I_m can be expressed in terms of the set I′, J′, . . ., the set I_a, J_a, . . ., and of invariants proper to lower ranks.

It therefore follows that the invariants, educed from the absolute irreducible invariants I, J, K, . . . proper to the rank n, can be expressed in terms of I′, J′, K′, . . . ; I_a, J_a, K_a, . . . proper to the rank $n+1$, and of invariants proper to lower ranks. All these educed invariants are, if n be greater than 3, linear in the partial differential coefficients, which are of order $n+1$, and so determine the rank of the invariants.

We know that, for values of n greater than 3, the number of irreducible invariants proper to the rank n is $n+1$, all of which can be made absolute on division by an appropriate power of A_0; hence, the number of invariants educed as above is $2(n+1)$, which must all be expressible in terms of the $n+2$ irreducible invariants proper to the rank $n+1$. But so far there is nothing to indicate which of them, or how many of them, are equivalent to irreducible invariants proper to the rank to which they belong.

28. Again, we have seen that there are four simultaneous invariants of two functions proper to the rank 2, and that there is a single invariant proper to the rank 1; so that

$$C_0 = A_0 J^{-2}, \quad C'_0 = A'_0 J^{-2}, \quad \mathfrak{C}_0 = \mathfrak{A}_0 J^{-2}, \quad \mathfrak{C}'_0 = \mathfrak{A}'_0 J^{-2},$$

are absolute irreducible invariants proper to the rank 2. Let

$$\frac{1}{J}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right)C_0 = F, \qquad \frac{1}{J}\left(q'\frac{\partial}{\partial x} - p'\frac{\partial}{\partial y}\right)C_0 = F';$$

$$\frac{1}{J}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right)C'_0 = G, \qquad \frac{1}{J}\left(q'\frac{\partial}{\partial x} - p'\frac{\partial}{\partial y}\right)C'_0 = G';$$

$$\frac{1}{J}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right)\mathfrak{C}_0 = \mathfrak{F}, \qquad \frac{1}{J}\left(q'\frac{\partial}{\partial x} - p'\frac{\partial}{\partial y}\right)\mathfrak{C}_0 = \mathfrak{F}';$$

$$\frac{1}{J}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right)\mathfrak{C}'_0 = \mathfrak{G}, \qquad \frac{1}{J}\left(q'\frac{\partial}{\partial x} - p'\frac{\partial}{\partial y}\right)\mathfrak{C}'_0 = \mathfrak{G}';$$

then F, F', G, G', $\mathfrak{F}$, $\mathfrak{F}'$, $\mathfrak{G}$, $\mathfrak{G}'$, are eight educed invariants proper to the rank three. But instead of q and p, or q' and p', we can substitute the first differential coefficients of any unchanging quantity, say of any one of the absolute invariants C_0, C'_0, $\mathfrak{C}_0$, $\mathfrak{C}'_0$, and thus educe new invariants. All these, however, can be expressed in terms of the set of eight already retained; for we at once have

$$\frac{\partial C_0}{\partial x} = p\,F' - p'\,F,$$

$$\frac{\partial C_0}{\partial y_s} = q\,F' - q'\,F\ ;$$

and therefore

$$\frac{1}{J}\left(\frac{\partial C_0}{\partial y}\frac{\partial}{\partial x} - \frac{\partial C_0}{\partial x}\frac{\partial}{\partial y}\right)C'_0 = J\,(F'G - F\,G'),$$

which proves the statement.*

Hence, through the present class of eductive operators we are able to derive from the simultaneous invariants proper to a rank n double the number of educed invariants proper to the next higher rank; but it is not to be inferred that they are all irreducible, or that they form the complete system of irreducible invariants proper to that rank.

29. The foregoing linear operators are not the only eductive operators; in fact, each new invariant suggests a new eductive operator. For the fundamental property of non-variation on the part of z, the differential coefficients of which are combined into invariants, enables us to substitute for z any other unchanging quantity, such as an absolute invariant. Thus, for instance, if I be any absolute invariant, then

$$\left(\frac{\partial I}{\partial y}\right)^2\frac{\partial^2 I}{\partial x^2} - 2\frac{\partial I}{\partial x}\frac{\partial I}{\partial y}\frac{\partial^2 I}{\partial x\,\partial y} + \left(\frac{\partial I}{\partial x}\right)^2\frac{\partial^2 I}{\partial y^2},$$

---

* Similarly for functions of $z + \lambda z'$; thus

$$\mathbb{F}(z + \lambda z') = F + \lambda(\mathfrak{F} + F') + \lambda^2(\mathfrak{G} + \mathfrak{F}') + \lambda^3(G + \mathfrak{G}') + \lambda^4 G'.$$

the same function of I as A'_0 is of z', is an invariant of index 2, and

$$q^2\frac{\partial^2 I}{\partial x^2} - 2pq\frac{\partial^2 I}{\partial x\,\partial y} + p^2\frac{\partial^2 I}{\partial y^2} + 2\left\{(qr - ps)\frac{\partial I}{\partial y} - (qs - pt)\frac{\partial I}{\partial x}\right\},$$

the same function of z and I as $\mathfrak{A}_0$ is of z and z', is an invariant of index 2.

30. The expressions for educed invariants can be applied as follows to obtain the expressions for the effects of the *operation of* Θ_1, Θ_2, Θ_3, Θ_4 (§ 20) *on differential coefficients of the invariants with regard to the variables.*

Let V be an invariant of index m, and let the operators $q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}$, $q'\frac{\partial}{\partial x} - p'\frac{\partial}{\partial y}$ be denoted by δ and δ' respectively. Then

$$\begin{aligned} V_1 &= J\delta V - mV\delta J, \\ V'_1 &= J\delta' V - mV\delta' J \end{aligned}$$

are invariants of index $m + 2$; they must satisfy the equations

$$\Theta_1 f = 0 = \Theta_2 f = \Theta_3 f = \Theta_4 f.$$

Hence

$$\begin{aligned} J\Theta\delta V &= mV\Theta\delta J, \\ J\Theta\delta' V &= mV\Theta\delta' J, \end{aligned}$$

are satisfied for each of the operators Θ, because J and V are themselves invariants. Now, actual substitution gives

$$\begin{aligned} \Theta_1\delta J &= 3qJ, & \Theta_1\delta' J &= 3q'J; \\ \Theta_2\delta J &= -3pJ, & \Theta_2\delta' J &= -3p'J; \\ \Theta_3\delta J &= 0, & \Theta_3\delta' J &= 0; \\ \Theta_4\delta J &= 0, & \Theta_4\delta' J &= 0; \end{aligned}$$

and therefore

$$\begin{aligned} \Theta_1\delta V &= 3qVm, & \Theta_1\delta' V &= 3q'Vm; \\ \Theta_2\delta V &= -3pVm, & \Theta_2\delta' V &= -3p'Vm; \\ \Theta_3\delta V &= 0, & \Theta_3\delta' V &= 0; \\ \Theta_4\delta V &= 0, & \Theta_4\delta' V &= 0. \end{aligned}$$

Now, since

$$\Theta_1(\delta V) = q\Theta_1\frac{\partial V}{\partial x} - p\Theta_1\frac{\partial V}{\partial y},$$

and

$$\Theta_1\delta' V = q'\Theta_1\frac{\partial V}{\partial x} - p'\Theta_1\frac{\partial V}{\partial y},$$

it follows from the first pair of equations that

$$\left.\begin{aligned} \Theta_1 \frac{\partial V}{\partial x} &= 3mV \\ \Theta_1 \frac{\partial V}{\partial y} &= 0 \end{aligned}\right\} .$$

Similarly, from the second pair

$$\left.\begin{aligned} \Theta_2 \frac{\partial V}{\partial x} &= 0 \\ \Theta_2 \frac{\partial V}{\partial y} &= 3mV \end{aligned}\right\} ;$$

from the third pair

$$\left.\begin{aligned} \Theta_3 \frac{\partial V}{\partial x} &= \frac{\partial V}{\partial y} \\ \Theta_3 \frac{\partial V}{\partial y} &= 0 \end{aligned}\right\} ;$$

and from the fourth pair

$$\left.\begin{aligned} \Theta_4 \frac{\partial V}{\partial x} &= 0 \\ \Theta_4 \frac{\partial V}{\partial y} &= \frac{\partial V}{\partial x} \end{aligned}\right\} .$$

And the general laws, of which these are particular examples, and which can be established by means of the successive educts of the invariant V, are

$$\left.\begin{aligned} \Theta_1 \frac{\partial^n V}{\partial x^s \partial y^{n-s}} &= s(3m+n-1)\frac{\partial^{n-1} V}{\partial x^{s-1} \partial y^{n-s}} \\ \Theta_2 \frac{\partial^n V}{\partial x^{n-s} \partial y^s} &= s(3m+n-1)\frac{\partial^{n-1} V}{\partial x^{n-s} \partial y^{s-1}} \\ \Theta_3 \frac{\partial^n V}{\partial x^r \partial y^{n-r}} &= r\frac{\partial^n V}{\partial x^{r-1} \partial y^{n-r+1}} \\ \Theta_4 \frac{\partial^n V}{\partial x^{n-r} \partial y^r} &= r\frac{\partial^n V}{\partial x^{n-r+1} \partial y^{r-1}} \end{aligned}\right\} .$$

From these the effect on V of any combinations in any order of the operators $\partial/\partial x$, $\partial/\partial y$, Θ_1, Θ_2, Θ_3, Θ_4, can be deduced.

31. The following is another application of the theory of eduction. The index of U_2 (§ 16) is 6, so that $U_2U_0^{-3}$ is an absolute invariant, and therefore

$$\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right)U_2U_0^{-3}$$

is an invariant, say

$$V = V_0\,\delta U_2 - 3U_2\,\delta U_0.$$

Now the quantities U are expressed in terms of the quantities A; and from the values of those quantities it at once follows that

$$\begin{aligned}\delta A_m &= A_{m+1} + (qr - sp)\frac{\partial A_m}{\partial p} + (qs - tp)\frac{\partial A_m}{\partial q}\\ &= A_{m+1} - \tfrac{1}{2}\left(\frac{\partial A_0}{\partial p}\frac{\partial A_m}{\partial q} - \frac{\partial A_0}{\partial q}\frac{\partial A_m}{\partial p}\right)\\ &= A_{m+1} - \tfrac{1}{2}J_{0m};\end{aligned}$$

and therefore in particular

$$\begin{aligned}\delta A_0 &= A_1,\\ \delta A_1 &= A_2 - \tfrac{1}{2}J_{01},\\ \delta A_2 &= A_3 - \tfrac{1}{2}J_{02}.\end{aligned}$$

Hence

$$\begin{aligned}\delta U_2 &= \delta(A_0A_2 - A_1^2)\\ &= A_0A_3 - A_1A_2 - \tfrac{1}{2}A_0J_{02} + A_1J_{01};\end{aligned}$$

and therefore

$$V = A_0^2A_3 - 4A_0A_1A_2 + 3A_1^3 - \tfrac{1}{2}A_0^2J_{02} + A_0A_1J_{01},$$

an invariant proper to the rank 5. But

$$U_3 = A_0^2A_3 - \tfrac{10}{3}A_0A_1A_2 + \tfrac{20}{9}A_1^3$$

is an invariant proper to the rank 5; hence

$$Z = V - U_3 = \tfrac{7}{9}A_1^3 - \tfrac{2}{3}A_0A_1A_2 - \tfrac{1}{2}A_0^2J_{02} + A_0A_1J_{01}$$

is an invariant, and it is evidently proper to the rank 4. It must, therefore, be expressible in terms of the irreducible invariants within the rank 4 given by u_4, Q_5, Q_6, Q_7, Q_9, . . . , Q_{13}. The verification of this inference is as follows.

32. We have

$$Q_{13} = u_4P_{13} - P_8^2;$$

when the values of P_{13} and of P_8—viz.:

$$\frac{1}{u_1'^2}(u_1^2u_{13} + 12u_1u_3u_8 + 36u_3^2u_4) \quad\text{and}\quad \frac{1}{u_1}(u_1u_8 + 6u_3u_4)$$

respectively—are substituted, we at once have

$$\begin{aligned}Q_{13} &= u_4u_{13} - u_8^2\\ &= A_0A_2 - (-A_1)^2 = A_0A_2 - A_1^2,\end{aligned}$$

thus identifying Q_{13} with U_2.

Again, we have

$$\begin{aligned} Q_{12} &= 18u_4{}^3P_{12} + 3u_4P_8P_{13} + 81u_4{}^4P_6 - 2P_8{}^3 \\ &= 3u_4X + 81u_4{}^4P_6 - 2P_8{}^3 ; \end{aligned}$$

so that X, denoting $6u_4{}^2P_{12} + P_8P_{13}$, includes all terms proper to the rank 4. When we substitute for P_8, P_{12}, and P_{13} their values we have

$$\begin{aligned} X &= \frac{6u_4{}^2}{u_1{}^3}(u_1{}^2u_{12} + 9u_1u_3u_7 - 12u_3{}^3 - \tfrac{3}{2}u_2v_8) + \frac{v_8}{u_1{}^3}(u_1{}^2u_{13} + 12u_1u_3u_8 + 36u_3{}^2u_4) \\ &= \frac{6u_4{}^2u_{12} + v_8u_{13}}{u_1} + X', \end{aligned}$$

where

$$X' = \frac{18u_4{}^2}{u_{13}}(3u_1u_3u_7 - 4u_3{}^3 - \tfrac{1}{2}u_2v_8) + \frac{12v_8}{u_1{}^3}(u_1u_3u_8 + 3u_3{}^2u_4).$$

Now, for the first part of X we have

$$\frac{6u_4u_{12} + v_8u_{13}}{u_1} = u_8u_{13} + \frac{6u_4}{u_1}(u_4u_{12} + u_3u_{13}) ;$$

and

$$u_4u_{12} + u_3u_{13} = \tfrac{1}{8}\left\{\left(p\frac{\partial u_4}{\partial p} + q\frac{\partial u_4}{\partial q}\right)\frac{\partial u}{\partial p} + \left(q\frac{\partial u_{13}}{\partial q} + p\frac{\partial u_{13}}{\partial p}\right)\left(-\frac{\partial u_4}{\partial p}\right)\right\} = -\tfrac{1}{8}u_1J_{02},$$

the former of the two last lines being obtained partly from the forms of u_3 and u_{12} and partly because u_4 and u_{13} are homogeneous in p and q. Hence

$$\begin{aligned} X &= X' + u_8u_{13} + \tfrac{3}{4}u_4J_{02} \\ &= X' - A_1A_2 - \tfrac{3}{4}A_0J_{02}, \end{aligned}$$

and X′ includes terms of rank not greater than 3. It thus appears that the aggregate of the terms proper to the rank 4 are functionally the same in Z as in Q_{12}; and we have

$$\begin{aligned} \frac{Q_{12}}{3u_4} - \frac{3Z}{2u_4} &= X' + 27u_4{}^3P_6 - \tfrac{2}{3}\frac{P_8{}^3}{u_4} - \tfrac{7}{6}\frac{A_1{}^3}{u_4} - \tfrac{3}{2}A_1J_{01} \\ &= X' + 27u_4{}^3P_6 - \tfrac{2}{3}\frac{P_8{}^3}{u_4} + \tfrac{7}{6}\frac{u_8{}^3}{u_4} + \tfrac{3}{2}u_8J_{01}. \end{aligned}$$

Now, from § 13 we have

$$J_{01} = \frac{6}{u_1}(u_3u_8 + u_4u_7) ;$$

and from the values of P_7 and P_8 it follows that

$$\begin{aligned} u_8 &= P_8 - \frac{6u_3u_4}{u_1}, \\ u_7 &= u_1P_7 + \frac{u_3u_4}{u_1} + \frac{2u_3{}^2}{u_1}. \end{aligned}$$

Substituting now in X′, in J_{01}, and for u_8 the values as given for u_7 and u_8 in the last two equations (so as to express all the aggregates of coefficients proper to the rank 3 in terms of P_6, P_7, P_8 and to leave the residue of terms—if there be such a residue—as a function of u_1, u_2, u_3, u_4) and gathering together like terms, we find

$$\begin{aligned} X' + 27u_4{}^3P_6 - \tfrac{2}{3}\frac{P_8{}^3}{u_4} + \tfrac{7}{6}\frac{u_8{}^3}{u_4} + \tfrac{3}{2}u_8J_{01} &= 27u_4{}^3P_6 + \tfrac{1}{2}\frac{P_8{}^3}{u_4} + 9u_4P_7P_8 \\ &= \frac{Q_6}{2u_4}. \end{aligned}$$

Hence we have

$$Z = \tfrac{2}{9}Q_{12} - \tfrac{1}{3}Q_6\,;$$

and it follows that *the first educt of* $U_2\ (= Q_{13})$ *when reduced by means of* U_3 *is functionally equivalent to the invariant* Q_{12}.

33. In the preceding investigation the Jacobian of the function A_0 and any other function A_m of the series in § 16 entered. The following formulæ, interesting in themselves, are of use in a verification that Z actually satisfies all the differential equations which are characteristic of an invariant:—

$$\left.\begin{array}{ll} \text{For } m > 2, & \Delta_1\dfrac{\partial A_{m-2}}{\partial p} = m(m-1)\,q\,\dfrac{\partial A_{m-3}}{\partial p} \\ & \Delta_1\dfrac{\partial A_{m-2}}{\partial q} = m(m-1)\left(q\,\dfrac{\partial A_{m-3}}{\partial q} + A_{m-3}\right) \\ \text{and, for } m = 2, & \Delta_1\dfrac{\partial A_0}{\partial p} = -\,2q^2 \\ & \Delta_1\dfrac{\partial A_0}{\partial q} = 2pq \end{array}\right\}\,;$$

$$\left.\begin{array}{ll} \text{and, for } m > 2, & \Delta_2\dfrac{\partial A_{m-2}}{\partial p} = -\,m(m-1)\left(p\,\dfrac{\partial A_{m-3}}{\partial p} + A_{m-3}\right) \\ & \Delta_2\dfrac{\partial A_{m-2}}{\partial q} = -\,m(m-1)\,p\,\dfrac{\partial A_{m-3}}{\partial q} \\ \text{and, for } m = 2, & \Delta_2\dfrac{\partial A_0}{\partial p} = 2pq \\ & \Delta_2\dfrac{\partial A_0}{\partial q} = -\,2p^2 \end{array}\right\}\,;$$

and, for all values of m,

$$\left.\begin{aligned} \Delta_3 \frac{\partial A_{m-2}}{\partial p} &= 0 \\ \Delta_3 \frac{\partial A_{m-2}}{\partial q} &= -\frac{\partial A_{m-2}}{\partial p} \end{aligned}\right\} ;$$

$$\left.\begin{aligned} \Delta_4 \frac{\partial A_{m-2}}{\partial p} &= -\frac{\partial A_{m-2}}{\partial q} \\ \Delta_4 \frac{\partial A_{m-2}}{\partial q} &= 0 \end{aligned}\right\} .$$

From these it follows that, if neither l nor m be zero,

$$\left.\begin{aligned} \Delta_1 J_{l,m} &= (l+1)(l+2)\left(qJ_{l-1,m} - A_{l-1}\frac{\partial A_m}{\partial p}\right) + (m+1)(m+2)\left(qJ_{l,m-1} + A_{m-1}\frac{\partial A_l}{\partial p}\right) \\ \Delta_2 J_{l,m} &= -(l+1)(l+2)\left(pJ_{l-1,m} + A_{l-1}\frac{\partial A_m}{\partial q}\right) - (m+1)(m+2)\left(pJ_{l,m-1} - A_{m-1}\frac{\partial A_l}{\partial q}\right) \\ \Delta_3 J_{l,m} &= 0 = \Delta_4 J_{l,m} \end{aligned}\right\} ;$$

and

$$\left.\begin{aligned} \Delta_1 J_{0,m} &= -2q(m+2)A_m + (m+2)(m+1)\left(qJ_{0,m-1} + \frac{\partial A_0}{\partial p}A_{m-1}\right) \\ \Delta_2 J_{0,m} &= 2p(m+2)A_m - (m+2)(m+1)\left(pJ_{0,m-1} - \frac{\partial A_0}{\partial q}A_{m-1}\right) \\ \Delta_3 J_{0,m} &= 0 = \Delta_4 J_{0,m} \end{aligned}\right\} .$$

Connexion with Theory of Binary Forms.

34. In connexion with the fact (§ 15) that the irreducible invariants proper to the rank 3 are expressible in terms of the simultaneous concomitants of A_0 and A_1, viewed as binary forms (quadratic and cubic) in q and $-p$ as variables, it is important to remark that the equations $\Delta_3 f = 0$ and $\Delta_4 f = 0$ are in fact the differential equations satisfied by all concomitants of binary forms which have q and $-p$ for their variables, and have

$$\begin{array}{lllll} r, & s, & t\,; \\ a, & b, & c, & d\,; \\ e, & f, & g, & h, & i\,; \\ \cdot & \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

for their coefficients, that is, of A_0, A_1, A_2, . . . , viewed as binary forms. Each form of a concomitant-system satisfies the differential equations characteristic of its concomitants; and it thus appears how $A_0, A_1, A_2, \ldots$ are (§§ 9, 16) simultaneous solutions of the two characteristic equations in question. Moreover, since the Jacobian of two binary forms occurs in their concomitant-system, and therefore satisfies the characteristic equations, it is now evident that the quantities denoted by $J_{l,m}$, being

$$\frac{\partial A_l}{\partial p}\frac{\partial A_m}{\partial q} - \frac{\partial A_l}{\partial q}\frac{\partial A_m}{\partial p},$$

must satisfy the equations $\Delta_3 f = 0 = \Delta_4 f$.

Hence, it appears that one method of obtaining the irreducible invariants, which are proper to the rank n and are additional to those proper to ranks less than n, is as follows:—(1) to obtain the concomitants of A_{n-2}, and the simultaneous concomitants of A_{n-2}, and of the concomitant-system of $A_{n-3}, A_{n-4}, \ldots, A_1, A_0$, viewed as binary forms; (2) to frame the combinations of these concomitants which will satisfy the remaining characteristic equations $\Delta_1 f = 0 = \Delta_2 f$; (3) to select from among these combinations such as are, from the supposed known algebraical relations among the concomitants, found to be irreducible.

35. Again, in the case of binary forms in two systems of variables, q and $-p$, q' and $-p'$, and with coefficients

$$\begin{array}{llll} r, & s, & t, & \\ a, & b, & c, & d, \\ \cdot & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{llll} r', & s', & t', & \\ a', & b', & c', & d', \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

the characteristic equations satisfied by their simultaneous concomitants are of the form

$$(\Delta_3 + \Delta'_3)\psi = 0 = (\Delta_4 + \Delta'_4)\psi,$$

that is,

$$\Theta_3\psi = 0 = \Theta_4\psi.$$

And every solution of these equations, with proper limitations as to degree and grade, is a concomitant. Hence, every functional invariant of the two dependent variables z and z' already considered can be expressed in terms of simultaneous concomitants of the set of quantities A_0, A'_0; A_1, A'_1; $\ldots$, viewed as binary forms in variables q and $-p$, q' and $-p'$.

Thus, for example, we have seen the simultaneous functional invariants, proper to the rank 2, are five in number, and they are—one, J, being the covariant $pq' - p'q$ in the variables alone; two, A_0 and A'_0, being the quadratic forms; and two, $\mathfrak{A}_0$ and $\mathfrak{A}'_0$, which can be exhibited in the respective forms

$$\tfrac{1}{2}\left(q\frac{\partial}{\partial q'} + p\frac{\partial}{\partial p'}\right)^2 A'_0 + \left(q'\frac{\partial}{\partial q} + p'\frac{\partial}{\partial p}\right)A_0,$$

and

$$\tfrac{1}{2}\left(q'\frac{\partial}{\partial q} + p'\frac{\partial}{\partial p}\right)^2 A_0 + \left(q\frac{\partial}{\partial q'} + p\frac{\partial}{\partial p'}\right)A'_0,$$

which are combinations of polar emanants of A_0 and A'_0, the fundamental quadratic forms. And, from the note to § 24, it follows that they can also be represented in the forms

$$\left(q'\frac{\partial}{\partial q}+p'\frac{\partial}{\partial p}+r'\frac{\partial}{\partial r}+s'\frac{\partial}{\partial s}+t'\frac{\partial}{\partial t}\right)A_0,$$

$$\left(q\frac{\partial}{\partial q'}+p\frac{\partial}{\partial p'}+r\frac{\partial}{\partial r'}+s\frac{\partial}{\partial s'}+t\frac{\partial}{\partial t'}\right)A'_0,$$

and also in the forms

$$\tfrac{1}{2}\left(q\frac{\partial}{\partial q'}+p\frac{\partial}{\partial p'}+r\frac{\partial}{\partial r'}+s\frac{\partial}{\partial s'}+t\frac{\partial}{\partial t'}\right)^2A'_0,$$

$$\tfrac{1}{2}\left(q'\frac{\partial}{\partial q}+p'\frac{\partial}{\partial p}+r'\frac{\partial}{\partial r}+s'\frac{\partial}{\partial s}+t'\frac{\partial}{\partial t}\right)^2A_0.$$

36. Returning now to the functional invariants of only a single dependent variable, we have seen that they are combinations of the simultaneous covariants of A_0, A_1, A_2, . . . , considered as binary forms in q and $-p$; and all these simultaneous covariants satisfy the equations $\Delta_3 f = 0$, and must, therefore, be expressible in terms of u_1, u_2, . . . , u_5, . . . , u_9, ... The actual expressions may be obtained as follows:—

From the values of the quantities u we have

$$\begin{cases} t = u_2, \\ u_1 s = u_3 + pu_2, \\ u_1^2 r = u_4 + 2pu_3 + p^2u_2\,; \end{cases}$$

$$\begin{cases} d = u_5, \\ u_1 c = -u_6 + pu_5, \\ u_1^2 b = u_7 - 2pu_6 + p^2u_5, \\ u_1^3 a = -u_8 + 3pu_7 - 3p^2u_6 + p^3u_5\,; \end{cases}$$

$$\begin{cases} i = u_9, \\ u_1 h = -u_{10} + pu_9, \\ u_1^2 g = u_{11} - 2pu_{10} + p^2u_9, \\ u_1^3 f = -u_{12} + 3pu_{11} - 3p^2u_{10} + p^3u_9, \\ u_1^4 e = u_{13} - 4pu_{12} + 6p^2u_{11} - 4p^3u_{10} + p^4u_{10}\,; \end{cases}$$

and so on. It thus appears that any differential coefficient of z, when multiplied by a power of u_1 equal to the x-grade of the differential coefficient, is linearly expressible in terms of the quantities u proper to its rank, the coefficients of these quantities u in the expression being powers of u.

But in the case of the function A_{n-2}, which is

$$(z_{n,0},\ z_{n-1,1},\ z_{n-2,2},\ \ldots \between q,\ -p)^n,$$

the weights of its concomitants are estimated by assigning to $z_{n,0}$, $z_{n-1,1}$, $z_{n-2,2}$, ... the weights 0, 1, 2, ... in succession, that is, the integers which represent the y-grade of these coefficients $z_{n-s,s}$.

If we have a covariant Ψ of order m which is simultaneous to A_{n-2}, $A_{n'-2}$, $A_{n''-2}$, ..., and of degrees l, l', l'', ... in their respective coefficients, and its leading term be $C_0 q^m$, then the weight of C_0 is

$$\tfrac{1}{2}(nl + n'l' + n''l'' + \ldots - m),$$

which is, therefore, the number representing the y-grade of C_0, considered as the leading term of a functional invariant. Since the grade of each of the coefficients of A_{n-2} is n, it follows that the grade of C_0, so far as it involves the coefficients of A_{n-2} is ln, and, therefore, the grade of C_0 is, in the aggregate,

$$nl + n'l' + n''l'' + \ldots .$$

But the aggregate grade of C_0 is the sum of the x-grade and the y-grade; hence, the x-grade of C_0 is

$$\tfrac{1}{2}(nl + n'l' + n''l'' + \ldots + m).$$

In order, then, to express Ψ in terms of the quantities u, we should proceed to substitute for the coefficients r, s, t, ... the values above obtained, and assuming

$$\Psi = C_0 q^m - C_1 q^{m-1} p + \ldots,$$

it is evident that the only term in Ψ from which terms independent of p can come is the first term. Moreover, since Ψ is expressible as a function of the quantities u alone, it follows that when these substitutions are carried out the terms involving p must disappear, for p is the only non-u quantity which enters into the expressions substituted; and the value of Ψ is, therefore, the aggregate of terms which survive, that is, the aggregate of terms independent of p arising from $C_0 q^m$.

Now, in C_0 this aggregate is obtained by replacing a coefficient, $z_{m,n}$, by a quantity $\pm u_s u_1^{-m}$; since C_0 is isobaric *qua* seminvariant, it is of uniform x-grade *qua* part of functional invariant; and therefore the result of these substitutions is to give a function of u_2, u_3, u_4, ..., divided by a power of u_1 equal to the x-grade of C_0, that is, divided by

$$u_1^{\frac{1}{2}(nl + n'l' + n''l'' + \ldots + m)}.$$

If, then, Γ_0 denote this function of u_2, u_3, u_4, ..., we have

$$\Psi = u_1^m \,.\, u_1^{-\frac{1}{2}(nl + n'l' + n''l'' + \ldots + m)},$$

or

$$u_1^{\frac{1}{2}(nl + n'l' + n''l'' + \ldots - m)}\Psi = \Gamma_0.$$

37. Hence we have the following theorem:

To express any simultaneous concomitant Ψ *of*

$$\begin{gathered}(r, s, t \between q, -p)^2, \\ (a, b, c, d \between q, -p)^3, \\ (e, f, k, h, i \between q, -p)^4, \\ \cdots\cdots\cdots \end{gathered}$$

in terms of the quantities $u_1, u_2, u_3, u_4, \ldots$, *which are the irreducible solutions of* $\Delta_3 f = 0$, *the equation characteristic of all these concomitants, it is sufficient to take the coefficient* C_0 *of the highest power of* q *in* Ψ, *to construct a new function* Γ_0, *which is the same combination of the coefficients of*

$$\begin{gathered}(u_4, u_3, u_2 \between q, -p)^2, \\ (-u_8, u_7, -u_6, u_5 \between q, -p)^3, \\ (u_{13}, -u_{12}, u_{11}, -u_{10}, u_9 \between q, -p)^4, \\ \cdots\cdots\cdots\cdots \end{gathered}$$

as C_0 *is of the coefficients of the binary quantics, and to divide* Γ_0 *by* $u_1^{\frac{1}{2}(nl + n'l' + n''l'' + \ldots - m)}$, *where* m *is the degree of* Ψ *in* q *and* $-p$, *and* $l, l', l'', \ldots$ *are the degrees of* C_0 *in the coefficients of* $A_{n-2}, A_{n'-2}, A_{n''-2}, \ldots$ *respectively.*

The theorem is illustrated by one or two examples which have already occurred in the reduction of Q_5, Q_6, Q_7. Thus, for $H_0 = rt - s^2$, we have only one quantic entering into its composition, viz., A_0; so that $n = 2$, $l = 2$; $l' = 0 = l'' \ldots$, and $m = 0$ hence,

$$u_1^{\frac{1}{2} \cdot 2 \cdot 2} H_0 = u_4 u_2 - u_3^2,$$

that is,

$$H_0 = (u_4 u_2 - u_3^2) u_1^{-2}.$$

Again

$$\begin{aligned} -\tfrac{1}{12} H_{01} &= r(pd - qc) - 2s(pc - qb) + t(pb - qa) \\ &= q(-cr + 2bs - at) + p(dr - 2cs + bt); \end{aligned}$$

so that two quantics enter. Thus, we have, for H_{01},

$$\begin{aligned} &n = 2, \ l = 1; \\ &n' = 3, \ l' = 1; \qquad l'' = l''' = \ldots = 0; \\ &m = 1. \end{aligned}$$

Hence

$$\begin{aligned} -\tfrac{1}{12} u_1^{\frac{1}{2}(2 \cdot 1 + 3 \cdot 1 - 1)} H_{01} &= -u_4(-u_6) + 2u_7 u_3 - u_2(-u_8) \\ &= u_4 u_6 + 2u_3 u_7 + u_2 u_8 \end{aligned}$$

as before; and so for others.

38. We might, if we pleased, carry the theorem further, for every simultaneous concomitant satisfies the two characteristic equations $\Delta_3 f = 0$, $\Delta_4 f = 0$, and is therefore expressible in terms of the simultaneous irreducible solutions of these two equations. Such irreducible solutions are necessarily functional combinations of $u_1, u_2, u_3, \ldots$ such as satisfy $\Delta_4 f = 0$; and, as in the earlier cases of §§ 14, 17, it is easy to show that these irreducible solutions within the rank three are equivalent to the set

$$\begin{aligned} u_4 &= A_0, \\ (u_2 u_4 - u_3^2)\, u_1^{-2} &= H_0, \\ u_8 &= -A_1, \\ (u_4 u_7 + u_3 u_8)\, u_1^{-1} &= J_{01}, \quad \text{(dropping a factor 6)} \\ (u_6 u_8 - u_7^2)\, u_1^{-2} &= H_1, \\ (u_5 u_8^2 - 3u_6 u_7 u_8 + 2u_7^3)\, u_1^{-3} &= Q, \end{aligned}$$

which are f, A_{ff}, ϕ, ϑ, Δ, Q respectively in GORDAN's notation. Thus, every simultaneous concomitant within the rank three can be expressed algebraically—though not necessarily rationally—in terms of these six quantities.

The actual expression can be obtained by a development of the method adopted in the preceding theorem. It is first necessary to replace the covariant by its value in terms of $u_1, u_2, u_3, \ldots$; then to substitute by means of the equations

$$\begin{aligned} u_4 &= A_0, \\ u_2 u_4 &= u_3^2 + u_1^2 H_0, \\ u_8 &= A_1, \\ u_4 u_7 &= u_1 J_{01} + u_3 A_1, \\ u_4^2 u_6 u_8 &= u_1^2 u_4^2 H_1 + (u_1 J_{01} + u_3 A_1)^2, \\ u_5 u_4^3 u_8^2 &= u_1^3 u_4^3 Q + (u_1 J_{01} + u_3 A_1)^3 + 3u_1^2 u_4^2 H_1 (u_1 J_{01} + u_3 A_1), \end{aligned}$$

for $u_2, u_4, u_5, u_6, u_7, u_8$. The result, we know, must appear as a function of $A_0, H_0, A_1, J_{01}, H_1, Q$; and, therefore, the terms involving u_3 will disappear, and the factors u_1 will cancel.

For example, in the case of H_{01}, $= p$ (GORDAN) $= L_1$ (SALMON), we have, dropping the factor 12 and using L_1, which is $\frac{1}{2} H_{01}$,

$$\begin{aligned} u_1^2 L_1 &= -u_2 u_8 - 2u_3 u_7 - u_4 u_6 \\ &= \frac{A_1}{A_0}(u_3^2 + u_1^2 H_0) - 2\frac{u_3}{A_0}(u_1 J_{01} + u_3 A_1) + \frac{1}{A_0 A_1}\{u_1^2 A_0^2 H_1 + (u_1 J_{01} + u_3 A_1)^2\} \\ &= \left(\frac{A_1}{A_0} H_0 + \frac{A_0}{A_1} H_1 + \frac{J_{01}^2}{A_0 A_1}\right); \end{aligned}$$

so that

$$L_1 A_0 A_1 = A_1^2 H_0 + A_0^2 H_1 + J_{01}^2.$$

Similarly for others of the simultaneous concomitants of A_0 and A_1. And it is not difficult to show that all functional invariants within the rank 4, or, what is the equivalent, all the simultaneous concomitants of A_0, A_1, A_2, considered as three binary forms, can be expressed in terms of the foregoing six quantities A_0, H_0, A_1, J_{01}, H_1, Q, and the succeeding five, viz.:—

$$\begin{aligned} u_{13} &= A_2, \\ (u_{12}u_4 + u_3u_{13})\,u_1^{-1} &= J_{02}, \\ (u_{11}u_{13} - u_{12}^2)\,u_1^{-2} &= H_2, \\ (u_{10}u_{13}^2 - 3u_{11}u_{12}u_{13} + 2u_{12}^3)\,u_1^{-3} &= \Phi_2, \\ (u_9u_{13} - 4u_{10}u_{12} + 3u_{11}^2)\,u_1^{-4} &= I_2. \end{aligned}$$

Inferences can also be deduced as to the expressibility of the simultaneous concomitants of A_0 and A_2 alone as simultaneous quantics, and of the simultaneous concomitants of A_1 and A_3 alone, as simultaneous concomitants; but all such results are chiefly interesting from the point of view of the theory of binary forms, and are more useful in that theory than in the theory of functional invariants.

III. *Total Eclipse of the Sun observed at Caroline Island, on 6th May,* 1883.

By Captain W. DE W. ABNEY, *C.B., R.E., F.R.S.*

Received May 25,—Read June 16, 1887.
Revised June 4, 1888.

[PLATES 1, 2.]

OWING to the representations of the Committee on Solar Physics, who communicated with the Royal Society the desirability of observing this eclipse, an expedition was organised under the auspices of the latter body. The Council of the Royal Society having requested me to draw up a report on the Total Eclipse observed at Caroline Island, I undertook the task so far as relates to the results which were obtained with the same instruments which were employed in the observations of the Total Eclipse in Egypt in 1882.

Two observers, Mr. H. LAWRANCE and Mr. C. R. WOODS, who had both taken part in the Eclipse Expedition to Egypt as assistants to Professors LOCKYER and SCHUSTER, were entrusted with the arduous duty of making the observations. The expedition was devoted entirely to photographic work, the main object being to continue the photographic observations which had been carried on in Egypt, consisting of photographs of the corona taken on very rapid plates with varying exposure, photographs of the corona taken with a slitless spectroscope (the prismatic camera), and a photograph of the corona spectrum, the image of the moon and the corona being thrown on the slit cutting the diameter of the former. There is no occasion to describe the instruments which were employed for the first two classes of observations, as they have been fully described in the previous communication to the Royal Society by Professor SCHUSTER and myself which appears in the 'Philosophical Transactions' for 1884. The photographic spectroscope which was employed on this occasion differed in one detail, and in one detail only, in that the dispersion was doubled, two medium dense flint prisms of 62½° being employed instead of one prism of the same angle. The experience gained in Egypt seemed to show that, if the coronal light was equally bright in the two eclipses, the rapid plates used on both occasions would be amply adequate to secure photographs with the larger dispersion. Besides these observations several others were made, but did not meet with the success it was hoped they would have done. A photoheliograph, giving a 4-inch solar image, was attached to an equatorial mount, in addition to the wooden camera carrying a lens of 5 ft. 6 in. focus, with which the smaller-sized pictures of the corona were taken in Egypt. The pictures taken with the former, though sufficiently exposed, showed that a large image could be utilised. They were not, however, satisfactory, owing to a multiplicity of images being formed, due to the shake given to the instrument by the insertion of the slides in the smaller instrument, the large pictures requiring considerably more exposure than these latter. In the matter of spectroscopic analysis of the eclipse phenomena, Mr. Lockyer devised an ingenious contrivance for securing impressions of the bright lines seen immediately after and before totality. These photographs were only partially successful, and will not be considered in this report. The number of instruments to be used by the two observers and the assistants they hoped to obtain were nine, entailing the use of 11 cameras. Only two equatorial mountings accompanied the expedition, and it was impossible to mount all these instruments on them, had it been advisable, indeed, to do so. A siderostat, having a 12-inch silver-on-glass mirror, was therefore taken, four of the instruments being stationary, reflected light being utilised.

The following was the disposition of the instruments :—

On the 1st equatorial—

A. A finder of 3¾-inch aperture was attached to the above for viewing the eclipse.

B. A 7-prism spectroscope, with camera attached, for obtaining photographs near the sun's limb immediately before and after totality.

C. A 6-inch achromatic telescope by Cooke, of York, the eye-piece being withdrawn. Attached to it was a Rutherfurd grating of 17,200 lines to the inch, to be used for obtaining spectra of the corona in the 1st and 2nd order, two cameras being employed.

D. A slit spectroscope, having one prism of dense flint glass. The condenser throwing the image of the moon on the slit was a photographic lens by Dallmeyer, of 6-inch focus.

On the 2nd equatorial were mounted—

M. The photoheliograph for taking 4-inch pictures.

N. The corona camera, having a lens of 4-inch aperture, and 5 ft. 6 in. focal length.

The instruments used with the siderostat were—

F. A photographic spectroscope to be used without a condenser, consisting of one prism of white flint, a collimator 4½ feet long, and a lens attached to the camera of 3-inch aperture, and of about 9-inch focus. In this case the photographic plate was caused to move vertically during exposure of the plate by means of clock-work for the registration of bright lines immediately before and after totality.

G. A slit spectroscope of two prisms of the same dimensions as that used in Egypt in the eclipse of 1882, and described in Dr. Schuster's and my report and strongly resembled that of 1882. Mr. **Woods** states that the coronal light was more natural than in Egypt, and Mr. **Lawrance** describes it as not so violet as in Egypt.

Results.—Although photographs were taken successfully in nearly every instrument, it is to be regretted that the majority have so far proved to be of but little use. At present I have not been able to utilise for measurement more than the photograph of the spectrum of the corona taken with the two-prism slit spectroscope, and the corona photographs taken in the camera with the lens of 5ft. 6in. focus. These last had exposures given of 1 sec., 2 secs., 3 secs., 10 secs., 20 secs., and 120 secs.

The photographs taken with the slitless spectroscopes are good, but they possess no great features of interest. The prominences were of small height and few in number, and I have been unable to mark any distinction in the light they emitted. The rings of light due to 1474, D_3, and other substances which were noticed in the eclipse of 1882 are absent, probably because of the greater angular diameter of the moon. I have, therefore, not given either drawings or measurements of these photographs.

The negatives of the corona were placed at my request in the hands of Mr. **Wesley**, Assistant Secretary of the Royal Astronomical Society, and he has made two drawings from them, in one of which the coronal detail near the limb is shown, being taken from the photographs which had but short exposure, and in the other the coronal detail further from the sun, being sketched from the photographs to which long exposure had been given. The general features of the corona are those which might be expected from the sun-spot period in which the eclipse took place, a matter which was discussed in the Report of the Egyptian Eclipse, and which scarcely need be restated here. The corona spectrum has been carefully measured by Mr. **Lawrance** and myself. The method we adopted was as follows:—First, I took some measurements of the most prominent lines and recorded them, taking out the wave-lengths by the same method employed in measuring the photograph in the Egyptian eclipse, the reference spectra taken after the eclipse on the same plate being utilised for the purpose. Mr. **Lawrance** then carefully and independently measured the photograph three separate times. All lines were rejected which he did not measure in all these sets of measurements. I then measured it myself in the same way, and rejected all lines which did not appear in each of my measures. Finally, the lines taken as absolutely present were those which appeared in my expurgated measures and in Mr. **Lawrance's**. By this means it is believed that every line of which there can be no doubt has been recorded, whilst there are many others whose existence is doubtful but which are probably present. Lists of each sets of measures are given, which may be useful in comparing the lines obtained in this photograph with the Egyptian negatives, and those which may be obtained in future eclipses. That a large number are coronal lines is a fact, and the coincidences between those found in the photograph now under discussion and the Egyptian one, in which all the lines given were undoubtedly coronal, is important. It will be better in future eclipse expeditions to place the slit of the spectroscope tangential to the moon's limb in preference to normally. This has been done in the recent eclipse observed in the West Indies (August, 1886), with most satisfactory results. The chief point to attain is to separate all prominence light from the coronal light, as it tends to mask the true spectrum of the latter. From the photographs I have examined I have come to the conclusion that not much more is to be learnt at present from them. It may be that as more eclipses come to be observed with the same instruments, or at all events on the same lines, the photographs of the Caroline Island station will prove to be of greater value than they seem to be now.

If we compare the corona of this eclipse with that of the eclipse in Egypt, perhaps the most striking feature is the absence of the hydrogen lines. In Egypt the photograph shows, besides the lines which may be presumed to be hydrogen at H, at least two other lines of hydrogen, λ 4340 and λ 4101. In the Caroline Island photographs these lines are entirely absent. It may be well to draw attention to the fact that in the former eclipse the prominences were very marked, and in the prismatic (slitless) spectrum the hydrogen rings were very powerfully shown. In the eclipse now under consideration the prominences were very small, and the prismatic (slitless) spectrum gave no result other than rings at H and K. It would seem, then, that the corona at the time of the Egyptian eclipse was illuminated more or less by the prominence light. If this be admitted, we ought to find that the corona during the Caroline Island eclipse was illuminated by the light which emanated from the matter which gave H and K so strongly in the ring spectrum. Looking at the list of lines, we find that such is the case. Calcium was evidently present in the light, more especially near the limb of the moon. We find that three calcium lines are shown *reversed* across the dark moon, and two iron lines. It is somewhat hard to see how these reversed lines made their appearance in such a locality. It is quite evident that they must be due to reflected light. I can find no trace of FRAUNHOFER lines about G outside the corona, such as Dr. SCHUSTER and myself found in the Egyptian eclipse photograph, and which would be the first to appear in the photographic plate were any reflected sunlight as it reaches us present in those regions. It should be remarked that the reversed lines across the moon are extremely faint, but perfectly distinguishable and measurable. Most of the lines in the spectrum of the corona lie near the moon's limb, and have quite a different aspect to those delineated in the Egyptian eclipse negative, and some of them are probably prominence lines, and I think it would be dangerous to found any theory on the discovery of new lines in the coronal spectrum from the list of lines here recorded.

In conclusion, I think I may say that the two English observers, Mr. H. A. LAWRANCE and Mr. C. R. WOODS, deserve every credit for the amount of work they did. The large number of instruments they were called upon to utilise during the eclipse, and which they evidently most skilfully manipulated, could only have been done by those who were thoroughly competent, and who possessed a freedom from a tendency to excitement, which occasions such as that on which they were engaged is apt to create, more especially when they have a heavy responsibility resting upon them. The results they brought home show how assiduously they worked, and how completely they carried out the programme with which they were entrusted.

CORONA Spectrum, 1883.

ABNEY'S measures, in wave-lengths.		LAWRANCE'S measures, in wave-lengths.		Adopted lines.	Remarks.
1	2	1	2		
3837	3837	3836 ± 2	3835	3836	
		3883			
3898		3898			
3934	3934	3934	3934	3934	K reversed across moon Fe Ca
	3954				
3969	3969	3969	3969	3969	H reversed across moon Fe Ca
3986		3986			
3998		3997		3998	Reversed across moon Fe
4015	4016	4014	4018	4016	E 4015
		4030	4031		
		4038	4036		
4045				4045	E 4044
4057		4055		4056	E 4057
4065	4065		4063	4064	E 4067
4071	4071			4071	Reversed across moon Fe
	4076		4074	4075	Ca (4077)
4081		4081			
4086		4085		4085	E 4085
4092	4092		4094	4092	Reversed across moon Ca
			4113		
			4125		
4131				4131	Reversed across moon Ca Fe
4137	4142	4142	4146	4144	
		4153	4158		
			4169	4169	E 4168 Ca 4167
4184	4185	4183	4187	4185	
		4191	4194	4192	E 4195
		4213		4213	E 4213 Ca Fe Sr (4215·5)
		4220			
4225	4227		4228	4227	E 4224 Ca Sr (4226·3)
			4237	4242	E 4241 Fe 4235·5
		4248	4246		Fe 4245·2
4253	4249	4254	4255	4255	E 4252
		4261			
		4274 ± 2			
4279	4279		4280	4279	
	4290	4291		4291	E 4289 Ca Cr Ce 4289·4
4310		4310			
	4313		4314		
	4331		4329		
4354	4354	4351	4353	4353	Fe 5352
		4360	4358		
4370	4370		4363 ± 3	4370	E 4370

CORONA Spectrum, 1883 (continued).

ABNEY's measures, in wave-lengths.		LAWRANCE's measures, in wave-lengths.		Adopted lines.	Remarks.
1	2	1	2		
		4377			
		4383			
		4400			
		4427			
4450		4448			
4464		4465			
4473		4473		4473	E 4473 4471·4 often in prominences
4490		4490			
4501		4502		4501	E 4501
	4518		4518		
4545		4546±1			
		4557	4555±3		
		4571			
		4577			
4602		4606			
4620	4636		4636		
		4642			
	4672		4674		
		4695			
			4706		
4720		4717	4717	4717	Fe 4717
4730	4730	4729±1		4730	Fe 4730·7
			4738		
4754		4754			
			4760		
			4764		
4780		4776			
		4798			
			4803		
			4818		
		4845			
	4949		4955		

E signifies lines found in the photograph of the corona spectrum taken in Egypt, 1882.
In the column marked 1 the lines were found in three different measurements.
In the column marked 2 the lines were found in two different measurements.

In the adopted spectrum only those lines which were each measured by the two computers on each limb of the moon have been taken as coronal, unless a coincidence was noted between lines measured on one and the coronal spectrum of 1882 taken in Egypt. It will be noted that lines occurring in Ca and Fe lie very close to those given in the adopted spectrum.

APPENDIX I.

Government Eclipse Expedition, 1883.

Instructions to Observers.

1. In case of any difficulty at any port, either on going out or coming home, Mr. LAWRANCE to hand Foreign Office letter herewith to the British Consul at that port, and ask his assistance.

2. On joining the American party, Mr. LAWRANCE and Mr. WOODS to report themselves to the astronomer in charge of the expedition, and to hand him the accompanying letter, taking his advice and following his instructions with reference to the transference of the instruments to the United States ship of war.

3. On arriving at the place of observation, the instruments to be erected on a site to be chosen by the American astronomer in charge.

4. Packing cases to be re-closed up as far as possible, and to be protected from damage and the weather. Care to be taken not to damage tin cases.

5. The gratings to be kept together, and special precautions to be taken with regard to them, as also with the silvering of the siderostat mirrors. Mr. LAWRANCE to give special attention to this point.

6. For as many days as possible before the eclipse all the instruments to be arranged as during the eclipse, and from 11.23 A.M., local mean time, to 11.48 A.M., local mean time, complete rehearsals of all the observations intended to be made during the eclipse to be most rigidly carried out.

7. A statement of the days on which these rehearsals have been made to be given in the report of the operations.

8. If the aforesaid times, derived from Mr. HIND, do not agree with the times determined by the American astronomers, the instructions of the astronomer in charge are to be taken.

9. Instruments to be focussed and trial plates taken, if possible, at least three days before totality. These trial plates to be carefully preserved.

10. The rehearsal on the day before the eclipse should be a complete rehearsal with photographic plates, exactly as during the eclipse itself; and these plates to be developed at once, and brought home.

11. The observers should confer with the American astronomer in charge regarding time signals before and after totality.

12. If additional observing power can be obtained from the American party, the additional observers to be trained to obtain photographs with the photoheliographs, and, if desirable, the time table for that instrument to be handed over to them, they being placed in entire charge of that part of the operations.

13. If such assistance cannot be afforded, then, if the photoheliograph programme cannot be carried out in its entirety, the large pictures to be alone attempted.

14. Special attention to the rating of the clocks, including the eclipse clock and siderostat, to be given at least three days before the eclipse.

15. A quarter of an hour before totality, clocks to be wound, and caps and stops which had been hitherto used to diminish the amount of light to be removed, if necessary.

16. The timekeeper should be asked to give these instructions in a loud voice, as experience has shown that this is apt to be forgotten.

17. In the observations and adjustments during the eclipse, no deviation from the time table and adjustments to be made except after consultation, and with the approval of the American astronomer in charge.

18. The clockwork of the integrating spectroscope to be so adjusted that the plate will fall through one inch in eight minutes.

19. The distance of plate from concave grating to be that given by Captain ABNEY for vertical distortion.

20. In equatorial, the slits to be parallel and vertical in the meridian, and their centres lying on the same part of the sun.

21. All slits to be $\frac{1}{500}$ in. = No. 2 on Captain ABNEY's screw, with the exception of the integrating spectroscope, which should be $\frac{1}{250}$ in.

22. At some convenient time—say 100 secs.—near the middle of totality, the slits of equatorial to be brought to the point of reappearance.

23. The plates to be developed and copied at the first convenient time after the eclipse is over.

24. Half the positives and half the negatives to be handed to the British Consul at Callao, to be forwarded to the Foreign Office for transmission to the Science and Art Department by the next mail after that by which the observers leave.

25. On arrival at Callao, a cypher telegram to be despatched to Secretary, Kensington Museum, London, giving the results obtained with each instrument, and stating any other matter of importance.

26. Great care to be taken in repacking the instruments after the eclipse. Tin cases to be re-closed.

27. A detailed report, to be prepared before arrival at Callao, of the general results to be posted to me immediately on arrival at Callao, in case of any delay *en route.*

28. If a convenient opportunity arises for sending this report from the Marquesas, this course to be followed as well as the other.

29. It is to be understood that the records of the eclipse are the property of the British Government.

30. In case no pictures are taken with the small photoheliograph, Mr. LAWRANCE is requested to ask the American astronomer in charge for an oriented positive of the corona to facilitate reference here.

31. Mr. LAWRANCE is empowered to hand to the American astronomer in charge positives of any of the pictures taken by the English party which he may require for a similar purpose, and to obtain a receipt for them.

W. SPOTTISWOODE, Pres. R.S.,
Feb. 16, 1883.

J. NORMAN LOCKYER,
Feb. 16, 1883.

APPENDIX II.

Adjustments.

B. Seven-prism spectroscope.
    F line in centre of plate.
C. Flat grating spectroscope.
    First order—F in centre of plate.
    Second order—F in centre of plate.
D. Dense prism.
    F in centre of plate.
F. Integrating HILGER (Flash).
    G in centre of plate.
G. Red end slit.
H. Red end prismatic camera.
K. First order blue. ROWLAND.
    F in centre of plate.
L. Second order blue. ROWLAND.
    H in centre of plate.
M. 4″ photoheliograph.
    See that sun runs along horizontal wire.
N. Small photoheliograph.

J. NORMAN LOCKYER,
Feb. 16, 1883.

APPENDIX III.

Time Table.

Time.	Siderostat.					Equatorial.				Photoheliographs.	
	F	K	L	H	G	B	C		D	M	N
	HILGER.	ROWLAND grating.		Prismatic camera.	Slit spectroscope. 2 prisms.	7 prisms.	Grating.		Dense prism. Quarter-inch slide.	Large photoheliograph.	Corona camera.
		1st order.	2nd order.				Red, 1st order.	Blue, 2nd order.			
10 min.	..	..	..	..	..	Expose	Expose	Expose			
9											
8	..	..	..	..	..	..	..	..	Expose		
7	..	..	..	..	..	..	..	..	Expose		
6	Ref. spectrum 30 secs.	..	..	..	..	..	..	..	Expose		
5	..	..	..	..	..	..	..	..	Expose		
4	..	..	..	..	..	..	..	..	Expose		
3	..	..	..	..	..	..	Expose	Expose	Expose		
2	..	..	..	..	..	Expose	..	..	Expose		
1	..	..	..	..	..	..	..	..	Expose		
40 secs.	..	Expose	Expose	..	..	..	Run $\frac{1}{4}$ inch				
30	..	..	..	..	..	..					
20	..	Expose	Expose								
3											
2	Expose and start clock										
1											
Totality	..	Expose	Expose	Expose col. plate	Expose	Expose	Expose	Expose	Expose	Expose	Expose 1 sec.
300 secs.											
290											
280	..	..	..	..	..	..	..	..	..	..	Expose 20 secs.
270											
260											
250											
240											
230	..	..	..	Shut							

APPENDIX III. (continued).

Time Table (continued).

Time.	Siderostat.					Equatorial.				Photoheliographs.	
	F HILGER.	K ROWLAND grating.	L	H Prismatic camera.	G Slit spectroscope. 2 prisms.	B 7 prisms.	C Grating.		D Dense prism. Quarter-inch slide.	M Large photo-heliograph.	N Corona camera.
		1st order.	2nd order.				Red, 1st order.	Blue, 2nd order.			
220	...	..	..	Expose gel. plate							
210	..	..	..	..	..	..	..	..	..	..	Expose
200	...	Expose	Expose	..	...	..	..	..	Expose	Expose	
190											
180											
170											
160											
150											
140											
130											
120	...	..	..	Shut							
110	..	..	..	Expose col. plate							
100											
90	..	..	..	..	..	..	..	..	..	..	Shut
80											
70	..	..	..	..	..	..	..	..	..	..	Expose 3 secs.
60											
50	..	..	..	..	..	..	..	..	..	Expose	
40 + x	...	..	..	..	..	..	..	..	..	..	Expose 10 secs.
30 + x											
20 + x	..	..	..	..	..	..	..	..	..	..	Expose 2 secs.
10 + x											
Just before end	..	Expose	Expose	Shut	Shut	Expose	Expose, run ¼ inch	Expose	Expose	Shut	

APPENDIX III. (continued).

Time Table (continued).

Time.	Siderostat.					Equatorial.				Photoheliographs.	
	F	K	L	H	G	B	C		D	M	N
	HILGER.	ROWLAND grating.		Prismatic camera.	Slit spectroscope. 2 prisms.	7 prisms.	Grating.		Dense prism. Quarter-inch slide.	Large photo-heliograph.	Corona camera.
		1st order.	2nd order.				Red, 1st order.	Blue, 2nd order.			
1 sec.											
2											
3											
4	..	Expose	Expose								
5											
10	..	Expose	Expose								
15											
20											
30											
40											
50											
1 min.	...	Shut	Shut	..	..	..	Expose	Expose	Expose		
2	Shut										
3	..	..	..	..	..	..	..	..	Expose		
4											
5	..	..	..	..	..	Expose	Expose	Expose	Expose		
6											
7	..	..	..	..	..	..	..	..	Expose		
8											
9	..	..	..	..	..	..	..	..	Expose		
10	Ref. 25 secs.	..	..	..	..	Shut	Shut	Shut	Shut		
Ref. spectra, say half hour after totality	1 sec.	..	..	..	2 secs.	10 secs.	10 secs.	10 secs. J. NORMAN *Feb.* 16,	1 sec. LOCKYER, 1883.	Run	Run

The following Report was written from U.S.S. " Hartford," at sea.

April 20th, at 7 o'clock in the morning, we came in sight of Caroline Island.

A boat was sent off under Lieutenant QUALTROUGH, and on his return we learnt that there were two empty frame-houses belonging to Messrs. HOLDER BROS., of Leadenhall Street, to whom the island is leased, and seven native inhabitants.

The disembarkation commenced that afternoon, and was concluded next day; but, as at nightfall a large part of the goods were still on the shore, the " Hartford " lay-by all night, and landed strong parties at daylight, to carry the boxes up to the site chosen by Professor HOLDEN for the observatory, while W. M. PEACOCK, the cooper, put in the foundations for our three piers.

The landing was very difficult, as the boats had to be run in through a narrow opening in the reef; then the boxes had to be carried through fifty yards or so of water, varying from two to three feet deep; then over fifty yards of sharp irregular coral rock, that cut the men's shoes to pieces; and then along a soft sandy beach, up hill, for more than a quarter of a mile. Our best and most hearty thanks are due to Captain C. C. CARPENTER, who superintended the disembarkation; to Lieutenant-Commander E. WHITE, the executive officer, who saw personally to the lading of the boats; to Lieutenant-Commander J. W. MILLER, who received the goods on shore; to Lieutenant QUALTROUGH, the Cadets, and Warrant Officers, who looked after the working parties on shore.

The " Éclaireur " came in on the evening of the 22nd, just as the " Hartford " was leaving, with the French expedition, consisting of Messrs. JANSSEN, TROUVELOT, PALISA, and TACCHINI.

The landing party left with us consisted of—

Lieutenant EDWARD F. QUALTROUGH.
WILLIAM S. DIXON, Esq., M.D.
Cadet, W. B. FLETCHER.
Cadet, J. G. DOYLE.
Seaman-Gunner, H. R. YEWELL.
Carpenter, PETER MURPHY.
Carpenter's Mate, CHARLES EMMS.
Seaman, JAMES HAROLD.
O. Seaman, JOHN MACKINNON.
O. Seaman, C. H. PERKINS.
O. Seaman, J. SMITH (Cook).
Steward, P. BURNS.
Servant, T. BROOKS.
Servant, MORTIMER SPENCE.

By Saturday, the 28th of April, the siderostat, equatorial, and photoheliograph were erected and adjusted in position. The arrangement of the nest of spectroscopes for use with the siderostat was taken in hand, and the spectroscopes were attached to the equatorial.

We had a great deal of trouble with the photoheliograph, as the tube did not fit the cradle; the clock went badly, and the square box could not be perfectly adjusted for parallelism.

By Thursday evening, the 3rd of May, we were nearly ready for trial plates, which we hoped to take the following day; but it turned wet, and before noon on Friday over five inches of rain had fallen, and our dark room was destroyed, all the dye being washed out of our ruby curtain and window.

The early part of the week was taken up in arranging the various spectroscopes, which took up a good deal of time, and in rating the clockwork slide and equatorial and photoheliograph clocks.

At last the latter went fairly well, but that of the equatorial could not be made to go fast enough, so that recourse had to be made to the fine motions.

On the day previous to the eclipse the weather was very unsettled, and the rehearsals and final adjustments occupied so much time that we were unable to take trial plates.

The photoheliograph stand vibrated so badly that to steady it two cords were attached to the end of the polar axis and fastened to stakes driven in the ground.

The weather on the 6th was very unsettled till about 9 o'clock, when the sky commenced to clear and the instruments were uncovered; by 10 o'clock the sky was moderately clear. After first contact the lenses were dusted, slits cleaned, and the adjustments inspected. Forty minutes before totality the plateholders which had been filled during the night were served out.

The following are the reports of each observer of the work done during totality:—

Mr. H. A. Lawrance's Report of work done during the Eclipse.

About 40 minutes before totality Mr. Woods gave me the plateholders, which I put into the cameras, and examined the screens to see that the three instruments were in good adjustment, then I moved the slides ready for exposure and wound the clock. The slits of the spectroscopes were parallel and nearly tangential to the point of disappearance.

I commenced to expose 10 minutes before totality, and followed the time table, with the exception that 100 seconds after totality I shifted to the other side of the sun and made a new exposure on each plate; after totality, by mistake, I shifted the grating plates at 3 instead of 5 minutes. I took reference spectra 25 minutes after totality.

The corona, examined through the finder, was full of delicate detail near the limb, especially upon the preceding one.

With a pocket spectroscope, with lens in front of the slit, I only saw the green line 1474; and, taking off the slit and examining with the prism at mid-totality, I saw the 1474 ring very brilliant, while C and D_3 were faint, with a lot of continuous spectrum. F I could not see, although I looked for it.

H. A. Lawrance.

Mr. C. R. Woods' Report.

The instruments under my charge were arranged as proposed in England, the integrating spectroscope, slit spectroscope, and prismatic camera being adjusted and focussed with F in the centre of the plate. The Rowland grating was placed normal with the siderostat mirror, and the first and second order on the brightest side adjusted, with F and H respectively in the centres of the plates. Some difficulty was experienced in getting the clockwork to move the slide of the integrating spectroscope sufficiently slow, as the desired rate of speed had been changed too late before starting to enable the alteration to be made at home; during the 8 minutes' run of the clock the plate was moved through the space of 1¼ inches.

Five minutes previous to totality the siderostat mirror was finally adjusted and the clock wound up. A red end collodion plate, coated 15 minutes before, was then washed and placed in one of the prismatic camera slides. All other slides had been filled the night previously with gelatine plates. At one minute before totality (not 2 seconds, as stated, I believe *erroneously*, in the instructions), the clockwork of the integrating spectroscope slide was started. At 40 seconds before, total exposures were made in the Rowland grating cameras. At totality, the prismatic camera and slit spectroscope were opened. The three exposures in the former instrument were performed as arranged, the last being closed 5 seconds after the lapse of 300 seconds. The slit spectroscope was closed at the end of the 300 seconds. The exposures in the Rowland grating were carried out strictly to programme, except as to the last exposure during totality, when, owing to longer totality than was expected, the plates were moved up between 10 and 15 seconds after the lapse of the 5 minutes. The clock of the integrating spectroscope ran down at about 1½ minutes after totality, and the slit was covered over simultaneously with the stopping of the clock.

Several long intervals during exposures enabled me to look at the corona and my surroundings. The corona resembled that of 1882 in its general character, the streamers seeming to extend to a little over 2 diameters. Several stars were visible, but the amount of illumination of the sky seemed little less than that of the Egyptian eclipse; but, unlike the latter, its light was more natural, and the landscape lacked the weird colouring that was so noticeable during the eclipse last year.

Two minutes after totality I took the red end plate into the dark room to develop it. Having to manipulate it almost in the dark, it got torn in putting it in. On letting in orange light, half of it was still on the plate, but nothing appeared on that part, which, in spite of my utmost care, also tore into several pieces, leaving nothing on the plate save the gelatine edging.

Five minutes after the eclipse a cloud passed over the sun, and shortly after the sky clouded over.

The plates were developed in the evening, and the copies made on the two following nights.

The six photographs taken with the small photoheliograph are very good, the one with 2 minutes' exposure extending as far as those of JANSSEN, which were exposed during the whole totality. All taken with the large instrument show slight shifts, probably due to the changing of the slides in the smaller instrument; still they will be useful in making out the detail near the limb, and we believe that from the nine plates a drawing can be made that will show the whole structure from the limb to the furthest extension of the corona.

With the second order flat grating apparently we do not seem to have caught anything, but before stating that we have been unsuccessful we must examine the plate under better conditions of illumination; in the first order grating H and K are present as bright lines at the commencement and end of totality; the dense prism spectroscope also shows bright lines at the beginning and end, especially at the end, H, K, *h*, *f*, F being very marked.

Two gelatine red end plates in the prismatic camera were successful as photographs, but, owing to the comparative absence of prominences, will not be so fruitful in their results as the photograph obtained with this instrument in 1882. The slit spectroscope gave a good photograph from the ultra-violet to the green. The spectrum appears mainly continuous, but differs on the two sides of the disc. H and K are very marked, but do not extend across the interval, as they did in last year's. The hydrogen line near G extends out nearly a solar diameter; *h*, F, and 1474 also appear. There are other lines, but they are not so numerous as in the 1882 eclipse. The ROWLAND grating seems to have given no useful results, but this is due to the same cause as the indifferent results in the prismatic camera. The integrating spectroscope was successful. There was little or no perceptible change in the character of the spectrum till just before totality, when the brightest lines of the reversion spectrum were caught. H and K, 1474, and the hydrogen lines are most prominent. The flash was also caught at the end of totality. During totality no result appears to have been obtained.

We commenced packing up on the 7th; and on the 9th, at 5 P.M., we left Caroline Island for Honolulu.

IV. *On Evaporation and Dissociation.*—Part VIII. *A Study of the Thermal Properties of Propyl Alcohol.*

By Professor William Ramsay, *Ph.D., F.R.S., and Professor* Sydney Young, *D.Sc.*

Received June 14,—Read June 21, 1888.

[Plates 3–7.]

In continuation of our investigations of the thermal properties of pure liquids, we have now determined the vapour-pressures, vapour-densities, and expansion in the liquid and gaseous states, of Propyl Alcohol, and from these results we have calculated the heats of vaporization at definite temperatures. The range of temperature is from 5° to 280°, and the range of pressure from 5 mms. to 56,000 mms.

Preparation of pure Propyl Alcohol.—A sample of propyl alcohol was procured from Kahlbaum, of Berlin. It was dried with barium oxide, and then with small quantities of sodium; but in this case the results were not nearly so satisfactory as with methyl and ethyl alcohol, for propyl alcohol is soluble in water, forming a mixture or "hydrate," which boils constantly at a lower temperature than the pure alcohol. It is not completely decomposed by sodium, and can be separated only by repeated fractional distillation. This hydrate was first described by Chancel ('Comptes Rendus,' vol. 68, 1867, p. 659), who, observing that it boiled with perfect constancy, assumed that it possessed a definite composition, and gave it the formula C_3H_8O, H_2O. It has more recently been examined by Konowalow (Wiedemann's 'Annalen,' vol. 14, 1881, p. 34), who has determined the vapour-pressures of varying mixtures of propyl alcohol and water at definite temperatures. Konowalow finds that the composition of the mixture, the vapour of which exerts the greatest pressure, is not the same at different temperatures, but that the mixture contains more alcohol at high temperatures than at low. From this it has been concluded that the composition of the "hydrate" must depend on the pressure under which the liquid is distilled. We have proved experimentally that this is the case (but we reserve a discussion of this interesting substance for a future paper), and we give the results of our experiments in an Addendum to this paper.

After repeated fractionation we succeeded in obtaining a quantity of propyl alcohol which boiled almost constantly at 97°·6 (the rise of temperature during distillation being less than 0°·1) under a pressure of 763·8 mms. This sample was employed for the determination of vapour-pressures at low temperatures, and for the constants at high temperatures with quantities A, B, and D (see p. 140).

The specific gravity of the alcohol was determined several months later, and a fresh quantity of the alcohol was prepared from the hydrate into which the greater part of the alcohol had been converted. The hydrate was treated with dry potassium carbonate, when, as described by CHANCEL (*loc. cit.*), two layers were formed, the lower one being an aqueous solution of potassium carbonate, the upper one consisting of the partially hydrated alcohol. The alcohol was fractionated several times, potassium carbonate being each time added to the most volatile distillate, until a quantity was obtained, boiling from 97°·1 to 97°·15 under a pressure of 752 mms.

For the determinations of vapour-density at high temperatures with quantity C, the alcohol was refractionated. The portion employed boiled constantly at 97°·1 under a pressure of 750·6 mms.

Reduced to 760 mms. these temperatures would be (1) 97°·45; (2) 97°·4; (3) 97°·4.

The boiling-point of propyl alcohol has been determined by numerous experimenters, and the results obtained by several are very concordant; the most reliable appear to be the following:—

Observer.	Reference.	Pressure.	Temperature.	Temperature reduced to 760 mms.
BRÜHL	'Annalen der Chemie,' vol. 200, p. 173 . .	752·2	97–97·2	97·35
ZANDER . . .	" " " 214, p. 153 . .	760·0	97·40	97·40
LINNEMANN . .	" " " 161, p. 26 . .	760·0	97·41	97·41
SCHIFF . . .	" " " 220, p. 101 . .	752·4	97·10	97·35
KONOWALOW . .	*Loc. cit.*	749·2	97·00	97·37

PIERRE and PUCHOT ('Annales de Chimie,' vol. 22, 1871, p. 276) found 98°; and PERKIN ('Chem. Soc. Trans.,' vol. 45, p. 446) gives two determinations: 97°·5 to 98°·5 and 98°, but the boiling-point of the alcohol employed in the final determination of specific gravity is not stated.

Apparatus employed.—The apparatus employed was the same as that described in our memoir on Ethyl Oxide ('Phil. Trans.,' A, 1887, p. 57).

EXPERIMENTAL RESULTS.

Vapour-Pressures at Low Temperatures.—These were determined by our dynamical method. The thermometer had been standardized by determinations of the vapour-pressures of water; its zero point was redetermined.

TABLE I.

Pressure.	Temperature.	Pressure.	Temperature.	Pressure.	Temperature.
mms.	°	mms.	°	mms.	°
5·25	5·5	34·5	33·7	235·35	69·5
5·10	4·9	42·05	36·95	240·1	70·0
5·25	5·5	50·05	40·0	273·7	72·85
6·10	7·5	57·55	42·3	312·85	75·85
7·25	9·7	67·6	45·2	355·8	78·8
8·70	12·8	79·7	48·2	404·9	81·9
9·20	13·4	94·6	51·4	455·5	84·6
10·55	15·6	110·95	54·5	505·6	87·0
13·30	18·7	130·8	57·6	561·9	89·6
16·40	21·75	151·85	60·5	615·1	92·0
20·30	25·1	175·3	63·4	672·7	94·3
24·55	28·2	201·1	66·2	760·7	97·5
29·35	31·1	204·8	66·7		

These results were plotted on sectional paper, curves drawn through them, and the pressures corresponding to equal intervals of temperature read off.

Specific Gravity of Propyl Alcohol.—A Sprengel's tube of the form recommended by PERKIN was employed. The weighings were reduced to a vacuum:—

Weight of water at 16°·7	15·3169 grms.
Capacity of tube at 16°·7	15·3339 c.cs.
Weight of alcohol at 0°	12·5577 grms.
Capacity of tube at 0°	15·3275 c.cs.
Specific gravity at 0°	0·81929.
Volume of 1 grm. at 0°	1·22056 c.cs.

Weight of alcohol at 10°·72	12·4338 grms.
Capacity of tube at 10°·72	15·3316 c.cs.
Specific gravity at 10°·72	0·81099.
Volume of 1 grm. at 10°·72	1·23306 c.cs.

The results of other observers* are not very easy to compare with ours; some of them are given in terms of water at 0°, others at 4°, and others again at the same temperature as the alcohol. The best method of comparison appears to be to reduce the results of other observers to true densities, and to read from the curve constructed from our observations the densities at the corresponding temperatures. The most reliable results appear to be the following (the references are the same as before):—

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* A tabulated statement of determinations of the boiling-point and specific gravity of propyl alcohol is given by LOSSEN, 'Annalen der Chemie u. Pharmacie,' vol. 214, p. 105.

Observer.	Specific gravity $\frac{t^\circ}{t'^\circ}$	Density or $\frac{t^\circ}{4^\circ}$	Result from our curve.
BRÜHL	·8044 20°/4°	·8044 at 20°	·8035
ZANDER	·8069 17°/0°	·8068 at 17°	·8059
LINNEMANN	·8066 15°/15°	·8059 at 15°	·8076
PERKIN	·80883 15°/15°	·8082 at 15°	·8076
"	·80247 25°/25°	·8002 at 25°	·7995
SCHIFF	·7366 97°·1/4°	·7366 at 97°·1	·7353

It will be seen that our results are in close agreement with those of other observers; they are very slightly lower than most of the others, but at 15° our result is rather higher than LINNEMANN'S.

Constants at High Temperatures.—Four different amounts of propyl alcohol were employed for these determinations. The vapour-pressures and the orthobaric volumes and compressibilities of the liquid were determined with the largest quantity, A, the weight of which was calculated from its volume and specific gravity. The smaller quantities, B, C, D, were employed for the determinations of the volumes of a gram of vapour. The weight of B was ascertained by comparisons of its volume with that of A under similar conditions of temperature and pressure. The weight of C was obtained in a similar manner from that of B, and the weight of D similarly from that of C.

Weight of quantity A.—The actual volumes of this quantity at various temperatures were plotted on sectional paper, and a curve drawn through the points; the volumes at 0° and 10°·72 were read off, and the weight calculated from the densities at those temperatures.

Volume of A at 0° . .	0·32335.	Volume of 1 grm. at 0° . .	1·22056.
Volume of A at 10°·72 .	0·32675.	Volume of 1 grm. at 10°·72 .	1·23306.

	Grm.
Weight from volume at 0°	0·26492
Weight from volume at 10°·72	0·26499
Mean	0·26496

		Grm.
Weight of B.	Mean of numerous comparisons with A	0·03763.
Weight of C.	Mean of numerous comparisons with B . . .	0·006015.
Weight of D.	Mean of numerous comparisons with C . . .	0·005180.

Constants with the largest quantity, A.—The vapour-pressures at each temperature were, as usual, determined at the widest possible limits of volume.

Volume of 1 grm. at 23°·7 = 1·2493. Specific gravity 0·8004.

The pure liquids employed for obtaining constant temperatures were carbon bisulphide, ethyl alcohol, chlorobenzene, bromobenzene, aniline, methyl salicylate, and bromonaphthalene ('Chem. Soc. Trans.,' 1885, p. 640). We think it unnecessary to state in each case the liquid employed and the pressure under which it boiled.

Temperature.	Volume of 1 grm.	Specific gravity. (Weight of 1 c.c.)	Pressure.
°	c.cs.		mms.
30	1·2569	0·7956	800
	1·2522	..	32,590
40	1·2712	0·7866	800
	1·2659	..	22,910
	1·2612	..	53,480
50	1·2850	0·7782	800
	1·2796	..	19,730
	1·2743	..	53,480
60	1·2992	0·7697	800
	1·2934	..	23,730
	1·2873	..	53,480
70	1·3137	0·7612	800
	1·3070	..	26,210
	1·3017	..	53,480
80	1·3298	0·7520	800
	1·3274	..	6,750
	1·3207	..	29,090
	1·3180	..	53,480
90	1·3742	0·7423	800
	1·3412	..	17,070
	1·3346	..	37,970
100	..	..	842 843 842 843
		Mean vap.-press.	= 842·5
	1·3651	0·7326	800
	1·3618	..	11,510
	1·3550	..	31,910
	1·3483	..	53,480
110	..	..	1,204 1,207 1,205 1,207
		Mean vap.-press.	= 1,206
	1·3863	0·7214	1,200
	1·3822	..	7,415
	1·3754	..	27,910
	1·3687	..	47,480

Temperature.	Volume of 1 grm.	Specific gravity. (Weight of 1 c.c.)	Pressure.
°	c.cs.		mms.
120	..	..	1,681
			1,683
			1,684
			1,684
		Mean vap.-press.	= 1,683
	1·4028	0·7129	1,684
	1·3961	..	18,540
	1·3894	..	35,000
	1·3833	..	53,490
130	..	..	2,292
			2,293
			2,291
			2,302
		Mean vap.-press.	= 2,295
	1·4292	0·6997	2,300
	1·4231	..	12,300
	1·4163	..	26,100
	1·4097	..	40,290
	1·4031	..	54,680
140	..	..	3,077
			3,071
			3,076
			3,072
		Mean vap.-press.	= 3,074
	1·4572	..	1,160
	1·4546	0·6875	3,074
	1·4505	..	7,305
	1·4439	..	20,190
	1·4371	..	33,430
	1·4303	..	48,270
150	..	..	4,053
			4,062
			4,067
			4,054
		Mean vap.-press.	= 4,059
	1·4835	0·6741	4,022
	1·4775	..	11,250
	1·4708	..	23,010
	1·4642	..	35,230
	1·4576	..	44,690
160	..	..	5,270
			5,257
			5,272
			5,259
		Mean vap.-press.	= 5,264
	1·5144	0·6603	5,294

Temperature.	Volume of 1 grm.	Specific gravity. (Weight of 1 c.c.)	Pressure.
°	c.cs.		mms.
170	..	..	6,657
			6,720
			6,702
			6,702
		Mean vap.-press.	= 6,695
	1·5518	0·6444	6,750
	1·5458	..	14,012
	1·5391	..	22,689
	1·5323	..	29,197
	1·5256	..	38,919
	1·5189	..	45,898
	1·5122	..	55,431
180	..	..	8,374
			8,388
			8,387
			8,398
		Mean vap.-press.	= 8,387
	1·5913	0·6284	8,560
	1·5865	..	11,810
	1·5797	..	18,710
	1·5730	..	24,260
	1·5663	..	31,430
	1·5596	..	38,000
	1·5529	..	45,750
	1·5497	..	52,510
190	..	..	10,488
			10,467
			10,468
			10,441
		Mean vap.-press.	= 10,466
	1·6356	0·6114	10,410
	1·6343	..	12,020
	1·6274	..	16,680
	1·6206	..	22,040
	1·6138	..	26,650
	1·6070	..	32,250
	1·6004	..	38,020
	1·5938	..	43,800
	1·5869	..	49,500
200	..	..	12,807
			12,817
			12,831
			12,856
		Mean vap.-press.	= 12,828
	1·6888	0·5921	12,829
	1·6820	..	16,540
	1·6753	..	20,030
	1·6617	..	27,840
	1·6481	..	36,170
	1·6346	..	45,710
	1·6278	..	50,607

Temperature.	Volume of 1 grm.	Specific gravity. (Weight of 1 c.c.)	Pressure.
°	c.cs.		mms.
210	..	..	15,525
			15,562
			15,614
			15,599
		Mean vap.-press.	= 15,575
	1·7489	0·5718	15,576
	1·7435	..	17,910
	1·7299	..	23,680
	1·7164	..	29,580
	1·7028	..	36,050
	1·6892	..	43,240
	1·6757	..	51,690
220	..	..	18,626
			18,653
			18,692
			18,712
		Mean vap.-press.	= 18,671
	1·8244	0·5481	18,695
	1·8121	..	21,690
	1·7985	..	25,710
	1·7849	..	29,860
	1·7713	..	34,680
	1·7576	..	39,970
	1·7440	..	45,540
	1·7304	..	51,710
230	..	..	22,108
			22,158
			22,144
			22,234
		Mean vap.-press.	= 22,161
	1·9081	0·5241	22,223
	1·8944	..	24,870
	1·8808	..	27,520
	1·8672	..	30,490
	1·8536	..	33,900
	1·8399	..	37,290
	1·8262	..	41,090
	1·8126	..	45,170
	1·7990	..	49,810
	1·7854	..	54,770
240	..	..	26,204
			26,224
			26,201
			26,204
		Mean vap.-press.	= 26,208
	2·0308	0·4924	26,202
	2·0182	..	27,160
	2·0044	..	28,640
	1·9907	..	30,140
	1·9632	..	33,860
	1·9360	..	38,050
	1·9085	..	43,430
	1·8812	..	49,340
	1·8677	..	53,170

Temperature.	Volume of 1 grm.	Specific gravity. (Weight of 1 c.c.)	Pressure.
°	c.cs.		mms.
250	..	..	30,729
			30,725
			30,822
			30,840
		Mean vap.-press.	= 30,779
	2·2093	0·4526	30,831
	2·1989	..	31,160
	2·1703	..	32,470
	2·1426	..	33,950
	2·1150	..	35,640
	2·0875	..	37,940
	2·0600	..	40,390
	2·0327	..	43,470
	2·0049	..	47,050
	1·9774	..	51,300
260	..	..	36,137
			36,087
			36,101
			36,115
		Mean vap.-press.	= 36,110
	2·5601	0·3906	36,115
	2·5468	..	36,221
	2·4770	..	36,880
	2·4072	..	38,060
	2·3375	..	39,560
	2·2680	..	42,050
	2·1994	..	45,380
	2·1294	..	50,560
	2·1018	..	53,260
263·15	..	..	37,950
263·5	..	..	38,100
263·54	3·0102	0·3322	38,110
263·64 (Critical Temperature)	4·1697	..	38,130
	3·4696	..	38,110
	3·0368	..	38,120
	2·7568	..	38,130
	2·6172	..	39,170
	2·4773	..	40,760
	2·3378	..	43,970
	2·2682	..	46,960
	2·1997	..	50,530
	2·1711	..	52,950
270	4·1702	..	41,540
	3·7496	..	41,940
	3·4701	..	42,150
	3·1770	..	42,620
	2·8973	..	43,340
	2·7571	..	44,000
	2·6172	..	45,480
	2·4776	..	47,760
	2·3381	..	52,130
	2·2686	..	55,630
280·1	4·1713	..	47,000
	3·7505	..	48,040
	3·4710	..	48,550
	3·1778	..	49,730
	2·8981	..	51,540
	2·7578	..	53,310
	2·6179	..	55,510

Second Quantity, B. Weight = 0·03763.

Temperature.	Volume of 1 grm.	Pressure.	Vapour-density.
°	c.cs.	mms.	
230	29·34	14,115	38·10
	26·38	15,254	39·14
	24·41	16,114	40·03
	22·35	17,120	41·16
	20·38	18,195	42·46
	19·40	18,793	43·20
	18·41	19,368	44·16
	17·43	20,037	45·09
	16·45	20,674	46·33
	15·48	21,373	47·61
	14·50	21,990	49·39
	Vapour-pressure	= 22,094	
240	29·35	14,602	37·48
	26·39	15,866	38·37
	23·34	17,356	39·65
	21·37	18,483	40·66
	19·40	19,742	41·93
	18·42	20,376	42·80
	17·43	21,073	43·72
	16·45	21,848	44·68
	15·48	22,606	45·89
	14·50	23,388	47·35
	13·54	24,229	48·97
	12·58	25,067	50·95
	11·62	25,869	53·44
	Vapour-pressure	= 26,102	
250	29·35	15,098	36·95
	26·39	16,414	37·80
	23·35	18,012	38·94
	21·38	19.209	39·87
	19·41	20,575	41·01
	17·44	22,126	42·44
	16·46	22,897	43·45
	15·49	23,811	44·41
	14·51	24,688	45·72
	13·54	25,657	47·13
	12·58	26,635	48·87
	11·62	27,658	50·94
	10·67	28,725	53·42
	9·72	29,649	56·81
	8·78	30,546	61·10
	Vapour-pressure	= 30 809	
260	29·36	15,647	36·32
	26·41	16,999	37·18
	23·35	18,710	38·20
	21·38	19,997	39·03
	19·41	21,424	40·13
	17·44	23,087	41·44
	15·49	24,939	43·20
	13·54	27,006	45·62
	12·58	28,178	47·07
	11·62	29,367	48·89
	10·67	30,586	51·12

Temperature.	Volume of 1 grm.	Pressure.	Vapour-density.
°	c.cs.	mms.	
260	9·72	31,867	53·85
	8·78	32,961	57·69
	7·83	34,138	62·45
	6·88	35,275	68·75
	6·41	35,646	73·06
	Vapour-pressure	= 35,955	
263·64 (Critical Temperature)	9·73	32,467	53·21
	8·78	33,854	56·54
	7·83	35,224	60·92
	6·88	36,433	67·01
	5·94	37,309	75·85
	4·99	37,940	88·66
	4·05	37,969	109·20
	3·11	38,068	141·80
	2·64	39,037	162·70
270	29·37	16,098	35·96
	26·41	17,548	36·69
	23·36	19,379	37·56
	21·39	20,744	38·32
	19·42	22,266	39·32
	17·44	24,002	40·60
	15·49	26,076	42·08
	13·55	28,373	44·23
	11·63	30,953	47·24
	9·73	33,807	51·70
	7·83	36,917	58·81
	5·94	39,765	72·00
	5·00	40,908	85·13
	4·05	41,753	98·17
	3·11	42,824	127·50
	2·64	45,250	142·00
	2·36	51,749	139·00
280·15	29·38	16,572	35·58
	26·41	18,097	36·23
	23·36	19,982	37·10
	21·39	21,417	37·80
	19·42	23,005	38·76
	17·45	24,907	39·84
	15·50	27,073	41·28
	13·55	29,609	43·16
	11·63	32,469	45·86
	9·73	35,803	49·72
	7·83	39,485	56·00
	5·94	43,429	67·14
	5·00	45,290	76·53
	4·05	47,200	90·50
	3·59	48,304	100·00
	3·11	50,451	110·20
	2·83	52,312	116·90

Third Quantity, C. The experiments with this quantity were made with a new pressure apparatus, a new volume-tube and new pressure-gauges. Fresh samples of methyl salicylate and bromonaphthalene were also employed for heating the tube.

At each temperature the propyl alcohol vapour was made to occupy the largest possible volume, and was left for several hours until the vapour-pressure of mercury had attained its maximum. Readings were taken every half hour to ascertain when the pressure had become constant. The subsequent readings were taken at diminishing volumes.

Temperature.	Volume of 1 grm.	Pressure.	Vapour-density.
°	c.cs.	mms.	
150	172·8	2,359	32·25
	162·4	2,497	32·41
	151·9	2,657	32·57
	141·4	2,838	32·77
	130·8	3,044	33·01
	120·2	3,281	33·33
	109·5	3,565	33·66
	98·5	3,891	34·21
	Vapour-pressure	= 4,053	
180	172·9	2,554	32·02
	152·0	2,884	32·25
	130·9	3,325	32·48
	115·0	3,745	32·84
	98·8	4,295	33·31
	82·6	5,043	33·95
	66·3	6,117	34·80
	55·4	7,091	36·02
	49·9	7,721	36·71
	Vapour-pressure	= 8,365	
200	173·0	2,683	31·67
	152·1	3,033	31·86
	131·0	3,492	32·14
	109·7	4,129	32·46
	93·5	4,786	32·85
	77·2	5,683	33·51
	60·9	7,013	34·43
	49·9	8,309	35·44
	44·5	9,136	36·20
	39·0	10,129	37·24
	33·5	11,362	38·63
	Vapour-pressure	= 12,691	
220	173·1	2,813	31·46
	152·2	3,182	31·64
	131·1	3,667	31·88
	109·7	4,335	32·20
	93·5	5,035	32·53
	77·2	5,995	33·09
	60·9	7,435	33·83
	49·9	8,850	34·66
	39·0	10,895	36·07
	33·5	12,318	37·12
	28·0	14,123	38·81
	22·4	16,454	41·56
	19·6	17,850	43·73
	Vapour-pressure	= 18,711	

Temperature.	Volume of 1 grm.	Pressure.	Vapour-density.
°	c.cs.	mms.	
230	173·2	2,881	31·34
	162·8	3,053	31·46
	147·0	3,367	31·60
	131·1	3,761	31·70
	115·1	4,252	31·93
	99·0	4,904	32·21
	82·7	5,783	32·69
	66·4	7,081	33·28
	55·4	8,317	33·90
	44·5	10,108	34·76
	33·5	12,784	36·48
	22·4	17,250	40·44
	16·8	20,587	45·09
	Vapour-pressure	= 22,183	
240	173·2	2,939	31·32
	152·3	3,327	31·47
	131·1	3,844	31·63
	109·8	4,552	31·90
	93·6	5,284	32·24
	77·3	6,304	32·73
	60·9	7,847	33·34
	50·0	9,380	34·02
	44·5	10,375	34·54
	39·0	11,606	35·21
	33·5	13,180	36·08
	28·0	15,215	37·46
	22·4	17,965	39·50
	19·6	19,674	41·26
	16·8	21,630	43·76
	14·1	23,894	47·43
	12·4	25,344	50·75
	Vapour-pressure	= 26,227	
260	173·3	3,061	31·23
	152·3	3,469	31·35
	131·2	4,006	31·52
	109·9	4,746	31·77
	93·6	5,521	32·04
	77·3	6,593	32·50
	61·0	8,228	33·02
	50·0	9,860	33·60
	44·5	10,929	34·05
	39·0	12,259	34·62
	33·5	13,977	35·34
	28·0	16,224	36·49
	22·4	19,320	38·23
	19·7	21,284	39·61
	16·9	23,612	41·63
	14·1	26,509	44·39
	12·4	28,499	46·87
	11·3	29,850	49·12
	Vapour-pressure	= 36,285	

Temperature.	Volume of 1 grm.	Pressure.	Vapour-density.
°	c.cs.	mms.	
279·9	173·4	3,184	31·13
	152·4	3,609	31·24
	131·3	4,168	31·41
	109·9	4,943	31·63
	93·7	5,754	31·87
	77·3	6,883	32·28
	61·0	8,596	32·77
	50·0	10,342	33·22
	39·0	12,904	34·10
	33·6	14,750	34·72
	28·0	17,196	35·69
	22·4	20,581	37·21
	19·7	22,810	38·32
	16·9	25,488	39·98
	14·1	28,886	42·25

Fourth Quantity, D. The experiments with this quantity were made with the old apparatus.

Temperature.	Volume of 1 grm.	Pressure.	Vapour-density.
°	c.cs.	mms.	
130	212·6	1,826	32·26
	191·1	2,014	32·54
	184·0	2,084	32·66
	176·9	2,158	32·80
	169·1	2,236	33·13
	Vapour-pressure	= 2,288	
150	212·7	1,940	31·86
	191·2	2,147	32·02
	169·2	2,402	32·35
	154·9	2,607	32·56
	140·6	2,847	32·84
	126·4	3,134	33·20
	112·2	3,484	33·63
	98·1	3,905	34·31
	Vapour-pressure	= 4,023	
180	212·9	2,097	31·68
	191·4	2,320	31·85
	169·3	2,612	31·98
	155·0	2,837	32·15
	140·7	3,102	32·39
	126·5	3,431	32·59
	112·3	3,821	32·96
	98·2	4,320	33·34
	84·3	4,953	33·88
	70·5	5,809	34·53

REDUCTION AND ARRANGEMENT OF RESULTS.

Vapour-Pressures.—The vapour-pressures experimentally determined, and also those calculated by the formula $\log p = a + b\alpha^t + c\beta^t$, are given in the following table:—

Temperature.	Pressure.							
°	Still method.	A.	B.	C.	D.	Mean.	Calculated.	Δp.
0	3·44	..	..	..	..	3·44	3·49	+ ·05
10	7·26	..	..	..	..	7·26	7·39	·13
20	14·50	..	..	..	..	14·5	14·78	·28
30	27·60	..	..	..	..	27·6	28·13	·53
40	50·20	..	..	..	..	50·2	51·12	·92
50	87·20	..	..	..	..	87·2	89·00	1·80
60	147·00	..	..	..	..	147·0	148·97	1·97
70	239·00	..	..	..	..	239·0	240·44	1·44
80	376·00	..	..	..	..	376·0	375·31	− 0·69
90	574·00	..	..	..	..	574·0	568·11	− 5·89
100	..	842·5	..	..	..	842·5	835·89	− 6·61
110	..	1206·0	..	..	..	1206	1198·2	− 7·8
120	..	1683·0	..	..	..	1683	1677·0	− 6·0
130	..	2295·0	..	..	2288	2293	2295·9	+ 2·9
140	..	3074·0	..	..	..	3074	3080·3	6·3
150	..	4059·0	..	4053	4023	4052	4057·1	5·1
160	..	5264·0	..	..	..	5264	5253·4	− 10·6
170	..	6695·0	..	..	..	6695	6697·8	+ 2·8
180	..	8387·0	..	8365	..	8383	8418·8	35·8
190	..	10466·0	..	..	..	10466	10445	− 21
200	..	12828·0	..	12691	..	12801	12809	+ 8
210	..	15575·0	..	..	..	15575	15539	− 36
220	..	18671·0	..	18711	..	18679	18667	− 12
230	..	22161·0	22094	22183	..	22154	22230	+ 76
240	..	26208·0	26102	26227	..	26194	26263	69
250	..	30779·0	30809	..	..	30785	30807	22
260	..	36110·0	35955	36285	..	36103	35908	− 195

In calculating the mean the greatest weight has always been given to the determinations with the largest quantity, A. The constants employed were calculated from pressures at 20°, 80°, 140°, 200°, and 260°. The constants for the formula are—

$$a = 4{\cdot}479370.$$

$$\log b = \bar{1}{\cdot}3915059. \qquad \log \alpha = 0{\cdot}001641423.$$

$$\log c = 0{\cdot}5509601. \qquad \log \beta = \bar{1}{\cdot}99657025.$$

$$c \text{ is negative.} \qquad t = t^{\circ}c - 20.$$

Determinations of the vapour-pressures of propyl alcohol by the statical method are given by KONOWALOW. They are reproduced in the following table, together with the pressures calculated from our constants for BIOT'S formula. It is to be noticed that our results at these temperatures were obtained by the dynamical method.

SERIES I.

Temperature.	Pressure.		Temperature.	Pressure.	
	KONOWALOW observed.	R. and Y. calculated.		KONOWALOW observed.	R. and Y. calculated.
°			°		
11·50	8·1	8·2	59·40	143·25	144·6
16·80	10·0	11·9	59·90	146·90	148·3
21·80	17·2	17·1	70·40	245·80	245·0
28·35	24·7	25·4	74·90	304·20	300·3
30·60	29·5	29·2	80·50	384·10	383·5
33·75	35·7	35·4	81·75	405·20	404·7
39·10	48·3	48·5	81·90	406·40	407·2
49·20	85·3	85·3	89·60	561·70	559·3
52·35	101·0	100·9	98·60	794·90	793·4

SERIES II.

Temperature.	Pressure.	
	KONOWALOW observed.	R. and Y. calculated.
°		
16·5	10·9	11·65
52·4	101·1	101·10
59·9	148·5	148·30
70·5	247·7	246·10
82·1	411·4	410·60

It will be seen that, with the single exception of the observation at 16°·8 in Series I., the agreement is extremely satisfactory. The vapour-pressures have also been determined by Dr. A. RICHARDSON by our method ('Chem. Soc. Trans.,' vol. 49, p. 763) with concordant results.

Orthobaric Volumes of 1 *Gram of Liquid.*

Temperature.	Volume.	Specific gravity.*	Temperature.	Volume.	Specific gravity.	Temperature.	Volume.	Specific gravity.
°	c.cs.		°	c.cs.		°	c.cs.	
0	1·221	0·8193	100	1·365	0·7325	200	1·689	0·5920
10	1·233	0·8110	110	1·385	0·7220	210	1·750	0·5715
20	1·245	0·8035	120	1·406	0·7110	220	1·823	0·5485
30	1·256	0·7960	130	1·430	0·6995	230	1·912	0·5230
40	1·270	0·7875	140	1·455	0·6875	240	2·032	0·4920
50	1·285	0·7785	150	1·484	0·6740	250	2·210	0·4525
60	1·299	0·7700	160	1·515	0·6600	260	2·561	0·3905
70	1·314	0·7610	170	1·550	0·6450	263·15	2·899	0·3450
80	1·330	0·7520	180	1·591	0·6285	263·50	2·959	0·3380
90	1·347	0·7425	190	1·637	0·6110	263·54	3·012	0·3320

* As in our former memoirs, the specific gravities are referred to water at 4°, and are therefore true masses of one cubic centimetre.

Orthobaric Volumes of 1 *Gram of Vapour.*

Temperature.	Volume of 1 grm.	Specific gravity (mass of 1 c.c.).	Vapour-density.	Temperature.	Volume.	Specific gravity.	Vapour-density.
°	c.cs.			°	c.cs.		
80	958·0	0·00104	30·50	180	44·50	0·0225	37·6
90	643·0	0·00156	30·90	190	35·40	0·0282	38·9
100	443·0	0·00226	31·30	200	28·30	0·0353	40·5
110	312·0	0·00320	31·80	210	22·65	0·0442	42·7
120	225·0	0·00443	32·40	220	18·00	0·0556	45·6
130	165·0	0·00605	33·00	230	14·21	0·0704	49·5
140	124·0	0·00805	33·70	240	11·06	0·0904	54·9
150	93·9	0·01060	34·50	250	8·50	0·1180	62·1
160	72·3	0·01380	35·45	260	6·20	0·1610	74·4
170	56·4	0·01770	36·45				

The following table gives the densities of the unsaturated vapour at equal intervals of temperature and pressure.

Pressure.	Temperatures.											
	130°	150°	180°	200°	220°	230°	240°	250°	260°	263°·64	270°	280°
mms.												
2,000	32·49	31·95	31·59	31·33	31·11	30·99	30·93	..	30·80	..	..	30·70
4,000	..	34·41	33·06	32·46	32·04	31·86	31·71	..	31·50	..	..	31·32
6,000	..	..	34·80	33·75	33·02	32·76	32·52	..	32·16	..	..	31·89
8,000	..	..	37·02	35·19	34·17	33·73	33·39	..	32·90	..	..	32·50
10,000	..	..	..	37·11	35·43	34·86	34·38	..	33·66	..	..	33·12
12,000	..	..	..	39·42	36·93	36·16	35·50	..	34·51	..	..	33·78
14,000	..	..	..	..	38·85	37·74	36·84	..	35·48	..	..	34·47
16,000	..	..	..	..	41·26	39·63	38·41	37·44	36·51	..	35·86	35·23
18,000	..	..	..	..	44·40	41·97	40·23	38·89	37·72	..	36·84	36·06
20,000	..	..	..	..	..	45·00	42·30	40·47	39·00	..	37·89	36·97
22,000	..	..	..	..	..	48·99	44·94	42·33	40·51	..	39·10	38·02
24,000	..	..	..	..	..	..	48·42	44·70	42·25	..	40·48	39·15
26,000	..	..	..	..	..	..	53·76	47·70	44·28	..	42·03	40·41
28,000	..	..	..	..	..	..	..	51·66	46·62	..	43·83	41·82
30,000	..	..	..	..	..	..	..	57·90	49·80	..	46·02	43·38
32,000	..	..	..	..	..	..	..	..	54·66	51·90	48·72	45·24
34,000	..	..	..	..	..	..	..	..	61·80	56·97	52·11	47·43
36,000	..	..	..	..	..	..	..	..	..	64·32	56·40	50·01
38,000	..	..	..	..	..	..	..	..	..	..	62·64	53·16
40,000	..	..	..	..	..	..	..	..	..	..	..	57·12
42,000	..	..	..	..	..	..	..	..	..	..	..	62·40

Heats of Vaporization.

The heats of vaporization are calculated from the thermo-dynamical formula,

$$\frac{\mathrm{L}}{s_1 - s_2} = \frac{dp}{dt} \cdot \frac{t}{\mathrm{J}}.$$

The values of dp/dt were calculated in the same manner as with the other liquids. The pressures for one-tenth of a degree above and below the required temperature were calculated by means of the equation $\log p = a + b\alpha^t + c\beta^t$, and the difference was multiplied by 5 to obtain the value for 1°. The pressures were reduced to grams per square centimetre, and the value of J was taken as 42,500.

Temperature.		dp/dt		$s_1 - s_2$	L.
° C.	° Abs.	mms.	grms.	c.cs.	cals.
80	353	15·99	21·7	957·00	173·0
90	363	22·73	30·9	642·00	169·0
100	373	31·16	42·4	442·00	164·0
110	383	41·70	56·7	311·00	159·0
120	393	54·45	74·0	224·00	153·0
130	403	69·75	94·8	164·00	147·0
140	413	87·60	119·1	123·00	142·4
150	423	108·20	147·1	92·40	135·3
160	433	131·50	178·8	70·80	129·0
170	443	157·80	214·5	54·90	122·8
180	453	187·00	254·2	42·90	116·3
190	463	219·00	297·7	33·80	109·6
200	473	254·00	345·3	26·60	102·2
210	483	292·50	397·7	20·90	94·5
220	493	334·00	454·0	16·20	85·3
230	503	379·00	515·0	12·30	75·0
240	513	428·00	582·0	9·03	63·4
250	523	481·00	654·0	6·29	50·6
260	533	540·00	734·0	3·64	33·5

The heat of vaporization of propyl alcohol at the boiling-point 97°·4 would be 165·2 calories.

Pressures and Temperatures of Propyl Alcohol at Definite Volumes.

In our previous papers we have given tables of the volumes of a gram of substance at definite temperatures and pressures.

We have recently shown, however, in two papers read before the Physical Society of London and published in the 'Philosophical Magazine' (May and August, 1887), that, when the volume of a stable liquid or gas is kept constant, a very simple relation exists between the pressure and the absolute temperature, which is expressed by the equation

$$p = bt - a,$$

where p is the pressure, t the absolute temperature, and b and a are constants depending on the substance and on the volume occupied by a gram of it.

We have, therefore, considered it better to construct lines of equal volume or "isochors," and to read temperatures and pressures from the isochors, rather than to read the volumes of a gram from isobars constructed from the isotherms.

Owing to the directions assumed by the isochors, it is most convenient to give temperatures (Centigrade) at definite pressures for volumes below the critical volume, and pressures at definite temperatures for larger volumes.

Volumes Smaller than Critical Volume.

Volume.	Pressure in metres of mercury.										
	5	10	15	20	25	30	35	40	45	50	55
1·26	32·40	33·05	33·70	34·35	35·00	35·70	36·35	37·00	37·65	38·30	39·00
1·28	48·10	48·80	49·50	50·20	50·90	51·60	52·30	53·00	53·70	54·40	55·10
1·30	62·80	63·55	64·30	65·05	65·80	66·30	67·05	67·80	68·55	69·30	70·00
1·32	75·40	76·20	77·00	77·75	78·55	79·30	80·10	80·90	81·65	82·45	83·20
1·34	87·55	88·35	89·15	90·00	90·80	91·65	92·45	93·25	94·10	94·90	95·75
1·36	98·85	99·70	100·55	101·40	102·30	103·15	104·00	104·85	105·60	106·50	107·35
1·38	109·00	109·90	110·80	111·70	112·60	113·50	114·40	115·30	116·20	117·10	118·00
1·40	118·40	119·35	120·30	121·25	122·20	123·15	124·10	125·05	126·00	126·95	127·90
1·42	127·05	128·05	129·05	130·05	131·05	132·05	133·05	134·05	135·05	136·05	137·05
1·44	134·75	135·80	136·85	137·90	138·90	139·95	141·00	142·05	143·10	144·10	145·15
1·46	142·00	143·10	144·20	145·25	146·35	147·40	148·50	149·60	150·65	151·75	152·80
1·48	148·55	149·70	150·85	151·95	153·10	154·20	155·35	156·50	157·60	158·75	159·85
1·50	..	155·85	157·05	158·20	159·40	160·55	161·75	162·95	164·10	165·30	166·45
1·52	..	161·85	163·10	164·30	165·55	166·75	168·00	169·25	170·45	171·70	172·90
1·54	..	167·35	168·60	169·90	171·15	172·45	172·70	174·95	176·25	177·50	178·80
1·56	..	172·40	173·70	175·05	176·35	177·70	179·00	180·30	181·65	182·95	184·30
1·58	..	177·30	178·70	180·05	181·40	182·80	184·20	185·55	186·95	188·30	189·70
1·60	..	..	183·15	184·55	186·00	187·40	188·85	190·30	191·70	193·15	194·55
1·62	..	..	187·65	189·10	190·60	192·05	193·55	195·05	196·50	198·00	199·55
1·64	..	..	191·95	193·45	195·00	196·50	198·05	199·60	201·10	202·65	204·15
1·66	..	..	195·50	197·05	198·65	200·20	201·80	203·40	204·95	206·55	208·10
1·70	..	..	202·40	204·10	205·80	207·50	209·20	210·90	212·60	214·30	216·00
1·75	..	..	209·80	211·65	213·45	215·30	217·15	219·00	220·85	222·65	224·50
1·80	..	..	..	218·05	220·00	222·00	224·00	226·00	228·00	229·95	231·95
1·85	..	..	..	..	225·75	227·85	230·00	232·15	234·30	236·45	238·55
1·90	..	..	..	..	230·60	232·90	235·20	237·50	239·80	242·10	244·40
1·95	..	..	..	..	..	237·00	239·50	241·95	244·40	246·90	249·35
2·00	..	..	..	..	..	240·60	243·25	245·90	248·55	251·20	253·80
2·10	..	..	..	..	..	..	248·80	251·80	254·80	257·80	260·80
2·20	..	..	..	..	..	..	..	256·10	259·50	262·90	266·30
2·30	..	..	..	..	..	..	..	259·40	263·25	267·05	270·90
2·40	..	..	..	..	..	..	..	261·70	266·00	270·25	274·55
2·50	..	..	..	..	..	..	..	263·20	267·95	272·75	277·50
2·60	..	..	..	..	..	..	..	264·30	269·40	274·50	279·60
2·70	..	..	..	..	..	..	..	265·25	270·45	275·60	
2·80	..	..	..	..	..	..	..	265·90	271·40	277·00	
2·90	..	..	..	..	..	..	..	266·00	272·00	278·10	
3·00	..	..	..	..	..	..	..	266·10	272·60	279·15	

It is, however, possible to give a few pressures at even volumes and temperatures, and these, and also the pressures in the following table for volumes up to 30 c.cs. are represented by crosses in the plates (Plate 3 and 6).

Volumes Larger than Critical Volume.

Volume.	Temperature.											
	130°	150°	180°	200°	220°	230°	240°	250°	260°	263°·64	270°	280°
c.cs.												
3	..	..	..	..	..	..	..	..	..	38,120	43,000	50,660
4	..	..	..	..	..	..	..	..	..	38,120	41,680	47,280
5	..	..	..	..	..	..	..	..	..	37,960	40,850	45,380
6	..	..	..	..	..	..	..	..	..	37,280	39,610	43,280
7	..	..	..	..	..	..	..	..	35,190	36,280	38,190	41,200
8	..	..	..	..	..	..	..	..	34,000	34,940	36,580	39,150
9	..	..	..	..	..	..	..	30,390	32,650	33,480	34,920	37,180
10	..	..	..	..	..	..	..	29,430	31,360	..	33,300	35,230
12	..	..	..	..	..	..	25,710	27,270	28,820	..	30,380	31,940
14	..	..	..	..	..	..	23,910	25,190	26,470	..	27,776	29,040
16	..	..	..	..	..	21,130	22,220	23,310	24,390	..	25,480	26,570
18	..	..	..	..	..	19,800	20,730	21,660	22,580	..	23,510	24,440
20	..	..	..	..	17,660	18,480	19,310	20,130	20,950	..	21,780	22,600
25	..	..	..	..	15,340	15,940	16,540	17,150	17,750	..	18,350	18,950
30	..	..	..	12,474	13,430	13,910	14,390	14,860	15,340	..	15,820	16,300
40	..	..	..	9,998	10,656	10,985	11,314	..	11,972	..	..	12,630
50	..	..	7,780	8,299	8,818	9,078	9,337	..	9,856	..	..	10,375
60	..	..	6,695	7,109	7,523	7,730	7,937	..	8,351	..	..	8,765
80	..	..	5,210	5,504	5,798	5,945	6,092	..	6,386	..	..	6,680
100	..	3,898	4,252	4,488	4,724	4,842	4,960	..	5,196	..	..	5,432
120	..	3,321	3,606	3,796	3,985	4,080	4,175	..	4,365	..	..	4,555
140	..	2,876	3,120	3,283	3,447	3,528	3,610	..	3,772	..	..	3,935
170	2,258	2,391	2,590	2,724	2,857	2,923	2,990	..	3,122	..	..	3,255
200	1,955	2,063	2,224	2,332	2,439	2,493	2,547	..	2,654	..	..	2,762

These pressures agree very well with those read directly from the isotherms, except near the condensing point at low temperatures, and to a much smaller extent at high temperatures. The greatest error is at the lowest pressure at 150°, and amounts to 1·45 per cent.

The approximate critical temperature of propyl alcohol is 263°·7, the approximate critical pressure 38,120 mms., and the approximate volume of 1 grm. 3·6 c.cs. The first two of these constants must be very nearly correct; the third cannot be determined with nearly the same accuracy.

ADDENDUM.

(Added February 11, 1889.)

The conclusions of KONOWALOW (*loc. cit.*) regarding the nature of the so-called hydrate of propyl alcohol have been fully confirmed. It was found that the composition of the mixture, which boiled constantly under a pressure of 198·7 mms., differed from that obtained under the ordinary atmospheric pressure; the lower the pressure, the higher is the percentage of water in the distillate.

The boiling-points of four different samples of the mixture were determined under pressures varying from 746 to 762 mms.; corrected to 760 mms., the temperatures observed were 87°·65, 87°·85, 87°·9, and 87°·6—mean 87°·75.

The vapour-pressures were determined by both the dynamical and statical methods, with the following results:—

Dynamical Method.

Pressure.	Temperature.	Pressure.	Temperature.	Pressure.	Temperature.
mms.	°	mms.	°	mms.	°
5·75	− 0·9	38·9	27·4	269·8	63·5
6·75	+ 1·25	46·0	30·2	304·3	66·1
8·15	4·1	57·1	33·6	347·9	69·1
9·0	5·5	66·8	36·3	358·7	69·8
9·2	5·85	78·9	39·2	405·7	72·6
10·1	7·2	92·55	42·2	466·4	75·9
12·55	9·8	109·5	45·3	526·6	78·9
15·8	13·3	123·5	47·65	590·9	81·9
19·05	16·1	143·7	50·5	658·8	84·5
22·75	18·7	167·4	53·6	760·6	87·9
27·55	21·7	200·6	57·4		
32·75	24·6	232·0	60·3		

Statical Method.

Temperature.	Pressure.	Pressure read from curve constructed from results by dynamical method.
°	mms.	mms.
13·3	16·6	15·7
15·1	18·2	17·7
15·7	19·2	18·45
25	35·5	33·8
30	47·9	45·4
40 *	83·8 and 83·95	81·1
50	142·1	138·9
60	231·7	226·5
70	365·4	361·5
75	452·7	450·0

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* The following pressures were also observed at 40° in the vapour-density tube:—83·5, 82·35, 82·55, 82·25.

It will be seen that the results by the statical method are uniformly a very little higher than by the dynamical method. The behaviour of the substance resembles that of an imperfectly purified stable substance more closely than that of a dissociating body.

A considerable number of determinations of vapour-density were made under varying conditions of temperature and pressure. It was proved, however, that condensation—probably of water—took place on the sides of the tube, and the results at the same temperature and pressure could be made to vary considerably by altering the conditions in such a manner as to increase or diminish the chance of such condensation taking place. The rise of vapour-density at low temperatures or high pressures was in no case greater than could be accounted for by premature condensation of liquid, and the only conclusion to be drawn from the results is that combination of propyl alcohol and water does not take place in the gaseous state.

The contraction on mixing propyl alcohol and water at 0°, in the ratio of 71·46 per cent. of alcohol to 28·54 of water, was ascertained by determining the specific gravities of the alcohol and of the mixture, that of water being known. For 1 grm. of the mixture the contraction was 0·0215 c.c., or 1·857 per cent. With ethyl alcohol and methyl alcohol the contraction is considerably greater.

It may be stated, in conclusion, that we have obtained no experimental evidence of chemical combination between propyl alcohol and water.

V. *The Radio-Micrometer.*

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Communicated by Professor A. W. Rücker, *F.R.S.*

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In the preliminary note on the Radio-micrometer which I had the honour to present to the Royal Society last year (1887), I promised to complete, as far as I might be able, the development of the instrument, and, in case of any great improvement in the proportions of the parts, to exhibit an instrument in the improved form. In the present paper I have shown how the best sizes of the several parts may be determined, and how the best result may be attained.

I must, however, first refer to the fact that on February 5, 1886, M. d'Arsonval showed, at a meeting of the Physical Society of France, an instrument called by him the Thermo-galvanometer, with which mine is in all essential respects identical. The invention of an instrument for measuring radiant heat, in which one junction of a closed thermo-electric circuit suspended in a strong magnetic field is exposed to radiation, is due entirely to M. d'Arsonval, and I need hardly say that it was in ignorance of the fact that he had preceded me that my communication was made to the Royal Society. As soon as I became acquainted with M. d'Arsonval's work, I took the earliest opportunity of admitting his claim to priority (see 'Nature,' vol. 35, p. 549).

I venture, however, to think that, although the differences between M. d'Arsonval's thermo-galvanometer and my radio-micrometer are essentially differences of detail, that even at the time of my original communication I had succeeded in producing the most sensitive instrument of practical utility, with the exception perhaps of the bolometer, which had up to that time been constructed for the measurement of radiant energy. As I hope to be able to show in the present communication that I have still further improved the proportions of the several parts, it may perhaps be fortunate that I was not aware that so able and ingenious a physicist had already made an instrument with the properties which I regarded as of so much importance, viz., the low resistance and small moment of inertia of the circuit, the small capacity for heat of the junction, the quickness and dead-beat character of the indications, and its freedom from extraneous influences. Had I known of M. d'Arsonval's work, I should probably have given no more attention to the idea.

It may, perhaps, be worth mentioning that I was led to attempt to make some improvement in the thermopile in consequence of the extraordinary sensibility which Professor LANGLEY had given to the bolometer, because it seemed incredible that, with so small a temperature coefficient of resistance as metals possess, an instrument depending on change of resistance should compare favourably with one equally well carried out, in which the comparatively large electromotive force of a thermo-electric junction is made use of.

The first object obviously was to reduce the mass of the exposed part of the pile. I was, therefore, naturally led to the use of fine wires or thin plates connected with a galvanometer in the usual way, but all the forms which such an instrument might take seemed to promise very little after the idea of the suspended circuit occurred to me. I, therefore, put them on one side, and devoted myself to the perfection of the instrument which forms the subject of the present paper. I notice, however, that there is an account of an instrument of such a kind in the January Number of the 'Philosophical Magazine' of this year (1888).

The considerations which led to the use of the general form of circuit, *i.e.*, one composed of a thin flat bar of antimony and bismuth, having its ends connected by a thin copper wire, which forms the remaining sides of a square or rectangle, are very simple.

A pair of metals must be chosen which have a high thermo-electric power, and which are not to any great extent magnetic, and which can be made exceedingly thin. Of ordinary metals antimony and bismuth so excel others in thermo-electric power that, unless they fail in other respects, they will be the best for the purpose. The diamagnetism is but a small disadvantage, and this may be overcome. The great density of the metals is objectionable; further, the difficulty of making the circuit increases as these metals are made thinner. The low conductivity for electricity of these metals is also a disadvantage, but this is balanced by their correspondingly low conductivity for heat. The disadvantages seem to be more than outweighed by the great thermo-electric power of the combination, if no attempt is made to complete the circuit with these metals only, but if copper is used for this purpose. In this way, the strongly diamagnetic metal is kept out of the intense part of the field; the high conductivity compared to its mass of copper, in which respect aluminium only is superior (but this cannot be soldered), is made use of to convey the current from one end of the bar to the other, round a circuit which may be of sufficient extent to enclose a large area in the magnetic field. Thus, the copper part of the circuit may have less weight and less resistance than it would have if made of antimony and bismuth.

As to the form of this hoop of copper, mechanical considerations determine that it shall be rectangular, for the pole pieces and central core could not conveniently be made to suit a circuit of other form; otherwise it would appear that a circular or elliptic form would be preferable.

I have attached the circuit to the lower end of a very thin capillary glass tube about 6 cm. long, which hangs at the end of a quartz fibre, made by the bow and arrow process described in the 'Philosophical Magazine' of June, 1887. Close to the top of the glass tube is fastened a very light galvanometer mirror, *m*, so that the heat which may fall upon it shall have no influence on the junction lower down (see fig. 1).

Fig. 1.

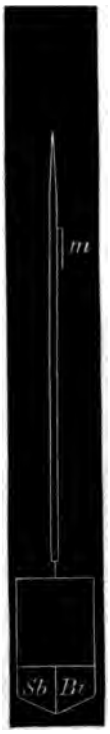


Fig. 2.

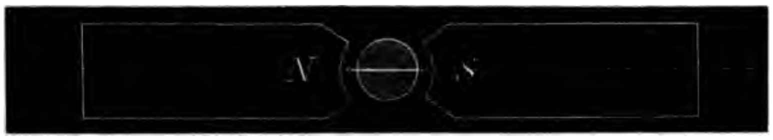
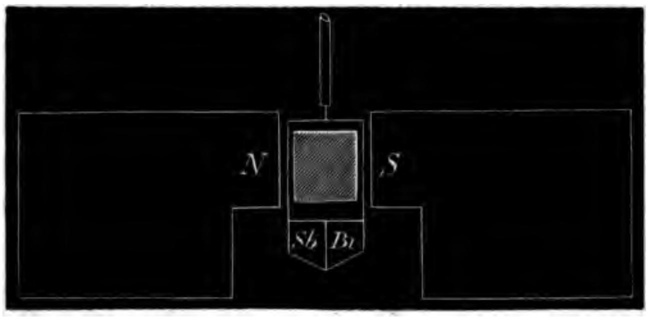


Fig. 3.



The form of pole pieces which I have used is shown in figs. 2 and 3, which are a plan and a vertical section through the dotted line respectively. The central drum of iron, which serves to intensify and make more uniform the magnetic field, is shown shaded in these figures.

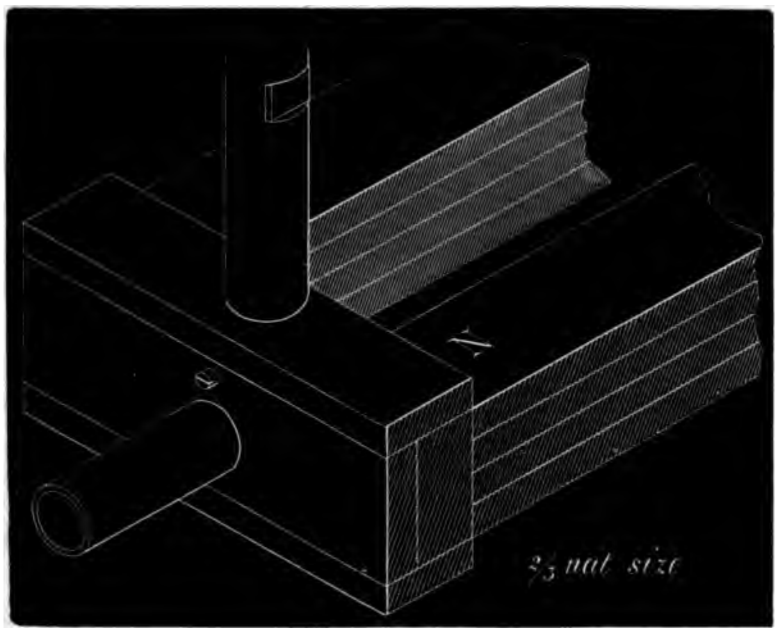
The copper hoop works in the annular space between the pole pieces and the central drum, where the field is most intense, while the active bar of antimony and bismuth hangs in the large cavity below the drum, where it is exposed to the radiation to be measured and where the weakness of the field prevents the diamagnetism from giving trouble. The pole pieces are screwed to massive brass plates, which cover the upper, front, and lower sides. The ends are left bare, as is the whole of the other side, which rests against the flat ends of a powerful horseshoe magnet. The remaining uncovered part between the poles of the magnet is covered with a piece of glass, which enables one, when levelling the instrument, to see if the circuit is free, or, when directing a spectrum upon the junction, to see that the desired colour is in its right place. It also protects the junction from air currents. Through the brass plate forming the front of the instrument, exactly opposite the junction, passes a brass tube open at the ends. This may carry a long tube, such as Professor Langley used in the bolometer, fitted with gradually diminishing diaphragms, which effectually prevent air-currents from penetrating into the chamber. It should also carry, just in front of the active plate, a vertical slit, so as to confine the received radiation to the line of junction and as small a distance on either side as may be desired. A screw passing through the front plate holds the iron drum securely and truly in its place. The upper brass plate is pierced by a brass tube, about 31 cm. long, in which there is a window at the level of the mirror. There is a simple form of torsion head at the upper end of the tube which carries the quartz fibre (see fig. 4).

I should here point out in what respects my instrument differs from that of M. d'Arsonval. He makes his circuit of a pair of wires with the two junctions in the axis of motion, one above and one below. The wires are made of palladium and silver. The circuit is hung by a fibre of silk, and is directed by the action of the magnet on a small piece of iron wire attached to the circuit. *One of the junctions is protected by fixing over it the mirror which reflects the beam of light on to the scale.* The radiation to be measured is concentrated on to the other junction. M. d'Arsonval does not appear to have used for this purpose carefully formed pole pieces, but has simply hung the junction between the legs of a vertical horseshoe magnet, using a central hollow drum of iron within the circuit. He sometimes uses no central drum, but then he places the two wires very much nearer together, forming a long, narrow rectangle, which thus has a very small moment of inertia. He speaks of the great sensibility and quickness of his instrument, which also is dead beat.

In the preliminary note on the radio-micrometer ('Roy. Soc. Proc.,' vol. 42, p. 191), I pointed out in a provisional manner how the instrument may be made as perfect as possible by so choosing the length of the rectangle, the thickness of the copper wire, the number of turns or of junctions, the strength of the field, or the torsion of the fibre, that, if any of them are made more or less, the value of the instrument will be diminished. As I have now completed the calculations up to a point beyond which it would be difficult to go, and where I believe but little on which the perfection of the instrument practically depends remains to be found, I wish, without further delay, to explain the formulæ I have obtained.

Owing to the large number of variables, and the complicated manner in which they are involved, it would be difficult by any direct mathematical process to find the best value for every one at the same time; what I have done is to take the variations, one or two at a time, in such an order that those taken later shall require little or no modification of the results previously found.

Fig. 4.



It is evident that the quickness of the instrument, whatever proportions may be given to it, will increase as the sensitive plate is diminished in thickness; and later on it will be shown that, besides the quickness, the ultimate sensibility, *i.e.*, the deviation for a given rate of radiation, may also be increased as the thickness is diminished. It may, therefore, be taken as a fact that, the thinner the plate, the more sensitive will be the instrument. When I began the calculations I did not think it likely that a plate composed of metals so difficult to work could be made much less than ¼ mm. thick, and, accordingly, that has been assumed as one starting point. I have found, however, not the slightest difficulty in producing plates thinner than this, but I have in what follows taken this quantity as the thickness when wishing to find the numerical values given by the formulæ.

The next thing to determine upon is the general size of the circuit. If it is made large, the enclosed magnetic field will be increased; but, supposing the period of oscillation to remain invariable, the moment of torsion must be increased, and, further, the resistance may be increased. Both of these actions will reduce the sensibility. Let it be supposed that there are two instruments in all respects the same, except that the circuit of one is n times the length and is n times the width of that of the other, and let it be supposed that the thickness of the bar and of the wire of the circuit is the same in each case; the value of the enclosed magnetic fields will be as $n^3 : 1$; the moment of inertia, and therefore the moment of torsion, will be as $n^3 : 1$; and the resistance as $n : 1$. Therefore, $\frac{\text{included magnetic field}}{\text{torsion} \times \text{resistance}}$ will be as $1 : n^2$; that is, the angle of deflection will be n^2 times as great in the smaller as in the larger instrument. But there is a limit to the smallness, owing to two causes. The moment of inertia is made up of two parts, the circuit and the mirror, on which account, when the moment of inertia of the mirror becomes comparable with that of the circuit, the smaller instrument will have a greater moment of inertia, and, therefore, its sensibility on this account will be less than that given by the above rule.

The second reason why there is a superior limit to the sensibility as the instrument is reduced in size is due to the fact that the sensitive plate conducts more heat from the hot to the cold junction in the case of the shorter plate, and thus, for a given rate of radiation, the junction will not be so hot; and, hence, the current will be less than it would be if the diminished resistance were the only cause of change. It is, therefore, necessary to choose some length of plate or width of rectangle which can conveniently be made in practice, and, assuming this as a constant, to find by calculation the best values of all the variables, and so, as it were, to fit to the plate the best possible instrument. Something must be assumed as a starting point, and this seemed, on the whole, the most convenient. I have assumed the plate, then, to be composed of two squares of antimony and bismuth, each 5 mm. in the side and $\frac{1}{4}$ mm. thick, soldered edge to edge, thus forming a plate $10 \times 5 \times \frac{1}{4}$ mm. This assumption I have made simply for the sake of numerical calculation; any size may be equally well taken as the basis of operations, provided that in the equations which follow the proper numerical values are assigned to the constants.

Of the three dimensions of the plate, the length, the thickness, and the breadth, the first two only need be considered as assumptions which can in any way affect the result, for it matters not how the breadth be varied, provided that the sectional area of the wire and the moment of torsion are varied in the same proportion, *i.e.*, if the whole breadth is exposed to the radiant energy.

It will first be convenient to see how the circuit can be formed, so as to give the best results when the magnetic field is supposed constant; it will be seen later that this best is not the ultimate best when the field and the circuit are adapted to one another.

The circuit may be considered best from more than one point of view. It may be with respect to weight, or with respect to moment of inertia.

The best circuit with respect to weight would be that which would give the greatest deflection when supported by a particular bifilar arrangement free from torsion. The strength of the field, the thermo-electric power of the junction, and the value of the radiation, being constant factors, may be omitted for the present.

Let the following be the meanings of the several symbols that will be used :—

(·1481)	W.	The weight of the plate, mirror, and stem, the invariable or dead weight.
	w.	The weight of the hoop of copper, the variable weight.
($6·742 \times 10^6$)	C.	The resistance of the plate, the invariable or dead resistance.
	r.	The resistance of the hoop of copper, the variable resistance.
	l.	The length of the rectangle.
	n.	The number of turns of wire.
(·00895)	u'.	The weight of a unit piece of copper (1 × ·1 × ·01 cm.).
($1·642 \times 10^6$)	v.	The resistance of a unit piece of copper (1 × ·1 × ·01 cm.).
	a.	The sectional area of the wire (·1 × ·01 being considered unit area).

The reason for taking 1/1000th of a square centimetre as the unit of sectional area for the wire is that this is not very different from the actual sizes that will be required, and that it avoids the absurdity of supposing a hoop of wire of such a size made of wire of 1 sq. cm. in sectional area. It is a matter of convenience, and nothing more. The mirror that I have used is 6 mm. in diameter, and weighs ·04 grm. The glass stem may be taken as ·005 grm. The weights and resistance of the three metals are taken from LUPTON'S Tables, and the numerical values of the several quantities so calculated are enclosed in parentheses before their respective symbols. All the quantities, except when otherwise specified, are in C.G.S. units.

It is necessary to bear in mind that a certain excess of wire may be required, over and above that actually necessary to reach the active bar. Though in the numerical examples which follow I have not allowed for any, I have taken care in the equations to introduce a symbol, p, which must be explained. Imagine a circuit made of one turn 1 cm. wide and l cm. long: then, if there is no excess, the amount of wire will be $2l + 1$; but, if there is an excess, then the amount will be $2l + 1 +$ excess. I have used the symbol p in the following calculations for $1 +$ excess, and I have assumed that both the resistance and the weight vary with the total length. The only possible discrepancy can be due to the manner in which the current leaves the wire for the plate.

Considering, first, the case of a circuit of only one turn of wire, the variable resistance r and weight w of the wire will be

$$r = (2l + p)\,v \div a,$$
$$w = (2l + p)\,u' \times a.$$

The conductivity G of the whole circuit will be

$$G = \frac{a}{(2l + p)\,v + aC}.$$

The efficacy of the circuit with respect to its weight, E_{wt}, *i.e.*, the moment which it can exert upon a unit field for every gramme that it weighs when unit E.M.F. is acting at the junction, is

$$E_{wt} = \frac{l \times G}{W + w} = \frac{la}{\{(2l + p)\,v + aC\}\{W + (2l + p)\,au'\}}.$$

Now, it is evident that there must be a maximum value for E_{wt}, both when l and when a is varied, for, if either is made very great or very small, more is lost than is gained. If, therefore, the expression for E_{wt} is treated in the usual way to find the two maxima, the result will be found to be

$$\text{best } a = \sqrt{\frac{Wv}{u'C}} \quad \dots\dots\dots\dots\dots \quad (1),$$

$$l^2 = \frac{(W + pau')(pv + aC)}{4vau'};$$

or, substituting the value of a above, it will be found that

$$2l = p + \sqrt{\frac{CW}{u'v}} \quad \dots\dots\dots\dots \quad (2).$$

If, further, the circuit is supposed to have n turns, the corresponding expressions will be

$$a = \sqrt{\frac{Wv}{u'C}},$$

$$2nl = (2n - 1)\,p + \sqrt{\frac{CW}{u'v}} \quad \dots\dots\dots\dots \quad (3).$$

Thus, the size of wire that is most suitable is independent of the length of the rectangle or of the number of turns. The numerical values of these quantities for 1 . . . 5 turns are as follows:—

n.	Best l.	E_{wt}.	Best a.
1	4·621	$9·202 \times 10^{-7}$	2·0075.
2	2·811	7·565 „	
3	2·207	6·423 „	
4	1·905	5·580 „	
5	1·724	4·933 „	

Thus, it appears that one turn is better than any other number, provided that the circuit may have sufficient length, and, of course, that the magnetic field is sufficiently extended. This result is really self-evident, for, whatever may be the efficacy of a circuit of say two turns of the best length, one of one turn of twice the length must be better, as in this case the value of the enclosed magnetic field will be the same, while the resistance and the weight will each be less, and, further, the circuit of twice the length will not have the best length for one of one turn only.

If the length of the circuit is limited to 1 cm., then two turns are better than one or three. The series of figures found on this supposition are not of sufficient importance to be worth giving.

It is interesting, however, to notice how slowly the efficacy changes when the length of the circuit is not the best. The following figures show this :—

l.	E_{wt}.	
3	$8·742 \times 10^{-7}$	
3·5	9·036 „	
4	9·152 „	
4·5	9·200 „	
4·621	9·202 „	max.
5	9·190 „	

Thus, it is not a matter of much consequence whether the circuit is very near the best length or not.

If it is desired to find the numerical value of the resistance or the weight of the wire part of the circuit when it is of the best length and sectional area, the following expressions, which have been obtained by substituting in those for r and w the best values found for a and l, may be used to save time :—

$$\text{best } r = \mathrm{C}\left(1 + 2p\sqrt{\frac{u'v}{\mathrm{CW}}}\right) \quad \ldots\ldots \quad (4),$$

$$\text{best } w = \mathrm{W}\left(1 + 2p\sqrt{\frac{u'v}{\mathrm{CW}}}\right) \quad \ldots\ldots \quad (5).$$

From these it is seen that that length and size of wire is best of which the weight as much exceeds the dead weight as the resistance exceeds the dead resistance. In the same way, the efficacy of the best circuit may be shown to be

$$\text{best } \mathrm{E}_{wl} = \frac{1}{8\sqrt{(\mathrm{CW}u'v)} + pu'v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

If, finally, the breadth be made a variable, the following equations will give the best conditions :—

As before,

$$\text{best } a = \sqrt{\frac{wv}{u'\mathrm{C}}};$$

and this is true whatever length, number of turns, width, or even shape the circuit may have. When b is greater than 1,

$$\text{for the best length, } 2l = 2b - 1 + e + \sqrt{\frac{\mathrm{CW}}{u'v}},$$

$$\text{for the best breadth, } 2b = 2l - 1 + e + \sqrt{\frac{\mathrm{CW}}{u'v}},$$

where e is the excess used for soldering.

Therefore, for any breadth the length must exceed the breadth by as much as the breadth should exceed the length when that is given. In other words, the circuit is improved by adding to it *ad infinitum*, so that a square of infinite size is the best rectangle. If it should happen that $\sqrt{(\mathrm{CW}/u'v)} + e < 1$, then the circuit in the same way would be made worse by increasing the dimensions. As a matter of fact, in the particular case, taking e as 0, this quantity is equal to 8·24.

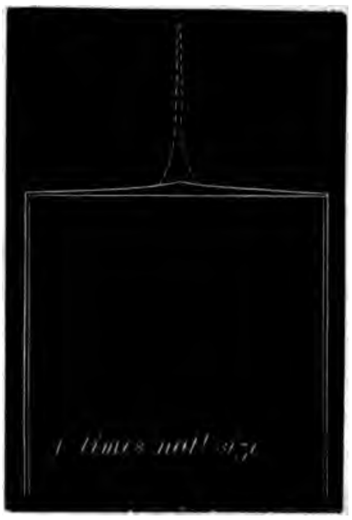
If, on the other hand, b is less than 1, then the quantity $2b - 1 + e$ in the two equations above must be replaced by $b + 1 + e$, a quantity necessarily positive; hence, whether b is less than or greater than 1, the circuit cannot be too large.

If the same process that has been followed in finding the best conditions with respect to weight be employed to find them with respect to moment of inertia, a difficulty arises in consequence of the fact that the upper end or cross wire of the circuit has a resistance which depends upon its length simply, while it has a moment of inertia which is only one-third of what it would have if it were placed alongside of one of the side wires of the circuit. Thus, while the expression for r remains as before, viz., $(2l + 1 + e)\, v \div a$, that for the moment of inertia of the wire will be $\{(6l + 3c + 1)/12\}\, u'a$. If with these values the attempt is made to find the best values for a and l with respect to moment of inertia, a complicated cubic equation results and symmetrical expressions can no more be obtained. If, however, the two coefficients in parentheses had the same value, there would be no difficulty.

While trying to find a remedy for this difficulty, I noticed that a wire of uniform section is not the best form of conductor when the moment of inertia is taken into account, *i.e.*, of that part of the wire which crosses the axis. That this is so is evident in two ways. Since a wire of uniform section has the least resistance for a given length and weight, it cannot have the least resistance for a given length and moment of inertia, except when it is parallel to the axis of rotation. Or, considering the cross piece only, since the uniform section is that which gives the least resistance for its weight, it is clear that a very small change of form, such for instance as would occur if a thin skin were removed from the outer and were transferred to those portions nearer the axis of rotation, would produce no appreciable change in resistance, but it would reduce the moment of inertia; therefore, there is an advantage in having those parts near the axis thicker than those more distant from it. If we suppose that the resistance of any element of the wire between the axis and the ends is inversely proportional to the cross section, and that the resistance of the whole piece is the sum of the resistances of the several elements, which is true as long as the rate of change of section is not so rapid as to make the stream lines notably inclined, then the best variation of section will be that in which no change will be made in the resistance ÷ the moment of inertia if a thin skin be transferred from one part to another. Taking the axis as origin of rectangular coordinate, and calling the sectional area y, then for all values of x, $1/y \div yx^2$, *i.e.*, $1/y^2x^2$ must be constant or y must vary inversely as x. It thus appears that, if the hoop is cut out of sheet copper instead of wire, that part which forms the upper bar should be bounded by two hyperbolas of such dimensions that the width at the ends of the cross bar is the same as that of the side pieces, which are uniform. If the metal is chosen of such a thickness, t, that the best section of the sides hereafter to be found $= t \times \cdot 1$ mm., then the error due to the inclination of the stream lines already pointed out will not be appreciable, except within about ½ mm. on either side of the axis. It will be well at once to point out that the value of this supposed form for the cross bar does not at all lie in the fact that it is the best form, for nothing worth consideration would be gained by adopting it, but the reason for bringing it forward is this: the total resistance of either half of the cross bar will be, if the resistance of the same length of the wire be called 1, $= \int_0^1 x\,dx = \frac{1}{2}$; the moment of inertia of either half will be, if that of the same length of the side be called 1, $= \int_0^1 x\,dx = \frac{1}{2}$. That is, not only has the best form been found for the cross bar, but the coefficients for moment of inertia and for resistance have at the same time been made identical, and thus all the expressions found with respect to weight will equally apply with respect to moment of inertia.

It may be objected, since, as already mentioned, the resistance of these parts close to the axis is greater than is supposed, on account of the inclination of the stream lines and the consequent concentration of the current in the more direct line, or again, since the weight of a piece of metal filling the space between two hyperbolas is infinite, that the solution for the difficulty thus put forward is not correct, and that any further calculations based upon this result will not be trustworthy. The answer evidently is that, if the axial spike be removed altogether, as shown in fig. 5, the actual resistance and moment of inertia would not be so much as 1 per cent. different from the value ($\frac{1}{2}$) already found, and that, as these calculations are merely a guide to direct the design of a practical instrument, an assumption which is not more than 1 or 2 per cent. in error in what is after all a small fraction of the total resistance and moment of inertia—one which, in fact, is so true that neither the power of the instrument maker nor our knowledge of the constants will prevent our introducing in practice variations of far greater magnitude—may be allowed to pass, especially as without something equivalent it would be impossible to find expressions which would apply to the more complicated conditions which will be considered later.

Fig. 5.



The same considerations apply exactly to the sensitive plate, which, when the greatest efficacy with respect to moment of inertia is required, should no longer be made prismatic, but should become more narrow towards the ends, as shown in fig. 1. I may mention that in the original instrument, which I made before I had arrived at any of these results, I did, as a matter of fact, make the active bar of the shape shown, because I felt that the lower corners were doing more harm by their moment of inertia than they were doing good by their conductivity.

Since the active plate is taken as one of the invariables of the circuit, any proportions that may be desired may be taken for it, and the resulting resistance and moment of inertia made use of in calculating numerical values. I have in the arithmetical work which follows taken the active plate to be of the original shape and size, that is, a rectangular prism, 10 × 5 × ·25 mm. I have further not made any allowance for excess of metal for soldering, *i.e.*, p has been taken as the coefficient of resistance and moment of inertia of the upper bar only, which has been found to be equal to ½, just as, when the conditions with respect to weight were considered, p was taken as equal to 1.

As in the rest of this paper the best conditions with respect to weight will no more be considered, but only those relating to moment of inertia, it will be well here to specify the meaning of the several symbols that will be used. The arithmetical values of the fixed quantities are, as before, included within parentheses.

$(9{\cdot}49 \times 10^{-3})$	K.	The moment of inertia of plate, mirror, and stem, the dead moment of inertia.
	k.	The moment of inertia of the copper hoop, the variable moment of inertia.
$(6{\cdot}742 \times 10^{6})$	C.	The resistance of the plate, the dead resistance.
	r.	The resistance of the copper hoop, the variable resistance.
	l.	The length of the rectangle.
	n.	The number of turns of wire.
$(2{\cdot}2375 \times 10^{-3})$	u.	The moment of inertia of the unit piece of copper $(1 \times {\cdot}1 \times {\cdot}01)$ at 5 mm. from the axis.
$(1{\cdot}642 \times 10^{6})$	v.	The resistance of the unit piece of copper.
	a.	The sectional area of the wire (${\cdot}1 \times {\cdot}01$ being considered unity).

Since the formulæ already given are now equally true with respect to moment of inertia, it will be unnecessary to do more than barely state them here for future reference.

$$\text{For best sectional area } (n \text{ turns}) \; a = \sqrt{\frac{\mathrm{K}v}{u\mathrm{C}}} \quad \ldots\ldots \quad (7),$$

$$\text{for best length } (n \text{ turns}) \; 2nl = (2n-1)p + \sqrt{\frac{\mathrm{KC}}{uv}} \quad \ldots\ldots \quad (8),$$

$$\text{for best } r \text{ (one turn) } r = \mathrm{C}\left(1 + 2p\sqrt{\frac{uv}{\mathrm{CK}}}\right) \quad \ldots\ldots \quad (9),$$

$$\text{for best } k \text{ (one turn) } k = \mathrm{K}\left(1 + 2p\sqrt{\frac{uv}{\mathrm{CK}}}\right) \quad \ldots\ldots \quad (10),$$

$$\text{for best } \mathrm{E}_k \text{ (one turn) } \mathrm{E}_k = \frac{1}{8\,\{\sqrt{(\mathrm{CK}\,uv)} + puv\}} \quad \ldots\ldots \quad (11).$$

$p = \frac{1}{2}$ + any excess allowed for soldering.

The following arithmetical results are obtained from some of these expressions, making $p = \frac{1}{2}$:—

n.	Best *l*.	E_t.	Best *a*.
1	2·3366	$7·1863 \times 10^{-6}$	1·0174.
2	1·4183	3·4222 „	
3	1·1122	2·1208 „	
4	·9591	1·5014 „	
5	·8673	1·1500 „	

Thus, the best length and the best sectional area with respect to moment of inertia are each just over half the values found with respect to weight. One turn, of course, gives a greater efficacy than any other number, just as it did before, and for the same reason, but in this case one turn is more than twice as good as two, whereas it was only about one-fifth in excess.

The result at present obtained, then, is this: provided a torsion fibre of convenient length can be obtained which shall in every case produce a certain pre-determined period, say 10 seconds, which, with my process for quartz, is possible even when the hanging body is far lighter and smaller than any galvanometer mirror, then the moment of torsion and the moment of inertia will always have the same ratio; therefore, the circuit which produces the greatest couple for its moment of inertia will also produce the greatest couple compared with the torsion, and will, therefore, give the greatest deflection. This circuit has been found above to have a length of 2·3366 cm., and to be made of copper, having a sectional area of ·0010174 sq. cm. at the sides.

Though this circuit has the best length and the best sectional area under certain magnetic conditions, it is not necessarily the best when these are varied.

It has, of course, been understood that in the comparison between long and short circuits the magnetic field is in both cases the same, and is uniform throughout the whole of the area enclosed by the circuit. This may or may not be true when an internal pole piece is arranged as shown in figs. 2 and 3. There must be some clearance above and below; thus, the internal drum is always less than the actual length. It is therefore probable that the effective area is proportional to the length, less a small constant quantity. If this defect is equal to the sum of the clearances above and below the drum, then the length of the drum must be called l in the preceding formulæ, and four times the clearance added to p. It is not possible to say exactly what is the error thus introduced into the calculations, but in no case can it be important. This, however, is a mere detail compared with the effect of very strong magnetism.

When a closed circuit oscillates under the conditions in which that in the instrument is placed, induced currents are formed which oppose the motion, and thus, if the field is strong enough, or the conductivity great enough, the oscillatory character ceases, and the circuit slowly moves towards its resting place, more slowly as it approaches it. It is a great advantage in an instrument that it should in this way be dead beat; but, if the field is more than strong enough, the extra resistance to the motion so increases the time of coming to the resting place that the loss of time more than counterbalances any advantage given by the increased ultimate sensibility. If the ultimate sensibility is required to be made a maximum, then the expressions found are the best, and, the stronger the field, the better; but, if the best combination which is dead beat (and no more than dead beat) is required—and this is what any one who has used both would require—then the length and the sectional area already found are not the best, unless the strongest field which can conveniently be employed is not strong enough to make the motion dead beat. In the particular case it is more than strong enough.

It is necessary, therefore, to introduce the effect of another variable, the strength of the field, the relation between it and the rest of the circuit being such that the motion is just dead beat.

It is well known that motion ceases to be oscillatory when half the coefficient of resistance to the motion is equal to the square root of the acceleration when the angular displacement is unity. In the investigations depending on these relations the following symbols will have the meanings attached to them:—

S. Sensibility, *i.e.*, efficacy × enclosed magnetic field.

A. Area enclosed by the circuit.

ω. Angular velocity.

T. Time of a complete undamped oscillation.

G. Conductivity of the whole circuit.

α. Angle included between the plane of the circuit and the direction of the lines of force, supposed parallel to one another.

κ'. Moment of inertia of the whole circuit, *i.e.*, $K + k$.

H′. The minimum strength of magnetic field for which the motion is dead beat. This will hereafter be called the dead beat magnetic field.

The resistance to the motion of the circuit

$$= G \times H^2 \times A^2 \times \omega \times \cos^2 \alpha;$$

half the coefficient of resistance $= GH^2A^2 \cos^2 \alpha/2\kappa'$.

Since the magnetic field is radial and is everywhere cut normally by the side wires of the circuit, the factor $\cos^2 \alpha$ ought to be omitted. Owing to the small possible angular deflection, it could not in any case differ appreciably from 1.

$$\sqrt{(\text{acceleration at unit angle})} = \frac{2\pi}{T}.$$

Since, when the motion is just dead beat, half the coefficient of resistance is equal to $\sqrt{(\text{acceleration at unit angle})}$,

$$\frac{GH'^2A^2}{2\kappa'} = \frac{2\pi}{T},$$

or

$$H' = \frac{2}{A}\sqrt{\frac{\pi}{T}}\cdot\sqrt{\frac{\kappa'}{G}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12),$$

$$S = \frac{GAH'}{\kappa'} = 2\sqrt{\frac{\pi}{T}}\cdot\sqrt{\frac{G}{\kappa'}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (13).$$

On differentiating the expression for S with respect to a (the sectional area of the wire) and equating to 0, the maximum sensibility will be found when a has the same value $\sqrt{(Kv/uC)}$, which gave the maximum efficacy; on the other hand, the differential coefficient with respect to l (the length of the rectangle of copper) is negative for all positive values of l, that is, however short l may be, provided that the magnetic field may be made strong enough to keep the motion dead beat, a still greater sensibility will be given by a shorter circuit. If the strongest magnetic field conveniently available is not sufficient to make the circuit having maximum efficacy dead beat, then that circuit is still the best. If, however, as will be the case with an ordinary magnet, the field is more than strong enough, then the length of rectangle must be reduced until the motion is deat beat. Taking the particular arrangement already referred to, in which the circuit is 1 cm. wide and is made of wire having the best sectional area (·001017 sq. cm.), the following are the values of the dead beat magnetic field for various lengths of circuit:—

								Best.
Length . .	·2	·4	·6	·8	1·0	1·5	2·0	2·3366
H' . . .	1722	929	664	532	455	347	294	271·8

It is interesting here, in the expression for H' (12), to substitute those values of κ' and G which belong to the circuit of greatest efficacy. If this is done, there finally results the equation

$$H'' = 8\sqrt{\frac{\pi}{T}}\sqrt{uv} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (12a),$$

where H'' is the dead beat magnetic field for the circuit of greatest efficacy. It thus appears that, no matter what the resistance of the antimony-bismuth bars, or what the moment of inertia of these bars, the mirror, and stem may be, provided that the circuit is so formed as to produce the greatest ultimate sensibility in any given field, the motion will be dead beat for one particular strength which simply depends upon the specific gravity and the specific resistance of the material with which the circuit is completed.

The sensibility obtained by such a combination is

$$S = \sqrt{\frac{\pi}{T}}\cdot\frac{1}{\sqrt{CK} + p\sqrt{uv}} \quad . \; . \; . \; . \; . \; . \; . \quad (13a).$$

As has already been shown, this is not the best arrangement to make use of. That is best in which the circuit is the shortest which will remain dead beat in the strongest magnetic field available.

In the original instrument, of which figs. 2 and 3 show the pole pieces, a strong compound horseshoe magnet was used. The working field was tested by replacing the active circuit by one composed of 50 turns of 1 sq. cm. each of the finest insulated copper wire. This was mounted so that it could be suddenly twisted through a definite angle by moving an arm between a pair of stops. The ends of the coil were connected with a ballistic galvanometer in a distant room, and the throw observed. The resistance of the whole circuit was measured. Then, in the place of the coil, a condenser of known capacity and a cell of known E.M.F. were arranged with a key so that a definite discharge of electricity could be sent through the galvanometer. The absolute value of that sent by each oscillation of the coil could thus be determined, and hence the value of the field found. In this way it was found that without the keeper a field of 1342 units existed in the working space.

It appears, then, from the table on the previous page, that with such a field the circuit should be about 3 mm. long. I did not actually alter the shape of the circuit, but adjusted the field by means of a sliding armature, until the sensibility, or the resistance to the motion, produced a convenient result. As has been shown, this is not so good a plan as adapting the circuit to the strongest field that is available, though less is lost by reducing the field than might be expected, as will be explained later.

If the breadth b that will give the greatest efficacy with respect to moment of inertia is required, there is no difficulty in finding the best a and l in terms of b, but the best b is involved in an expression of such complexity that it can only be found by arithmetical means.

It is not worth while to give at length the table showing the successive values of the efficacy, as b varies; it is sufficient to state that not only is the efficacy diminished by increasing the breadth beyond that of the active bar, but it is even increased as the breadth diminishes down to 2 mm., and probably far beyond.

The conclusion, then, is obvious, that not only should the rectangle be as narrow as possible, but the junction should be arranged also in a correspondingly narrow form. Further, in consequence of the extreme narrowness of the circuit, the resistance and moment of inertia of the cross wire may practically be neglected in comparison with the now much greater length of the rectangle. I must here remark that I have thus been brought to the adoption of the excessively narrow form which M. D'ARSONVAL has used. I do not know whether he was aware that the narrow form is not only far quicker than a wide form, which is the reason he gives for adopting it, but that it is in addition, when a convenient period is arranged, also more sensitive. I certainly did not expect to find it so.

Making use of the same methods and symbols, but neglecting the now infinitesimal effect of the cross wire, the following equations will be found to hold:—

$$\text{best } a = \frac{1}{b}\sqrt{\frac{\mathrm{K}v}{u\mathrm{C}}} \quad \ldots\ldots\ldots \quad (14),$$

$$\text{best } l = \frac{1}{2b}\sqrt{\frac{\mathrm{KC}}{uv}} \quad \ldots\ldots\ldots \quad (15),$$

$$\text{greatest } \mathrm{E}_k = \frac{1}{8\sqrt{(\mathrm{KC}uv)}} \quad \ldots\ldots\ldots \quad (16).$$

Further, it will be found that

$$\mathrm{C} = \frac{2lv}{a} \quad \text{and} \quad \mathrm{K} = 2\,laub^2,$$

and thus the copper hoop must be so proportioned that its resistance may be equal to the dead resistance, and its moment of inertia to the dead moment of inertia. It is also found that under the supposed circumstances, namely, that the cross wire is of no account, the efficacy is independent of the breadth, and the only effect of an increased breadth is to require a diminished sectional area of wire and a diminished length; thus, during variation of the breadth neither the resistance nor the moment of inertia of the copper wire is changed.

The expression for the dead beat magnetic field, when the circuit of greatest efficacy is used, is, as before,

$$\mathrm{H}'' = 8\sqrt{\frac{\pi}{\mathrm{T}}}\sqrt{uv} \quad \ldots\ldots\ldots \quad (17);$$

and this seems to be very generally true.

The sensibility under these conditions is

$$\mathrm{S} = \sqrt{\frac{\pi}{\mathrm{T}}} \cdot \frac{1}{\sqrt{\mathrm{KC}}} \quad \ldots\ldots\ldots \quad (18),$$

which shows that, no matter what the material of the hoop may be, if the dimensions which give the greatest sensibility in any given magnetic field are employed, and if the field is so strong that the motion is dead beat, the sensibility will always be the same, and this will only depend on the resistance and moment of inertia of the invariable part of the circuit.

Since, as in the case of the wide circuit, the magnetic field that will make the motion of the best narrow circuit dead beat is far less than that which is available, it will, as before, be best to employ a circuit so much smaller than the best as will just make the motion dead beat.

The circuit may be made smaller by reducing either factor, the length l or the breadth b. As has been shown, it is indifferent whether one or other factor is altered, provided that b is small compared with l. Let the product be made N times as small, so that bl becomes bl/N, then the first factor $2/bl$ in the expression for H′, which it is convenient to repeat here in another form,

$$\left(\mathrm{H}' = \frac{2}{bl}\sqrt{\frac{\pi}{\mathrm{T}}}\sqrt{\kappa'\mathrm{R}}\right),$$

will become $2\mathrm{N}/bl$. But when bl is changed, both κ', the total moment of inertia, and R, the total resistance, are changed also. In the circuit of greatest efficacy the resistance and moment of inertia of the hoop are each equal to the fixed or invariable resistance and moment of inertia; and thus, when bl becomes bl/N, one half of each κ' and R also becomes one-N$^{\text{th}}$ of what it was, the other half of each remaining unchanged, and thus the new expression for H′ becomes $\frac{\mathrm{N}+1}{bl}\sqrt{\frac{\pi}{\mathrm{T}}}\sqrt{\kappa'\mathrm{R}}$, that is $\frac{1}{2}(\mathrm{N}+1)$ times what it was. Therefore, if it is desired to make the dead beat magnetic field M times that which has been found for the circuit of greatest efficacy, the product bl must be reduced until it is $2\mathrm{M} - 1$ times as small, for $\mathrm{N} = 2\mathrm{M} - 1$.

By this process the actual sensibility is increased, and the amount of increase may be found as follows:—The sensibility of any combination varies directly as the magnetic field and as the product bl, and inversely as the total moment of inertia and the total resistance; of these four quantities it has just been shown, that when bl is divided by N, the dead beat magnetic field is multiplied by $\frac{1}{2}(\mathrm{N}+1)$, and at the same time R and κ' are each multiplied by $(\mathrm{N}+1)/2\mathrm{N}$; therefore, the sensibility of the arrangement becomes

$$\frac{\mathrm{N}+1}{2} \times \frac{1}{\mathrm{N}} \times \frac{2\mathrm{N}}{\mathrm{N}+1} \times \frac{2\mathrm{N}}{\mathrm{N}+1}, \text{ that is, } \frac{2\mathrm{N}}{\mathrm{N}+1},$$

or $2 - (1/\mathrm{M})$ times what it was, so that

$$\mathrm{S} \text{ becomes } \left(2 - \frac{1}{\mathrm{M}}\right)\sqrt{\frac{\pi}{\mathrm{T}}}\frac{1}{\sqrt{\mathrm{KC}}}.$$

Since the dead beat magnetic field for the circuit of greatest efficacy is about 272 units, N must be so chosen as to make H′ four or five times as great. Assuming that H′ is to be increased to four times its original value, or that $\mathrm{M} = 4$, the sensibility will only become $1\frac{3}{4}$ times what it was. Even in the case of an infinite field, it cannot be more than double that due to a field of 272 units if the motion is only just dead beat. From this it appears that, as long as the circuit has dimensions which at all approximate to those which theoretically are best, the sensibility obtained by moving the pole pieces until the dead beat conditions or the desired logarithmic decrement are produced is practically the highest which is possible.

Owing to the fact that, with increase in the breadth of the circuit, the cross piece becomes increasingly mischievous, both on account of its moment of inertia and of its resistance, it is clear that the circuit cannot be too narrow until the increased length becomes such that it is inconvenient to provide a magnet and pole pieces which will enclose so great a length. In another way the thick wire which the narrow circuit requires is advantageous, as will appear shortly.

Having thus found the best relations between the variable copper and the arbitrary junction and mirror, it remains to see how these may be modified with advantage.

As, with the narrow form of circuit, the smallest galvanometer mirror has a moment of inertia many times as great as that of the active bars, and since the copper must have a moment of inertia equal to their sum, it is evident that it will be advantageous to reduce the dimensions of the mirror until it again becomes small in comparison. By this reduction the defining power of the mirror, supposed optically perfect, is also reduced, and thus there must be a limit at which as much is lost by the increasing want of definition as is gained by the diminishing moment of inertia.

The defining power of a perfect mirror—and the smaller the mirror the more likely it is to be perfect—varies with its diameter, while the moment of inertia is proportional to the fourth power of the diameter when the thickness is constant, or to the fifth power if the thickness is also proportional to the diameter. To find the best diameter it is necessary to remember that the fixed moment of inertia K is the sum of the moment of inertia of the junction K_j and of the mirror K_m. Thus, the accuracy of observing a deflection $= K_m^{1/n}/2(K_m + K_j)$, where $n = 4$ or 5 as the case may be. The best size of mirror then will be such that

$$K_m = \frac{K_j}{3}; \qquad k = \frac{4K_j}{3} = 4K_m, \text{ when } n = 4,$$

or that

$$K_m = \frac{K_j}{4}; \qquad k = \frac{5K_j}{4} = 5K_m, \text{ when } n = 5.$$

Using the thinnest microscope cover glass, about ·1 mm. thick, it will be found that the size of mirror which gives the best result when the antimony-bismuth bars have the dimensions which will be assigned to them hereafter is one having a diameter of $2\frac{3}{4}$ mm.

I have picked out a number of discs sufficiently thin, silvered them, cut pieces of the proper size, and then examined them by reflection. With mirrors as small as this, the eye itself takes the place of the usual telescope, and it is easy to choose those which allow the eye to see by reflection fine distant lines as clearly as if the light came direct. The only difficulty I have had in attaching these mirrors to the stem arises from the bending of the glass under the action of even the smallest quantity of cement. If sealing-wax is used—and this is the least magnetic of all the cements I have examined—a speck less than 1 mm. in diameter will, by its capillarity when melted, so distort the glass, even though it only touches near one edge, as to make a double image clearly visible. All difficulty is overcome by using instead the smallest visible quantity of shellac varnish, and applying heat to a certain extent. Of course the light reflected from so small a mirror is not sufficient when the usual paraffin lamp is employed, but, with oxygen at its present price, there is no reason why a small lime light should not be used. I have found that with even a small supply of oxygen the light on the scale is abundant, and there is no difficulty in observing a deviation of $\frac{1}{4}$ mm. The theoretical defining power upon a scale a metre distant of a mirror $2\frac{3}{4}$ mm. in diameter is about ·23 mm.

The image of the cross wire given by a mirror of this size that was used in the radio-micrometer shown to the Royal Society was a sharp line which could be read with an accuracy of $\frac{1}{10}$ mm.

As the little mirror is plane, I have cemented a plano-convex lens of a convenient focus in the place of the usual plane glass window which must be used to protect the moving parts from currents of air. This is preferable to a double convex lens, because the flat surface is more convenient for cementing, but especially because this surface by reflection also throws an image on the scale which is invariable in position, and which may be used as a reference mark if the scale is moved.

As the definition of the mirror is still so good that the power of reading a deflection in the ordinary way is not materially reduced, no change will be practically necessary in the series of equations 14–18, which are only strictly applicable when the defining power is not affected by change of K.

The junction is the only part of the suspended portion of the instrument which now remains arbitrary. I have provisionally assumed, for the sake of arithmetical results, a pair of bars of antimony and bismuth $5 \times 1 \times \frac{1}{4}$ mm. fixed parallel to one another at a mean distance apart of 1 mm. A less mean distance is impracticable, though it would be an advantage; but the length and sectional area may be modified if found necessary.

Before considering the effect of varying the proportion of the antimony-bismuth bars, it will be convenient at this point to find numerically the value of the combination (see fig. 6) which has thus been developed.

They are as follows :—

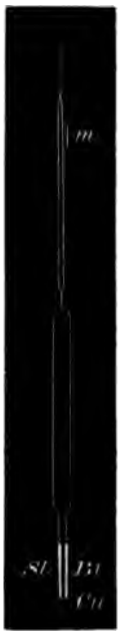
$$\left.\begin{aligned} a &= \frac{1}{b}\sqrt{\frac{\mathrm{K}v}{u\mathrm{C}}} = \cdot 000387 \text{ sq. cm.} \\ l &= \frac{1}{2b}\sqrt{\frac{\mathrm{KC}}{uv}} = 3\cdot 970 \text{ cm.} \end{aligned}\right\} \text{if } b = 1 \text{ mm.,}$$

$$\mathrm{E}_k = \tfrac{1}{8}\frac{1}{\sqrt{\mathrm{KC}uv}} = 4\cdot 284 \times 10^{-5},$$

$$\mathrm{H}'' = 8\sqrt{\frac{\pi}{\mathrm{T}}}\sqrt{uv} = 271\cdot 8.$$

Now, assuming that the working field is four times that found, then the product bl must be made one-seventh of that stated above, and the product $E_t \times H''$, which represents the available sensibility, must be made 1¾ times as great. It is satisfactory to find that the greatest efficacy, and therefore sensibility, is with the narrow circuit about six times as great as that found for the wide circuit first considered.

Fig. 6.



There is no reason why the exact dimensions assigned to the active bars should be employed; it will be well, therefore, to consider what will be the effect of using bars of other dimensions.

Let the sectional area be supposed increased in the ratio $1 : n$; then K will become nK, and C will become C/n; therefore, the greatest efficacy which depends on the product of these will be unchanged, but this assumes a constant difference of temperature between the ends of the bars. Now, in the case of the increased sectional area, the radiation of heat upon the warm junction will be unchanged, while for a given temperature difference the flow of heat to the cool junction, both on account of ordinary heat conduction and the PELTIER action of the current, will be increased, and, thus, the warm junction will not become so warm; thus, the actual sensibility will be less. The loss of heat by radiation from the bars can only be less in consequence of the warm junction being cooler, and, thus, the conclusion remains true, that there is no limit, except that imposed by the difficulty of working the materials, to the smallness of the sectional area that should be used. I may say here that I have found no great difficulty in making the bars as little as ⅛ mm. thick, and in making perfect soldered joints where the weight of solder used does not exceed the fifth part of a milligram.

But to do this the ordinary methods must be discarded, and special means and special tools devised, when the difficulty becomes greatly reduced. Though full instructions would be interesting to the few who would ever probably care to make circuits of this extreme fineness, I do not think a detailed account of the manipulation would be suitable for insertion in this paper. I may, however, mention two causes, ordinarily of no great moment, which become, under the peculiar circumstances, of the first importance. One is the apparently instantaneous conduction of heat, and the other the surface tension of melted solder, which, unless provided against, will produce troublesome and unexpected results. There is no need to do more than mention the fusibility of the bismuth in the presence of the melted solder.

If the length of the bars be supposed increased in the ratio of $1 : n$, K will become nK, and C will become nC; thus, since $E \propto 1/\sqrt{CK}$, E will become n times as small. But here again less heat will reach the cool junction with the larger bars, both by conduction and by the PELTIER action of the current. Thus, the warm junction will become warmer, and the cool junction colder: now, should the temperature difference become also n times as great, the actual sensibility of the circuit would remain unchanged. If no heat were radiated from the bars, then the temperature difference would be proportional to n, and the actual sensibility independent of n; but, on account of the radiation which must occur, the temperature difference would vary in a less ratio than $1 : n$, and therefore the bars could not be too short until the cold junction became sufficiently near the hot junction for it to be impossible to prevent the radiant heat from falling on it also.

On the other hand, on account of the increased flow of heat with the shorter bars, the cool junction would be made warmer, and the whole junction would therefore become warmer, and so there would be an increased loss by radiation from the warm junction. On this account the temperature difference would be less.

I may mention here that, in the narrow pattern of instrument, I have found it advantageous to make a special heat-receiving surface of the thinnest copper, of the size and shape suited to the purpose for which the instrument is made, and to keep the whole of the bars screened from the radiant heat altogether. This an adaptation to the radio-micrometer of the copper-faced thermopile, which Lord ROSSE has found, and which is, obviously, so far preferable to the ordinary construction.

Though it is impossible to find by calculation the exact relative value of the three sources of equalisation of temperature in the circuit—namely, conduction of heat, PELTIER effect, and radiation—it will be some guide to find, as far as data will allow, what their values are. It is most convenient to express them all by giving the time that would elapse before all the heat which is transferred to the cold junction by either of the first two actions, or which escapes in consequence of the third, would be sufficient to raise the pair of bars to the temperature of the warm junction, or, in the case of radiation, to raise it about half as much.

It is easy to show that this time is for the PELTIER effect equal to $JRs/t\theta$, where

J is JOULE'S equivalent ($4·2 \times 10^7$),
$R = 2C$ is the resistance of the whole circuit ($67·42 \times ·10^6$),
s is the heat capacity of the two bars (·000798),
t is the mean absolute temperature (taken as 290),
θ the thermo-electric power (taken as 10,000).

The time of equalisation at a supposed constant rate is on this account 77·9 seconds.

This is true of the circuit of greatest efficacy; if the circuit of reduced size is employed, that is, one with a length one-N^th^ of this, then the total resistance R and the time of equalisation will be $(N + 1)/2N$ times as great. This can never be less than $\frac{1}{2}$.

The corresponding time for the equalisation by conduction may be taken as equal to sl/Da, where

l is the length of the bars,
a their sectional area (separately),
D the sum of the conductivities of antimony and bismuth, *i.e.*, ·0607.

The value of this time is 2·63 seconds.

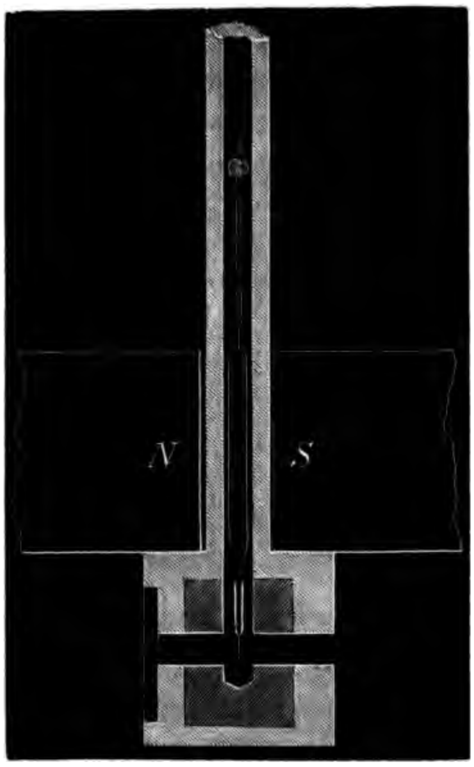
Thus, conduction appears to be far more important than the PELTIER effect, which, practically, may be left out of account. It can only become comparable when so much heat is lost by radiation that the rate of conduction at the cool end is far less than at the hot end, but in this case they would neither be of any practical importance in comparison with the radiation.

The data for finding the time for the escape of half the quantity of heat by radiation, contact of air, &c., are of doubtful value, on account of the very small size of the bars. But taking Professor TAIT'S figure, given in LUPTON'S Tables, for a black surface, the time would be about 29 seconds. Though no great value must be attached to this figure, it would appear that conduction is the main cause of the equalisation of the temperature of the circuit.

The PELTIER effect is involved in another manner in the action of the instrument. It must make a difference in the value of the least magnetic field which is necessary for the dead beat conditions. Thus, during motion of the circuit, currents are induced which oppose the motion; but these currents set up differences of temperature, which oppose the currents. Therefore, a stronger field may be employed before the dead beat conditions are reached. It is hardly necessary to do more than state that, as the result of calculation, no appreciable change is made in the value of the dead beat magnetic field on this account.

I must here mention the only difficulty which is apt to be found in practice. It arises from the magnetic properties of many materials which, insignificant though they are under ordinary methods of observation, become of serious importance when the extremely feeble force due to the torsion of the fibre is taken into account. I have found it absolutely necessary to sink the antimony and bismuth into a little well made by drilling a hole, no larger than necessary, in a piece of soft iron which is buried in the brasswork. A small lateral hole allows the radiation to fall on the heat-receiving surface. This effectually screens off the part which produces the greatest disturbance (see fig. 7, in which the iron is represented by the darker shading). The copper must be exposed to the magnetism; therefore, it must be carefully examined to see that it is neutral, and it must then be kept away from emery or magnetic cleaning materials.

Fig. 7.



It is finally necessary to show that advantage is gained by employing the antimony-bismuth-copper combination, instead of the plain pair of wires used by M. d'Arsonval. While antimony-bismuth wires would, on account of their great thermo-electric power, be superior to palladium-silver, they would, on account of their magnetic qualities, disturb the natural period of the circuit. By the combination which I have employed, I am able to make use of this great thermo-electric power at the same time that the magnetic disturbance is avoided.

Some of the conclusions enunciated in the preliminary note require modification in view of the more extended investigation described in this paper. In a note, added March 23rd, I had concluded that more than one junction would be advantageous, but this is not the case.

But the point that requires special correction is the estimate formed of the greatest possible sensibility. This was calculated on the assumption that an instrument could be used practically when a particular circuit had a natural vibration period of 20 seconds, and was suspended in a field of 10,000 units, produced by an electro-magnet. Without entering on the question whether an electro-magnet could be used, it is sufficient to say that the resistance to the motion would be so great, the instrument would be so much more than just dead beat, that the time of coming to rest would be enormously prolonged. Thus, though the figure given is correct, the conditions to which it refers would not practically be advantageous.

It is, then, with some satisfaction that I turn to the result given by an instrument of the narrow form, having the best proportions. I have now taken quantities which can not only be separately obtained, but which can be used together, and which I have actually used with success.

Under these conditions, the least difference of temperature that could be observed with certainty, that is, one giving a movement of the light of $\frac{1}{5}$ mm. on the scale, would be due to a temperature difference of less than one two-millionth of a degree Centigrade. This figure is obtained by putting in the values of the quantities in the formula for temperature, which may be expressed in a variety of ways. The following are convenient :—

$$\text{Temperature difference} = \frac{128\pi^2\alpha'\sqrt{\text{KC}uv}}{3\text{T}^2\text{H}\theta},$$

or, if the dead beat magnetic field for such a circuit be employed as well,

$$\text{Temperature difference} = \tfrac{16}{3}\frac{\alpha'}{\theta}\frac{\pi}{\text{T}}\sqrt{\text{KC}}.$$

α' is the least observable angle of deflection (supposed $\frac{1}{10,000}$),

θ, the thermo-electric power (supposed 10,000),

T, the natural vibration period (supposed 10 seconds),

K, C, u, and v as before.

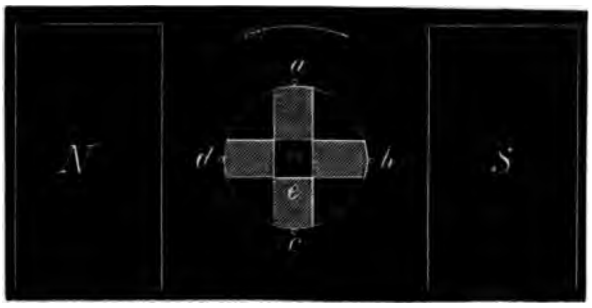
The temperature difference $8{\cdot}06 \times 10^{-7}$, found from the equation above, must be multiplied by $\frac{4}{7}$ to give the corresponding figures for the circuit of reduced length.

Tests made with an instrument of the narrow form, in which it was evident that the magnet still acted to a slight extent on the materials of the suspended portion, show it to be in practice exceedingly sensitive, and, what is of even more importance, the equilibrium of the moving parts remains perfectly stable. The surface which receives the radiant heat is in a particular case a disc only 2 mm. in diameter, and when the scale is 30 inches from the mirror, the hand held about a yard from the instrument produces at once a deflection of 16 cm. A candle flame at 9 feet produced a deviation of 45 mm. every time a small shutter close to the candle was pulled on one side with a piece of cotton, nothing else being allowed to move. Making a strict comparison between this result and those referred to in the preliminary note, this would give 1530 feet as the distance to which a halfpenny might be taken from a candle flame before the heat which it would receive would, if concentrated on the sensitive surface, be too small to produce a deflection of $\frac{1}{4}$ mm. This figure is considerably in excess of that obtained before, even though the fibre is 10 instead of 38 cm. long, the scale is only 30 inches from the screen, and a deflection of $\frac{1}{4}$ instead of $\frac{1}{10}$ mm. is here assumed as the least that could be observed with certainty.

With regard to the rotating pile described at the end of the preliminary note, I ought to say that something very similar is mentioned in NOAD's 'Electricity and Magnetism,' but I was not aware of this at the time of publication. There is, however, a curious difference, which is worth pointing out. So far as I have been able to learn, the wire frames described in NOAD's book only rotate one way when on one pole, and the other way when on the other pole of a magnet. Now, my arrangement will not rotate at all when placed over a pole; it will only rotate when between two poles, and then it will go either way when the heat is applied on one side, but will be prevented from moving when the heat is applied on the other side. Though the matter is of little importance, I may perhaps explain the reason for the peculiar behaviour of my arrangement.

The cross seen in fig. 8 is made with bismuth arms and an antimony centre. At the ends of the arms four copper wires, *a*, *b*, *c*, *d*, are soldered at right angles to the plane of the cross, and lower down to a ring of copper. The whole is balanced on a point between the poles of a magnet, and is free to turn.

Fig. 8.



If heat is applied to the point *e*, an up current in the wire is produced at *c*, and a down current at *a*. Hence, the position of the cross is one of unstable equilibrium. Whichever way the cross begins to move, it will be kept moving in this direction. Suppose it to start in the direction of the arrow, that side of the antimony centre which faces the north pole will become the hottest, though it is gaining heat most rapidly when passing *e*; hence, the up current at *d* and the down current at *b* will each be urged on in the same direction. If, however, the cross started in the other direction, the side nearest the south pole would become the hottest, the current would pass the other way, and the reversed motion would still be kept up. Thus, whichever way it moves, it is kept moving in the same direction. If, however, the heat is applied on the opposite side, the first current produced is in a position of stable equilibrium; hence, the circuit shows no disposition to move unless there is want of symmetry, when it moves 45°, and then the equilibrium is absolutely stable. Whichever way the circuit is now made to turn, the direction of the currents will be such as stop the motion, for the same reason that in the previous case they maintained it.

In conclusion, the principal advantages in the instrument the development of which is described in this paper are:—

Extreme quickness and sensibility.
Freedom from extraneous thermal and magnetic influence.
The sensibility can be varied at will.
The instrument may be made dead beat, or its logarithmic decrement may be varied at will.
By the use of the quartz fibre, difficulties caused by the uncertain behaviour of silk under varying conditions of temperature and moisture—difficulties that would be far greater than in the case of a galvanometer—are completely obviated.

On the other hand, a disadvantage inherent in the instrument is that it must, like a galvanometer, be fixed in position; it is inferior to the thermopile or bolometer in the ease with which they can be pointed in any desired direction.

VI. *The Waves on a Rotating Liquid Spheroid of Finite Ellipticity.*

By G. H. BRYAN, *B.A.*

Communicated by Professor G. H. DARWIN, *F.R.S.*

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1. THE hydrodynamical problem of finding the waves or oscillations on a gravitating mass of liquid which, when undisturbed, is rotating as if rigid with finite angular velocity, in the form of an ellipsoid or spheroid, was first successfully attacked by M. POINCARÉ in 1885. In his important memoir, "Sur l'Équilibre d'une Masse Fluide animée d'un Mouvement de Rotation,"* POINCARÉ has (§ 13) obtained the differential equations for the oscillations of rotating liquid, and shown that, by a transformation of projection, the determination of the oscillations of any particular period is reducible to finding a suitable solution of LAPLACE'S equation. He then applies LAMÉ'S functions to the case of the ellipsoid, showing that the differential equations are satisfied by a series of LAMÉ'S functions referred to a certain auxiliary ellipsoid, the boundary-conditions, however, involving ellipsoidal harmonics, referred to both the auxiliary and actual fluid ellipsoid. At the same time, POINCARÉ'S analysis does not appear to admit of any definite conclusions being formed as to the nature and frequencies of the various periodic free waves.

The present paper contains an application of POINCARÉ'S methods to the simpler case when the fluid ellipsoid is one of revolution (MACLAURIN'S spheroid). The solution is effected by the use of the ordinary tesseral or zonal harmonics applicable to the fluid spheroid and to the auxiliary spheroid required in solving the differential equation. The problem is thus freed from the difficulties attending the use of LAMÉ'S functions, and is further simplified by the fact that each independent solution contains harmonics of only one particular degree and rank.

By substituting in the conditions to be satisfied at the surface of the spheroid we arrive at a single boundary-equation. If we are treating the forced tides due to a known periodic disturbing force, this equation determines their amplitude and, hence, the elevation of the tide above the mean surface of the spheroid at any point at any time. If there be no disturbing force, it determines the frequencies of the various free waves determined by harmonics of given order and rank. Denoting by κ the ratio of the frequency of the free waves to twice the frequency of rotation of the liquid about its axis, the values of κ are the roots of a rational algebraic equation, and depend only on the eccentricity of the spheroid, as well as the degree and rank of the harmonic, while the number of different free waves depends on the degree of the equation in κ. At any instant the height of the disturbance at any point of the surface is proportional to the corresponding surface harmonic on the spheroid, multiplied by the central perpendicular on the tangent plane, and is of the same form for all waves determined by harmonics of any given degree and rank, whatever be their frequency; but the motions of the fluid particles in the interior will differ in nature in every case.

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* 'Acta Mathematica,' vol. 7.

Taking first the case of zonal harmonics of the n^{th} degree, we find that, according as n is even or odd, there will be $\frac{1}{2}n$ or $\frac{1}{2}(n+1)$ different periodic motions of the liquid. These are essentially oscillatory in character and symmetrical about the axis of the spheroid. In all but one of these the value of κ is essentially less than unity, that is, the period is greater than the time of a semi-revolution of the liquid.

Taking next the tesseral harmonics of degree n and rank s, we find that they determine $n - s + 2$ periodic small motions. These are essentially tidal waves rotating with various angular velocities about the axis of the spheroid, the angular velocities of those rotating in opposite directions being in general different. All but two of the values of κ are numerically less than unity, the periods of the corresponding tides at a point fixed relatively to the liquid being greater than the time of a semi-revolution of the mass. The mean angular velocity of these $n - s + 2$ waves is less than that of rotation of the mass by $2/\{s(n - s + 2)\}$ of the latter.

In the two waves determined by any sectorial harmonic, the relative motion of the liquid particles is irrotational. The harmonics of degree 2 and rank 1 give rise to a kind of precession, of which there are two.

I have calculated the relative frequencies of several of the principal waves on a spheroid whose eccentricity is $\frac{1}{2}\sqrt{2}$.

The question of stability is next dealt with, it being shown that in the present problem, in which the liquid forming the spheroid is supposed perfect, the criteria are entirely different from the conditions of secular stability obtained by POINCARÉ for the case when the liquid possesses any amount of viscosity, and which latter depend on the energy being a minimum. In fact, for a disturbance initially determined by any harmonic (provided that it is symmetrical with respect to the equatorial plane, since for unsymmetrical displacements the spheroid cannot be unstable), the limits of eccentricity consistent with stability are wider for a perfect liquid spheroid than for one possessing any viscosity. If we assume that the disturbed surface initially becomes ellipsoidal, the conditions of stability found by the methods of this paper agree with those of RIEMANN.

The case when the ellipticity and, therefore, the angular velocity are very small is next discussed, it being shown that all but two of the waves, or all but one of the oscillations for any particular harmonic, become unimportant, their periods increasing indefinitely. In the case of those whose periods remain finite for a non-rotating spherical mass, the effect of a small angular velocity ω of the liquid is to cause them to turn round the axis with a velocity less than that of the liquid by ω/n.

Finally, the methods of treating forced tides are further discussed. The general cases of a "semi-diurnal" forced tide, or of permanent deformations due to constant disturbing forces, are mentioned in connection with some peculiarities they present; and these are followed by examples of the determination of the forced tides due to the presence of an attracting mass, first, when the latter moves in any orbit about the spheroid, secondly, when it rotates uniformly about the spheroid in its equatorial plane. The effects of such a body in destroying the equilibrium of the spheroid where the forced tide coincides with one of the free tides form the conclusion of this paper.

POINCARÉ'S *Differential Equations for Waves or Oscillations of Rotating Liquid.*

2. Suppose a mass of gravitating liquid is in relative equilibrium when rotating as if rigid about a fixed axis with angular velocity ω, and that it is required to determine the waves or small oscillations due to a slight disturbance of the mass.

Let the motion be referred to a set of orthogonal moving axes, of which the axis of z is the fixed axis of rotation, while the axes of x, y rotate about it with angular velocity ω. In the steady or undisturbed motion the positions of the fluid particles relative to these axes will remain fixed. In the oscillations, let U, V, W be the small component velocities of the fluid at the point (x, y, z) *relative* to the axes. The actual component velocities referred to axes fixed in space and coinciding with our axes of x, y, z, at the time considered, will be $\mathrm{U} - \omega y$, $\mathrm{V} + \omega x$, W, and the equations of hydrodynamics may be written [*]

$$\begin{aligned}
&\frac{\partial \mathrm{U}}{\partial t} - \omega(\mathrm{V} + \omega x) + \mathrm{U}\frac{\partial \mathrm{U}}{\partial x} + \mathrm{V}\left(\frac{\partial \mathrm{U}}{\partial y} - \omega\right) + \mathrm{W}\frac{\partial \mathrm{U}}{\partial z} = \frac{\partial}{\partial x}\left(\mathrm{V}_1 - \frac{p}{\rho}\right),\\
&\frac{\partial \mathrm{V}}{\partial t} + \omega(\mathrm{U} - \omega y) + \mathrm{U}\left(\frac{\partial \mathrm{V}}{\partial x} + \omega\right) + \mathrm{V}\frac{\partial \mathrm{V}}{\partial y} + \mathrm{W}\frac{\partial \mathrm{V}}{\partial z} = \frac{\partial}{\partial y}\left(\mathrm{V}_1 - \frac{p}{\rho}\right),\\
&\frac{\partial \mathrm{W}}{\partial t} + \mathrm{U}\frac{\partial \mathrm{W}}{\partial x} + \mathrm{V}\frac{\partial \mathrm{W}}{\partial y} + \mathrm{W}\frac{\partial \mathrm{W}}{\partial z} = \frac{\partial}{\partial z}\left(\mathrm{V}_1 - \frac{p}{\rho}\right),
\end{aligned}$$

V_1 being the potential due to the attraction of the liquid and any forces which may act on it, p the pressure, and ρ the density.

For small disturbances we may neglect squares and products of the relative velocities U, V, W (as is usual in wave problems), and, therefore, the above equations reduce to

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[*] BASSET, 'Hydrodynamics,' vol. 1, § 23; or GREENHILL, 'Encyclopædia Britannica,' article "Hydromechanics."

$$\left.\begin{aligned} \frac{\partial \mathrm{U}}{\partial t} - 2\omega \mathrm{V} &= \frac{\partial \psi}{\partial x}, \\ \frac{\partial \mathrm{V}}{\partial t} + 2\omega \mathrm{U} &= \frac{\partial \psi}{\partial y}, \\ \frac{\partial \mathrm{W}}{\partial t} \qquad\quad &= \frac{\partial \psi}{\partial z}, \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1),$$

where

$$\psi = \mathrm{V}_1 - \frac{p}{\rho} + \tfrac{1}{2}\omega^2 (x^2 + y^2) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2).$$

We have also the equation of continuity,

$$\frac{\partial \mathrm{U}}{\partial x} + \frac{\partial \mathrm{V}}{\partial y} + \frac{\partial \mathrm{W}}{\partial z} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3).$$

Eliminating U, V, W from equations (1), (3), we obtain the differential equation

$$\frac{\partial^2}{\partial t^2} \nabla^2 \psi + 4\omega^2 \frac{\partial^2 \psi}{\partial z^2} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4),$$

where, as usual, ∇^2 stands for LAPLACE's operator $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

3. Let us now consider separately the simple harmonic oscillations of one particular period. Assume that U, V, W, and ψ all vary as $e^{2\iota\omega\kappa t}$, so that the ratio of the period of oscillation to the time of a complete revolution of the liquid mass about its axis is $1/2\kappa$. The equations (1), (4) reduce to

$$\left.\begin{aligned} 2\omega(\iota\kappa \mathrm{U} - \mathrm{V}) &= \frac{\partial \psi}{\partial x}, \\ 2\omega(\iota\kappa \mathrm{V} + \mathrm{U}) &= \frac{\partial \psi}{\partial y}, \\ 2\omega\iota\kappa \mathrm{W} \qquad &= \frac{\partial \psi}{\partial z}, \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5),$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \left(1 - \frac{1}{\kappa^2}\right) \frac{\partial^2 \psi}{\partial z^2} = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (6).$$

Put

$$1 - \frac{1}{\kappa^2} = \tau^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7),$$

and

$$z = \tau z' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8).$$

Equation (6) now becomes

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z'^2} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9).$$

If κ be greater than unity, τ and, therefore, also z' will be real. We may take (x, y, z') to be the coordinates of a point corresponding to the point (x, y, z) of the liquid. We thus obtain a new region of points derivable from the original region by homogeneous strain parallel to z or by projection. This region may be called the auxiliary region, and the surface formed by points corresponding to points on the fluid surface, the auxiliary surface. Our problem thus reduces to that of finding a suitable value of ψ satisfying LAPLACE'S equation (9) within the space bounded by the auxiliary surface.

But we must revert to the original system in order to satisfy the boundary-conditions, which must hold at the actual surface of the liquid, not at the auxiliary surface. If the surface of the liquid be free, p must be constant over it, and, therefore, the condition to be satisfied all over the *disturbed* surface of the liquid is

$$\psi = V_1 + \tfrac{1}{2}\omega^2(x^2 + y^2) + \text{const.} \quad . \; . \; . \; . \; . \; . \; . \quad (10).$$

In forming the expression for V_1 we must remember that the gravitation potential is due to the disturbed configuration of the liquid mass.

If κ be less than unity, τ will be imaginary, and, therefore, the auxiliary surface will also be imaginary. But the results arrived at by this method in the case where τ is real will still hold good even if τ be imaginary, provided that the expression obtained for ψ is a real function of the coordinates x, y, z. The method breaks down if $\kappa = \pm 1$, when τ vanishes; this must be treated as a limiting case.

Solution for the Spheroid by Spheroidal Harmonics.

4. Let the liquid be in the form of a MACLAURIN'S spheroid the equation of whose surface is

$$\frac{x^2 + y^2}{c^2(\zeta_0^2 + 1)} + \frac{z^2}{c^2\zeta_0^2} \equiv \frac{x^2 + y^2}{c^2 \operatorname{cosec}^2 \alpha} + \frac{z^2}{c^2 \cot^2 \alpha} = 1 \quad . \; . \; . \; . \; . \; . \quad (11),$$

so that

$$\zeta_0 = \cot \alpha \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12),$$

and $\sin \alpha$ is the eccentricity of the spheroid, c being the radius of its focal circle.

The locus of the corresponding point (x, y, z') is the auxiliary quadric

$$\frac{x^2 + y^2}{c^2 \operatorname{cosec}^2 \alpha} + \frac{\tau^2 z'^2}{c^2 \cot^2 \alpha} = 1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13).$$

This quadric will be a prolate spheroid if τ^2 lies between zero and $\cos^2 \alpha$, that is, if κ^2 lies between unity and $\operatorname{cosec}^2 \alpha$. If τ^2 is greater than $\cos^2 \alpha$, or κ^2 greater than $\operatorname{cosec}^2 \alpha$, the spheroid will be oblate. If τ^2 be negative, or κ^2 less than unity, equation (13) represents a hyperboloid of one sheet, but the part corresponding to the liquid surface is the imaginary portion for which $x^2 + y^2$ is less than $c^2 \operatorname{cosec}^2 \alpha$, and z'^2 is negative; this is the imaginary auxiliary spheroid.

We shall take as our standard case that in which equation (13) represents a prolate auxiliary spheroid. Let it be written in the form

$$\frac{x^2 + y^2}{k^2 (\nu_0^2 - 1)} + \frac{z'^2}{k^2 \nu_0^2} = 1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (14),$$

so that

$$k^2 (\nu_0^2 - 1) = c^2 \operatorname{cosec}^2 \alpha,$$

$$k^2 \nu_0^2 = \frac{c^2 \cot^2 \alpha}{\tau^2} = \frac{c^2 \cot^2 \alpha \, . \, \kappa^2}{\kappa^2 - 1}.$$

Solving for ν_0, k, we find

$$\nu_0 = \frac{\kappa \cos \alpha}{\sqrt{(1 - \kappa^2 \sin^2 \alpha)}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (15),$$

$$k^2 = c^2 \frac{\operatorname{cosec}^2 \alpha - \kappa^2}{\kappa^2 - 1} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (16).$$

The solution of the differential equation (9) must be effected by means of spheroidal harmonics applicable to the auxiliary spheroid (14), whilst the expressions for the gravitation potential of the liquid mass and the boundary-conditions will involve spheroidal harmonics referred to the actual liquid spheroid (11). We must, therefore, use two different sets of orthogonal elliptic coordinates for the auxiliary and the actual systems. Let these coordinates be denoted by (μ', ν, ϕ) (μ, ζ, ϕ) respectively, and let them be connected with the rectangular coordinates in the two systems by the relations

$$\left.\begin{aligned}
x &= k \sqrt{(\nu^2 - 1)} \sqrt{(1 - \mu'^2)} \cos \phi = c \sqrt{(\zeta^2 + 1)} \sqrt{(1 - \mu^2)} \cos \phi \\
y &= k \sqrt{(\nu^2 - 1)} \sqrt{(1 - \mu'^2)} \sin \phi = c \sqrt{(\zeta^2 + 1)} \sqrt{(1 - \mu^2)} \sin \phi \\
z' &= k\nu\mu' \qquad\qquad\qquad\qquad\qquad = c\zeta\mu/\tau \\
&\text{and, therefore,} \\
z &= k\nu\mu'\tau \qquad\qquad\qquad\qquad\quad\; = c\zeta\mu
\end{aligned}\right\} \quad . \quad (17).$$

The surfaces of the spheroids will be given by the equations

$$\zeta = \zeta_0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (18),$$

or

$$\nu = \nu_0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (18\text{A});$$

moreover, all over these surfaces, at corresponding points,

$$\mu = \mu' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (19).$$

The angular coordinate ϕ is the same in both systems; but, except over the surfaces, μ will not be equal to μ', nor will any other two of the surfaces $\nu =$ constant and $\zeta =$ constant coincide.

Put $\mu' = \cos\theta$. On transforming to (θ, ν, ϕ), equation (9) becomes

$$\frac{\partial}{\partial\nu}\left\{(\nu^2 - 1)\frac{\partial\psi}{\partial\nu}\right\} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\nu^2 - \cos^2\theta}{(\nu^2 - 1)\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = 0 \quad . \quad . \quad (20),$$

of which a solution, finite and continuous at all points within the spheroid (14), is

$$\psi = A_n{}^s T_n{}^{(s)}(\mu') \, T_n{}^s(\nu) \, e^{\iota s\phi} e^{2\iota\omega\kappa t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21),$$

where $A_n{}^s$ is any constant, and

$$T_n{}^{(s)}(\mu') = (1 - \mu'^2)^{s/2}\left(\frac{d}{d\mu'}\right)^s P_n(\mu') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22),$$

$$T_n{}^s(\nu) = (\nu^2 - 1)^{s/2}\left(\frac{d}{d\nu}\right)^s P_n(\nu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23),$$

P_n denoting the zonal harmonic of degree n.

In our standard case ν is real and greater than unity, and in every case μ' lies between the limits $+ 1$ and $- 1$, and is real. I have adopted the above notation (according to which the functions $T_n{}^{(s)}$, $T_n{}^s$ differ in form by the constant factor $(- 1)^{s/2}$) in order to avoid introducing imaginary coefficients unnecessarily.*

It is easy to see that the solution (21) is applicable in every case. For, if τ^2 be greater than $\cos^2\alpha$, both k and ν are purely imaginary; whilst, if τ^2 be negative, we may show that k will be imaginary, but ν will be real and less than unity. In any case $T_n{}^s(\nu)$ will be either real or purely imaginary, so that $A_n{}^s T_n{}^{(s)}(\mu) \, T_n{}^s(\nu)$ can be always made a *real* function of the coordinates (x, y, z). Moreover, the right-hand side of (21) is finite, single valued, and continuous throughout the liquid spheroid, and satisfies the differential equation (6). It therefore only remains to investigate the boundary-conditions which must be satisfied by ψ at the surface of the liquid.

5. The spheroidal harmonics referred to the liquid spheroid, required for these boundary-conditions, will be formed as follows :—

Let

$$p_n(\zeta) = \frac{1}{2^n n!}\left(\frac{d}{d\zeta}\right)^n (\zeta^2 + 1)^n \equiv (- 1)^{n/2} P_n(\iota\zeta) \quad . \quad . \quad . \quad (24),$$

$$t_n{}^s(\zeta) = (\zeta^2 + 1)^{s/2}\left(\frac{d}{d\zeta}\right)^s p_n(\zeta) \equiv (- 1)^{n/2} T_n{}^s(\iota\zeta) \quad . \quad . \quad . \quad (25),$$

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* The tesseral harmonics may be replaced by the associated functions of the first kind of HEINE without any change in the formulæ, the constant coefficients being supposed included in $A_n{}^s$.

and let

$$q_n(\zeta) = p_n(\zeta) \int_\zeta^\infty \frac{d\zeta}{(\zeta^2 + 1)\{p_n(\zeta)\}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26),$$

$$u_n^s(\zeta) = t_n^s(\zeta) \int_\zeta^\infty \frac{d\zeta}{(\zeta^2 + 1)\{t_n^s(\zeta)\}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

Then the expressions

$$v_1 = \mathrm{B}_n^s e^{2\iota\omega\kappa t} e^{\iota s\phi} \mathrm{T}_n^{(s)}(\mu)\, t_n^s(\zeta)/t_n^s(\zeta_0) \quad . \quad . \quad . \quad . \quad . \quad (28),$$

$$v_0 = \mathrm{B}_n^s e^{2\iota\omega\kappa t} e^{\iota s\phi} \mathrm{T}_n^{(s)}(\mu)\, u_n^s(\zeta)/u_n^s(\zeta_0) \quad . \quad . \quad . \quad . \quad . \quad (29),$$

are solutions of LAPLACE's equation which are finite and continuous, the former throughout the interior of the spheroid ($\zeta = \zeta_0$), the latter throughout all space outside the spheroid, and vanishing at infinity; while at the surface ($\zeta = \zeta_0$) both expressions become equal to

$$[v] = \mathrm{B}_n^s e^{2\iota\omega\kappa t} e^{\iota s\phi} \mathrm{T}_n^{(s)}(\mu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30).$$

The Elevation of the Waves on the Surface.

6. Let h be the normal displacement at any point of the liquid surface, *i.e.*, the height of the wave above the level of the undisturbed spheroid.

Let ϖ be the central perpendicular on the tangent plane, and $d\mathrm{N}$ an element of the outward drawn normal to the surface of the *liquid* spheroid (11). We readily find

$$\frac{\partial x}{\partial \zeta} = \frac{\zeta x}{\zeta^2 + 1}, \quad \frac{\partial y}{\partial \zeta} = \frac{\zeta y}{\zeta^2 + 1}, \quad \frac{\partial z}{\partial \zeta} = \frac{\zeta z}{\zeta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31),$$

and at the surface, since the element $d\mathrm{N}$ is a tangent to the curve $\mu = \text{const.}$, $\phi = \text{const.}$; therefore,

$$\left(\frac{d\mathrm{N}}{d\zeta}\right)^2 = \left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2;$$

whence,

$$\frac{d\mathrm{N}}{d\zeta} = c\sqrt{\left(\frac{\zeta_0^2 + \mu^2}{\zeta_0^2 + 1}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32),$$

the differentials in (32) being *total*

Also

$$\varpi = c\zeta_0 \sqrt{\left(\frac{\zeta_0^2 + 1}{\zeta_0^2 + \mu^2}\right)},$$

so that

$$\varpi\, d\mathrm{N} = c^2 \zeta_0\, d\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33).$$

From equations (5) we obtain

$$\left.\begin{aligned} 2\omega(1-\kappa^2)\,\mathrm{U} &= \iota\kappa\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}\\ 2\omega(1-\kappa^2)\,\mathrm{V} &= \iota\kappa\frac{\partial\psi}{\partial y}-\frac{\partial\psi}{\partial x}\\ 2\omega(1-\kappa^2)\,\mathrm{W} &= \iota\kappa\tau^2\frac{\partial\psi}{\partial z} \end{aligned}\right\} \quad \ldots\ldots \quad (34).$$

Multiplying by $\partial x/\partial\mathrm{N}$, $\partial y/\partial\mathrm{N}$, $\partial z/\partial\mathrm{N}$, and adding, we find at the surface

$$2\omega(1-\kappa^2)\left(\mathrm{U}\frac{\partial x}{\partial\mathrm{N}}+\mathrm{V}\frac{\partial y}{\partial\mathrm{N}}+\mathrm{W}\frac{\partial z}{\partial\mathrm{N}}\right)$$
$$=\iota\kappa\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\mathrm{N}}+\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\mathrm{N}}+\tau^2\frac{\partial\psi}{\partial z}\frac{\partial x}{\partial\mathrm{N}}\right)+\left(\frac{\partial\psi}{\partial y}\frac{\partial x}{\partial\mathrm{N}}-\frac{\partial\psi}{\partial x}\frac{\partial y}{\partial\mathrm{N}}\right).$$

But $\mathrm{U}\partial x/\partial\mathrm{N}+\mathrm{V}\partial y/\partial\mathrm{N}+\mathrm{W}\partial z/\partial\mathrm{N}$ is the normal velocity of the liquid relative to the moving axes, and is therefore equal to $\partial h/\partial t$ or to $2\iota\omega\kappa h$. We have, therefore,

$$4\omega^2(1-\kappa^2)\kappa h$$
$$=\kappa\frac{d\zeta}{d\mathrm{N}}\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\zeta}+\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\zeta}+\tau^2\frac{\partial\psi}{\partial z}\frac{\partial z}{\partial\zeta}\right)-\iota\frac{\partial\zeta}{\partial\mathrm{N}}\left(\frac{\partial\psi}{\partial y}\frac{\partial x}{\partial\zeta}-\frac{\partial\psi}{\partial x}\frac{\partial y}{\partial\zeta}\right)$$
$$=\kappa c^2\zeta_0\frac{d\zeta}{d\mathrm{N}}\left(\frac{x}{c^2(\zeta_0^2+1)}\frac{\partial\psi}{\partial x}+\frac{y}{c^2(\zeta_0^2+1)}\frac{\partial\psi}{\partial y}+\frac{\tau^2 z}{c^2\zeta_0^2}\frac{\partial\psi}{\partial z}\right)-\frac{\iota\zeta_0}{\zeta_0^2+1}\frac{d\zeta}{d\mathrm{N}}\left(x\frac{\partial\psi}{\partial y}-y\frac{\partial\psi}{\partial x}\right)$$
$$=\kappa c^2\zeta_0\frac{d\zeta}{d\mathrm{N}}\left(\frac{x}{k^2(\nu_0^2-1)}\frac{\partial\psi}{\partial x}+\frac{y}{k^2(\nu_0^2-1)}\frac{\partial\psi}{\partial y}+\frac{z'}{k^2\nu_0^2}\frac{\partial\psi}{\partial z'}\right)-\frac{\iota\zeta_0}{\zeta_0^2+1}\frac{d\zeta}{d\mathrm{N}}\frac{\partial\psi}{\partial\phi}$$
$$=\zeta_0\frac{d\zeta}{d\mathrm{N}}\left\{\frac{\kappa c^2}{\nu_0 k^2}\frac{\partial\psi}{\partial\nu}-\frac{\iota}{\zeta_0^2+1}\frac{\partial\psi}{\partial\phi}\right\} \quad \ldots\ldots\ldots\ldots \quad (35).$$

Now, taking ψ as given by (21), we have

$$\frac{d\psi}{d\phi}=\iota s\psi \quad \ldots\ldots\ldots \quad (36);$$

moreover, since by (23)

$$\mathrm{T}_n^s(\nu)=(\nu^2-1)^{s/2}\,\mathrm{D}^s\mathrm{P}_n(\nu),$$

where the symbol D stands for differentiation with respect to ν, therefore

$$\mathrm{DT}_n^s(\nu)=(\nu^2-1)^{s/2}\{\mathrm{D}^{s+1}\mathrm{P}_n(\nu)+s\nu/(\nu^2-1)\,.\,\mathrm{D}^s\mathrm{P}_n(\nu)\} \quad . \quad . \quad (37).$$

Also at the surface μ' is equal to μ.

Hence, we find

$$4\omega^2(1-\kappa^2)\kappa h = \mathrm{A}_n{}^s\zeta_0\frac{d\zeta}{d\mathrm{N}}\left\{\frac{\kappa c^2}{\nu_0{}^2k^2}\nu_0\,\mathrm{D}^{s+1}\mathrm{P}_n(\nu_0)+\frac{s\kappa c^2}{k^2(\nu_0{}^2-1)}\mathrm{D}^s\mathrm{P}_n(\nu_0)+\frac{s}{\zeta_0{}^2+1}\mathrm{D}^s\mathrm{P}_n(\nu_0)\right\}$$
$$\times(\nu_0{}^2-1)^{s/2}\,\mathrm{T}_n{}^{(s)}(\mu)\,e^{\iota s\phi}e^{2\iota\omega\kappa t}$$
$$=\mathrm{A}_n{}^s\zeta_0\frac{d\zeta}{d\mathrm{N}}\left\{\frac{\kappa^2-1}{\kappa\zeta_0{}^2}\nu_0\mathrm{D}^{s+1}\mathrm{P}_n(\nu_0)+\frac{s(\kappa+1)}{\zeta_0{}^2+1}\mathrm{D}^s\mathrm{P}_n(\nu_0)\right\}$$
$$\times(\nu_0{}^2-1)^{s/2}\,\mathrm{T}_n{}^{(s)}(\mu)\,e^{\iota s\phi}e^{2\iota\omega\kappa t}$$

by (15), (16).

Whence

$$h=\mathrm{C}_n{}^s\varpi\,\mathrm{T}_n{}^{(s)}(\mu)\,e^{\iota s\phi}e^{2\iota\omega\kappa t}\quad\dots\dots\dots\quad(38),$$

where

$$\mathrm{C}_n{}^s=-\mathrm{A}_n{}^s\frac{\tan^2\alpha}{4\omega^2\kappa c^2}\left\{\frac{1}{\kappa}\nu_0\,\mathrm{D}^{s+1}\mathrm{P}_n(\nu_0)+\frac{s\cos^2\alpha}{\kappa-1}\mathrm{D}^s\mathrm{P}_n(\nu_0)\right\}(\nu_0{}^2-1)^{s/2}\quad.\quad(39);$$

moreover, the equation of the disturbed surface of the liquid is

$$\zeta=\zeta_0+\delta\zeta_0\quad\dots\dots\dots\dots\quad(40),$$

where

$$\delta\zeta_0=h\frac{d\zeta}{d\mathrm{N}}=\mathrm{C}_n{}^s\varpi^2\tan\alpha/c^2\,\mathrm{T}_n{}^{(s)}(\mu)\,e^{\iota s\phi}e^{2\iota\omega\kappa t}.\quad\dots\dots\quad(41).$$

The Boundary-Conditions.

7. Let V_0 be the potential of a mass of the liquid filling the spheroid

$$\zeta=\zeta_0\quad\dots\dots\dots\dots\quad(18),$$

and let v be the potential of a distribution of the liquid of thickness everywhere equal to h over the surface of the spheroid. The combination of these two distributions is equivalent to the liquid mass as disturbed by the waves, so that

$$\mathrm{V}_1=\mathrm{V}_0+v\quad\dots\dots\dots\dots\quad(42).$$

For the free waves, the boundary-equation (10) requires that

$$\psi=\mathrm{V}_0+v+\tfrac{1}{2}\omega^2(x^2+y^2)+\text{const.}\quad\dots\dots\quad(43),$$

all over the surface (40).

Now ψ, v, $\delta\zeta_0$ are small quantities of the first order. Hence, expanding by TAYLOR's theorem, we have to first order

$$[\psi] = [\mathrm{V}_0 + \tfrac{1}{2}\omega^2(x^2+y^2)] + [v] + \delta\zeta_0 \frac{d}{d\zeta}\{\mathrm{V}_0 + \tfrac{1}{2}\omega^2(x^2+y^2)\} + \text{const.} \quad (44),$$

where the square brackets indicate that ζ is to be put equal to ζ_0.

In the case of forced tides due partly to small disturbing forces whose potential at any instant is V_2, and partly to periodic variations of pressure p_2 over the surface of the liquid, the condition at the surface becomes

$$[\psi] = [\mathrm{V}_0 + \tfrac{1}{2}\omega^2(x^2+y^2)] + [v] + \delta\zeta_0 \frac{d}{d\zeta}\{\mathrm{V}_0 + \tfrac{1}{2}\omega^2(x^2+y^2)\}$$
$$+ \left[\mathrm{V}_2 - \frac{p_2}{\rho}\right] + \text{const.} \quad . \quad (44^*)$$

Equating to zero the non-periodic terms, we obtain the well-known condition for steady motion

$$[\mathrm{V}_0 + \tfrac{1}{2}\omega^2(x^2+y^2)] + \text{const.} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (45).$$

Here

$$\mathrm{V}_0 = \text{const.} - \tfrac{1}{2}\{\mathrm{A}(x^2+y^2) + \mathrm{C}z^2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (46),$$

where

$$\left.\begin{aligned} \mathrm{A} &= 4\pi\rho\gamma\,\zeta_0(\zeta_0^2+1)\int_{\zeta_0}^{\infty}\frac{d\zeta}{(\zeta^2+1)^2} = 4\pi\rho\gamma\,\operatorname{cosec}^2\alpha\cot\alpha\,\frac{u_1^1(\zeta_0)}{t_1^1(\zeta_0)} \\ \mathrm{C} &= 4\pi\rho\gamma\,\zeta_0(\zeta_0^2+1)\int_{\zeta_0}^{\infty}\frac{d\zeta}{(\zeta^2+1)\zeta^2} = 4\pi\rho\gamma\,\operatorname{cosec}^2\alpha\cot\alpha\,\frac{q_1(\zeta_0)}{p_1(\zeta_0)} \end{aligned}\right\} \quad . \quad (47),$$

γ being the constant of gravitation, and being put equal to unity if the density is expressed in astronomical units.

From (45) we have, in the usual manner,

$$(\mathrm{A}-\omega^2)(x^2+y^2) + \mathrm{C}z^2 = \mathrm{C}\zeta_0^2\{(x^2+y^2)/(\zeta_0^2+1) + z^2/\zeta_0^2\} \quad . \quad . \quad (48),$$

whence

$$\begin{aligned} \omega^2 &= \mathrm{A} - \mathrm{C}\zeta_0^2/(\zeta_0^2+1) \\ &= 4\pi\rho\gamma\,\zeta_0\{t_1^1(\zeta_0)\,u_1^1(\zeta_0) - p_1(\zeta_0)\,q_1(\zeta_0)\} \quad . \quad . \quad . \quad . \quad . \quad (49), \end{aligned}$$

which can also be put in the form

$$\begin{aligned} \omega^2 &= 4\pi\rho\gamma\,\zeta_0\{\tfrac{1}{2}(3\zeta_0^2+1)\cot^{-1}\zeta_0 - \tfrac{3}{2}\zeta_0\} \\ &= 4\pi\rho\gamma\,\zeta_0 q_2(\zeta_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50). \end{aligned}$$

From (44*), (45) the boundary-condition for the oscillations is

$$[\psi] = [v] + \delta\zeta_0\frac{\partial}{\partial\zeta}\{\mathrm{V}_0 + \tfrac{1}{2}\omega^2(x^2+y^2)\} + \left[\mathrm{V}_2 - \frac{p_2}{\rho}\right] \quad . \quad . \quad . \quad (51).$$

Now, by (48)

$$\begin{aligned}\delta\zeta_0 \frac{d}{d\zeta}\{\mathrm{V}_0 + \tfrac{1}{2}\omega^2(x^2+y^2)\} &= -\tfrac{1}{2}\mathrm{C}\zeta_0^{\,2}\,\delta\zeta_0\left\{\frac{\partial x}{\partial\zeta}\frac{\partial}{\partial x}+\frac{\partial y}{\partial\zeta}\frac{\partial}{\partial y}+\frac{\partial z}{\partial\zeta}\frac{\partial}{\partial z}\right\}\left(\frac{x^2+y^2}{\zeta_0^{\,2}+1}+\frac{z^2}{\zeta_0^{\,2}}\right)\\ &= -4\pi\rho\gamma(\zeta_0^{\,2}+1)\,\zeta_0^{\,4}\frac{q_1(\zeta_0)}{p_1(\zeta_0)}\left\{\frac{x^2+y^2}{(\zeta_0^{\,2}+1)^2}+\frac{z^2}{\zeta_0^{\,4}}\right\}\delta\zeta_0\\ &= -4\pi\rho\gamma(\zeta_0^{\,2}+1)\,\zeta_0^{\,2}p_1(\zeta_0)\,q_1(\zeta_0)\,c^4\,\delta\zeta_0/\varpi^2\\ &= -4\pi\rho\gamma\,\mathrm{C}_n^{\,s}c^2\cot\alpha\,\mathrm{cosec}^2\alpha\,.\,p_1(\zeta_0)\,q_1(\zeta_0)\,\mathrm{T}_n^{(s)}(\mu)\,e^{\iota s\phi}e^{2\iota\omega\kappa t}\quad .\quad .\quad (52).\end{aligned}$$

8. To find *v*.—Suppose that the values of this potential inside and outside the spheroid respectively are given by the formulæ

$$v_1 = \mathrm{B}_n^{\,s}e^{2\iota\omega\kappa t}e^{\iota s\phi}\mathrm{T}_n^{(s)}(\mu)\,t_n^{\,s}(\zeta)/t_n^{\,s}(\zeta_0)\quad .\;.\;.\;.\;.\;.\;.\;.\quad (28),$$

$$v_0 = \mathrm{B}_n^{\,s}e^{2\iota\omega\kappa t}e^{\iota s\phi}\mathrm{T}_n^{(s)}(\mu)\,u_n^{\,s}(\zeta)/u_n^{\,s}(\zeta_0)\quad .\;.\;.\;.\;.\;.\;.\;.\quad (29).$$

Since v_1 v_0 are due to a surface distribution of surface density *ρh*, therefore,

$$\begin{aligned}-4\pi\rho\gamma h = \left[\frac{\partial\mathrm{V}_0}{\partial\mathrm{N}}\right]-\left[\frac{\partial\mathrm{V}_1}{\partial\mathrm{N}}\right] &= \frac{d\zeta}{d\mathrm{N}}\left[\frac{\partial\mathrm{V}_0}{\partial\zeta}-\frac{\partial\mathrm{V}_1}{\partial\zeta}\right]_{(\zeta=\zeta_0)}\\ &= \mathrm{B}_n^{\,s}\frac{d\zeta}{d\mathrm{N}}e^{2\iota\omega\kappa t}e^{\iota s\phi}\mathrm{T}_n^{(s)}(\mu)\left\{\frac{\mathrm{D}u_n^{\,s}(\zeta_0)}{u_n^{\,s}(\zeta_0)}-\frac{\mathrm{D}t_n^{\,s}(\zeta_0)}{t_n^{\,s}(\zeta_0)}\right\}\\ &= -\mathrm{B}_n^{\,s}\frac{d\zeta}{d\mathrm{N}}e^{2\iota\omega\kappa t}e^{\iota s\phi}\mathrm{T}_n^{(s)}(\mu)\div\{(\zeta_0^{\,2}+1)\,t_n^{\,s}(\zeta_0)\,u_n^{\,s}(\zeta_0)\},\end{aligned}$$

and, therefore, at the surface, by (30)

$$\begin{aligned}[v] &\equiv \mathrm{B}_n^{\,s}e^{2\iota\omega\kappa t}e^{\iota s\phi}\mathrm{T}_n^{(s)}(\mu)\\ &= 4\pi\rho\gamma\,\mathrm{C}_n^{\,s}c^2\cot\alpha\,\mathrm{cosec}^2\alpha\,.\,t_n^{\,s}(\zeta_0)\,u_n^{\,s}(\zeta_0)\,\mathrm{T}_n^{(s)}(\mu)\,e^{\iota s\phi}e^{2\iota\omega\kappa t}\quad .\;.\;.\;.\quad (53).\end{aligned}$$

9. Lastly, in the forced oscillations, whatever be the variable conservative bodily forces or surface tractions producing them, we know that it is always possible to expand the value of $[\mathrm{V}_2 - p_2/\rho]$ over the surface of the spheroid and at all times in a series of the form

$$[\mathrm{V}_2 - p_2/\rho] = \Sigma_{\kappa=-\infty}^{\kappa=\infty}\Sigma_{n=1}^{n=\infty}\Sigma_{s=0}^{s=n}\mathrm{W}_{(n,\kappa)}^{s}\mathrm{T}_n^{(s)}(\mu)\,e^{\iota s\phi}e^{2\iota\omega\kappa t}\quad .\;.\;.\;.\;.\quad (54),$$

where $\mathrm{W}_{(n,\kappa)}^{s}$ is a constant, and the summations may extend to all possible values of *κ*, but only to integral values of *n* and *s*. The effect of each term may be considered separately. To do this, let us take the case when there is a single term only, *i.e.*, take

$$[\mathrm{V}_2 - p_2/\rho] = \mathrm{W}_{(n,\kappa)}^{s}\mathrm{T}_n^{(s)}(\mu)\,e^{\iota s\phi}e^{2\iota\omega\kappa t}\quad .\;.\;.\;.\;.\;.\;.\;.\quad (55).$$

In the waves produced the values of n, s, κ will be the same.

Substituting from (21), (52), (53), (55) in (51), we obtain

$$A_n^s(\nu_0^2-1)^{s/2}\,D^sP_n(\nu_0)+4\pi\rho\gamma\,C_n^s c^2\cot\alpha\,\mathrm{cosec}^2\alpha\,\{p_1(\zeta_0)\,q_1(\zeta_0)-t_n^s(\zeta_0)\,u_n^s(\zeta_0)\}$$
$$=W^s_{(n,\kappa)} \quad . \quad . \quad (56).$$

This equation, combined with (39), suffices to determine the unknown constants A_n^s, C_n^s in terms of the known coefficient $W^s_{(n,\kappa)}$, and thus the amplitude of the forced oscillation is determined in terms of that of the disturbing force.

10. The most interesting point is to determine C_n^s, in order to find the height of the corrugations on the surface. This plan has, moreover, the advantage that, in considering the effect of several disturbing forces of different periods, we may add together the elevations (h) due to the separate forces, whereas, in determining the value of ψ, the terms having different periods are referred to different auxiliary systems. Substituting for A_n^s in terms of C_n^s from (39) in (56), and writing, for brevity,

$$K_n^s(\zeta_0)=p_1(\zeta_0)\,q_1(\zeta_0)-t_n^s(\zeta_0)\,u_n^s(\zeta_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57),$$

$$M=\tfrac{4}{3}\pi\rho c^3\cot\alpha\,\mathrm{cosec}^2\alpha=\text{mass of spheroid} \quad . \quad . \quad . \quad (58),$$

we find, after several reductions, the required equation for C_n^s, viz.,

$$\frac{3M\gamma C_n^s}{c}\left\{K_n^s(\zeta_0)-\frac{4\kappa q_2(\zeta_0)\,D^sP_n(\nu_0)}{s\,D^sP_n(\nu_0)/(\kappa-1)+\sec^2\alpha\,.\,\nu_0\,D^{s+1}P_n(\nu_0)/\kappa}\right\}=W^s_{(n,\kappa)} \quad . \quad (59),$$

in which it must be remembered that

$$\nu_0=\frac{\kappa\cos\alpha}{\sqrt{(1-\kappa^2\sin^2\alpha)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

The Period-Equations for Free Waves.

11. If the oscillations of the liquid be free, we must put $W^s_{(n,\kappa)}$ equal to zero in (59), and we therefore obtain

$$K_n^s(\cot\alpha)-\frac{4\kappa q_2(\cot\alpha)\,D^sP_n(\nu_0)}{s\,D^sP_n(\nu_0)/(\kappa-1)+\sec^2\alpha\,.\,\nu_0\,D^{s+1}P_n(\nu_0)/\kappa}=0 \quad . \quad . \quad (60),$$

which, together with (15), determines the admissible values of κ and ν_0. In reducing (60) to a rational algebraic equation for κ we must distinguish three cases.

I. Let $s=0$, and let n be even. Then we know that $P_n(\nu_0)$ and $DP_n(\nu_0)$ contain only even and odd powers of ν_0 respectively, and, therefore, that $DP_n(\nu_0)$ is divisible by ν_0. Multiplying (60) by $DP_n(\nu_0)/\nu_0$, we find (writing K_n for K_n^0)

$$K_n(\cot\alpha)\,DP_n(\nu_0)/\nu_0-4q_2(\cot\alpha)\,(1-\kappa^2\sin^2\alpha)\,P_n(\nu_0)=0 \quad . \quad . \quad (61).$$

Expanding $P_n(\nu_0)$ and $DP_n(\nu_0)/\nu_0$ in powers of ν_0, substituting for ν_0 by means of (15), and multiplying the resulting equation throughout by $(1 - \kappa^2 \sin^2 \alpha)^{n/2-1}$, we find

$$\begin{aligned}4q_2(\cot\alpha)&\Big\{(\kappa\cos\alpha)^n - \frac{n(n-1)}{2.(2n-1)}(\kappa\cos\alpha)^{n-2}(1-\kappa^2\sin^2\alpha) + \dots\Big\}\\ &- nK_n(\cot\alpha)\Big\{(\kappa\cos\alpha)^{n-2} - \frac{(n-1)(n-2)}{2.(2n-1)}(\kappa\cos\alpha)^{n-4}(1-\kappa^2\sin^2\alpha) + \dots\Big\}\\ &= 0 \quad . \quad . \quad (62).\end{aligned}$$

This is a rational algebraic equation in κ of the n^{th} degree, involving only even powers of κ. It is, therefore, satisfied by n values of κ occurring in pairs corresponding to $\frac{1}{2}n$ values of κ^2.

II. Let $s = 0$, but let n be odd. Then $DP_n(\nu_0)$ is not divisible by ν_0. Hence, we must multiply the equation (60) throughout by $DP_n(\nu_0)$ and obtain

$$K_n(\cot\alpha)\,DP_n(\nu_0) - 4q_2(\cot\alpha)(1-\kappa^2\sin^2\alpha)\,\nu_0 P_n(\nu_0) = 0 \quad . \quad . \quad (63).$$

If this be developed in the same manner as in the preceding case, we shall obtain

$$\begin{aligned}4q_2(\cot\alpha)&\Big\{(\kappa\cos\alpha)^{n+1} - \frac{n(n-1)}{2.(2n-1)}(\kappa\cos\alpha)^{n-1}(1-\kappa^2\sin^2\alpha) + \dots\Big\}\\ &- nK_n(\cot\alpha)\Big\{(\kappa\cos\alpha)^{n-1} - \frac{(n-1)(n-2)}{2.(2n-1)}(\kappa\cos\alpha)^{n-3}(1-\kappa^2\sin^2\alpha) + \dots\Big\}\\ &= 0 \quad . \quad . \quad (64).\end{aligned}$$

This is satisfied by $n + 1$ values of κ, but, as before, the positive and negative roots are numerically equal, so that there will only be $\frac{1}{2}(n + 1)$ different values of κ^2.

III. Let s be different from zero. Multiplying by the expression

$$sD^sP_n(\nu_0) + \sec^2\alpha\,.\,(\kappa - 1)/\kappa\,.\,\nu_0 D^{s+1}P_n(\nu_0),$$

we find

$$\begin{aligned}&\sec^2\alpha\,K_n^s(\cot\alpha)(\kappa-1)/\kappa\,.\,\nu_0 D^{s+1}P_n(\nu_0)\\ &\qquad - \{4q_2(\cot\alpha)\,\kappa(\kappa-1) - sK_n^s(\cot\alpha)\}\,D^sP_n(\nu_0) = 0 \quad . \quad . \quad (65),\end{aligned}$$

which reduces to the following equation in κ —

$$\begin{aligned}&\{4q_2(\cot\alpha)\,\kappa(\kappa-1) - sK_n^s(\cot\alpha)\}\Big\{(\kappa\cos\alpha)^{n-s}\\ &\qquad - \frac{(n-s)(n-s-1)}{2.(2n-1)}(\kappa\cos\alpha)^{n-s-2}(1-\kappa^2\sin^2\alpha) + \dots\Big\}\\ &- (n-s)\sec\alpha\,.\,K_n^s(\cot\alpha)(\kappa-1)\Big\{(\kappa\cos\alpha)^{n-s-1}\\ &\qquad - \frac{(n-s-1)(n-s-2)}{2.(2n-1)}(\kappa\cos\alpha)^{n-s-3}(1-\kappa^2\sin^2\alpha) + \dots\Big\} = 0 \quad . \quad (66).\end{aligned}$$

This equation is of the degree $n - s + 2$, and involves both odd and even powers of κ. It therefore has $n - s + 2$ roots, but in the present case these roots do not occur in pairs of equal and opposite values.

Equations (62), (64), (66) are the period-equations of the various free harmonic waves or oscillations of the liquid spheroid. Their roots depend on the value of α or the eccentricity ($\sin\alpha$) alone. The periods of the waves are the corresponding values of $\pi/\omega\kappa$ and depend also on ω.

Nature of the Real Oscillations and Waves.

12. The periodic movements determined by zonal harmonics ($s = 0$) and those determined by tesseral harmonics differ in character considerably.

The former are symmetrical about the axis. Taking the solution

$$\begin{aligned} \psi &= A_n P_n(\mu') P_n(\nu) e^{2\iota\omega\kappa t}, \\ h &= C_n \varpi P_n(\mu) e^{2\iota\omega\kappa t}, \end{aligned}$$

another solution got by changing the sign of κ is given by

$$\begin{aligned} \psi &= A_n P_n(\mu') P_n(\nu) e^{-2\iota\omega\kappa t}, \\ h &= C_n \varpi P_n(\mu) e^{-2\iota\omega\kappa t}. \end{aligned}$$

Compounding these, we get the real motions of the liquid determined by

$$\left.\begin{aligned} \psi &= A_n P_n(\mu') P_n(\nu) \sin(2\omega\kappa t - \epsilon_n) \\ h &= C_n \varpi P_n(\mu) \sin(2\omega\kappa t - \epsilon_n) \end{aligned}\right\} \quad \ldots\ldots \quad (67),$$

ϵ_n being any constant.

These are *stationary oscillations* of the liquid about the spheroidal form. By what has already been shown, there are either $\frac{1}{2}n$ or $\frac{1}{2}(n + 1)$ such free oscillations, according as n is even or odd. In all of these oscillations the expression for h is of the same form, that is, the corrugations produced on the surface are similar in each. But this will not be the case with the values of ψ, because the auxiliary systems of spheroidal coordinates to which they are referred are different for each different value of κ. Thus, the motions of the fluid particles in the interior of the mass are different for each of the oscillations.

13. Taking next the case when s is different from zero, let us change the sign of $\sqrt{-1}$ *everywhere* that it occurs in our investigations. The results will still hold good when this is done. Hence, for every root of (66) we get two solutions of the equations of oscillation, giving respectively

$$\psi = \mathrm{A}_n^s \mathrm{T}_n^{(s)}(\mu')\, \mathrm{T}_n^s(\nu)\, e^{\iota(s\phi + 2\omega\kappa t)},$$
$$h = \mathrm{C}_n^s \varpi\, \mathrm{T}_n^{(s)}(\mu)\, e^{\iota(s\phi + 2\omega\kappa t)},$$

and also

$$\psi = \mathrm{A}_n^s \mathrm{T}_n^{(s)}(\mu')\, \mathrm{T}_n^s(\nu)\, e^{-\iota(s\phi + 2\omega\kappa t)},$$
$$h = \mathrm{C}_n^s \varpi\, \mathrm{T}_n^{(s)}(\mu)\, e^{-\iota(s\phi + 2\omega\kappa t)},$$

which combine to give the real motions

$$\left.\begin{aligned} \psi &= \mathrm{A}_n^s\, \mathrm{T}_n^{(s)}(\mu')\, \mathrm{T}_n^s(\nu) \sin(s\phi + 2\omega\kappa t - \epsilon_n^s) \\ h &= \mathrm{C}_n^s\, \varpi\, \mathrm{T}_n^{(s)}(\mu) \sin(s\phi + 2\omega\kappa t - \epsilon_n^s) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (68).$$

These represent a system of *waves* travelling round the axis of the spheroid with *relative* angular velocity $-2\omega\kappa/s$. But it must be remembered that the coordinate axes to which we have referred the wave motion are themselves rotating with angular velocity ω. Hence, the angular velocity of the waves in space is $\omega(1 - 2\kappa/s)$. According to our convention, positive values of κ give waves rotating more slowly than the liquid, and *vice versâ.*

There are $n - s + 2$ such waves determined by harmonics of degree n and rank s, and, since the values of κ are not equal and opposite in pairs, these waves do not combine into oscillations fixed relatively to the moving axes. As in the symmetrical oscillations, the form of the corrugations is the same for all the waves, but the motion of the fluid particles different in each.

14. If $\kappa_1, \kappa_2, \ldots, \kappa_{n-s+2}$ be the roots of (66), it is obvious that

$$\kappa_1 + \kappa_2 + \ldots + \kappa_{n-s+2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69).$$

Hence, the *mean* relative angular velocity of *all* the different harmonic waves of degree n and rank s is

$$\frac{2\omega}{(n - s + 2)s},$$

in direction opposite to that of rotation of the liquid, whilst their mean actual angular velocity in space is

$$\omega\left\{1 - \frac{2}{(n - s + 2)s}\right\}.$$

Analysis of the Period-Equations.

15. From POINCARÉ's investigations it appears that the spheroid will be secularly stable, even if the liquid be viscous, provided that the coefficients which are here denoted by $\mathrm{K}_n^s(\zeta_0)$ or

$$p_1(\zeta)\, q_1(\zeta) - t_n^s(\zeta)\, u_n^s(\zeta)$$

are positive for all values of n greater than unity.* From our equations (49), (50) we have

$$-\mathrm{K}_1^{\,1}(\zeta) = q_2(\zeta) = \omega^2/(4\pi\rho\gamma\zeta),$$

so that $\mathrm{K}_1^{\,1}(\zeta)$ is essentially negative and $q_2(\zeta)$ positive.

In accordance with this, we shall now show that, if $\mathrm{K}_n^{\,s}(\zeta)$ and $q_2(\zeta)$ be both positive, the roots of the period-equation for harmonic waves of degree n and rank s are all real, and we shall find their situations.

In the first place, let us suppose s is different from zero. The period-equation (66), as it stands, may be written

$$\mathrm{F}(\kappa) = 0,$$

where

$$\mathrm{NF}(\kappa) \equiv (1-\kappa^2\sin^2\alpha)^{(n-s)/2}\{4q_2(\cot\alpha)\,\kappa(\kappa-1) - s\mathrm{K}_n^{\,s}(\cot\alpha)\}\,\mathrm{D}^s\mathrm{P}_n(\nu)$$
$$-(1-\kappa^2\sin^2\alpha)^{(n-s-1)/2}\sec\alpha\,\mathrm{K}_n^{\,s}(\cot\alpha)(\kappa-1)\,\mathrm{D}^s\mathrm{P}_n(\nu),$$

if we write for brevity

$$\mathrm{N} \equiv (2n)!/\{2^n n!\,(n-s)!\}.$$

We know that the roots of the equation

$$\mathrm{D}^s\mathrm{P}_n(\nu) = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (70)$$

are all real, and lie between $+1$ and -1; also they are separated by those of

$$\mathrm{D}^{s+1}\mathrm{P}_n(\nu) = 0.$$

Let $\kappa_1, \kappa_2, \ldots \kappa_{n-s}$ be the values of κ (taken in descending order of magnitude) corresponding to the roots of (70). These values of κ all lie between $+1$ and -1, also ν decreases as κ decreases. Moreover, if κ is put in turn equal to $\kappa_1, \kappa_2, \ldots \kappa_{n-s}$, the corresponding values of $\mathrm{D}^{s+1}\mathrm{P}_n(\nu)$ are alternately positive and negative.

We are now in a position to trace the changes in $\mathrm{F}(\kappa)$ as κ decreases from $+\infty$ to $-\infty$.

When κ is greater than cosec α, ν is imaginary. But $\mathrm{F}(\kappa)$ when written in the form of the left-hand side of (66) is obviously real; also when $\kappa = \infty$ the sign of $\mathrm{F}(\kappa)$ is that of the coefficient of κ^{n-s+2}. It is therefore *positive.*

When κ passes through the value cosec α, ν becomes infinite and then becomes real, but $\mathrm{F}(\kappa)$ does not in general change sign.

When $\kappa = 1$, $\nu = 1$, $\mathrm{D}^s\mathrm{P}_n(\nu)$ is positive, and $\mathrm{F}(\kappa)$ is negative.

When $\kappa = \kappa_1$, $\mathrm{F}(\kappa)$ is positive.

When $\kappa = \kappa_2$, $\mathrm{F}(\kappa)$ is negative.

When $\kappa = \kappa_3$, $\mathrm{F}(\kappa)$ is positive.

and so on; thus, when $\kappa = \kappa_{n-s}$, $\mathrm{F}(\kappa)$ has the same sign as $\overline{-1}|^{n-s+1}$.

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* We shall in future leave out the suffixes in ζ_0 and ν_0, using ζ, ν to denote the surface values, as these surface values alone occur in the remainder of our investigations.

In general $F(\kappa)$ does not change sign when $\kappa = -\operatorname{cosec}\alpha$, but when $\kappa = -\infty$, $F(\kappa)$ has the same sign as $\overline{-1}|^{n-s+2}$.

Hence, the equation $F(\kappa) = 0$ must have one real root between each of the following values :—

$$\infty,\ 1,\ \kappa_1,\ \kappa_2,\ \kappa_3,\ \ldots\ \kappa_{n-s},\ -\infty.$$

Thus, if $K_n^s(\zeta)$ is positive, all the roots of (66) are real. Let us now examine what happens when $K_n^s(\zeta)$ vanishes and becomes negative. POINCARÉ proves* that, if $t_n^s(\zeta)$ is divisible by ζ, the equation

$$K_n^s(\zeta) = 0$$

has no real root; we must, therefore, have $n - s$ even, so that $t_n^s(\zeta)$ is not divisible by ζ.

When $K_n^s(\zeta)$ vanishes, the equation

$$F(\kappa) = 0$$

reduces to

$$\kappa(\kappa - 1)(1 - \kappa^2 \sin^2\alpha)^{\frac{1}{2}(n-s)} D^s P_n \{\kappa \cos\alpha (1 - \kappa^2 \sin^2\alpha)^{-\frac{1}{2}}\} = 0,$$

of which the roots are

$$0,\ 1,\ \kappa_1,\ \kappa_2,\ \ldots,\ \kappa_{n-s}.$$

Since $n - s$ is even, the equation

$$D^s P_n(\nu) = 0$$

has not zero for one of its roots. Thus, the roots of the period-equation are all real and different. Therefore, when $K_n^s(\zeta)$ changes sign and becomes negative, the period-equation must, at any rate at first, continue to have all its roots real. If it have a pair of complex roots, the ratio of $K_n^s(\zeta) : q_2(\zeta)$ must not only be negative, but numerically greater than some *finite* limit.

16. The roots of the period-equations for the oscillations that are symmetrical about the axis of the spheroid are to be separated in exactly the same way. It will be sufficient to state the results here. We suppose $\kappa_1, \kappa_2, \kappa_3, \ldots, \kappa_n$ are the n values of κ which make

$$P_n(\nu) \equiv P_n\{\kappa \cos\alpha (1 - \kappa^2 \sin^2\alpha)^{-\frac{1}{2}}\} = 0 \quad \ldots \quad (71).$$

Let n be odd. One of the above values, viz., $\kappa_{\frac{1}{2}(n+1)}$ will be zero, whilst $\kappa_n = -\kappa_1$, $\kappa_{n-1} = -\kappa_2$, and so on. Also, the ratio $K_n(\zeta) : q_2(\zeta)$ *must* be positive. It will be found that the period-equation (64) has one root between each of the following values of κ;

$$\infty,\ \kappa_1,\ \kappa_2,\ \kappa_3,\ \ldots,\ \kappa_{\frac{1}{2}(n-1)},\ 0,\ \kappa_{\frac{1}{2}(n+3)},\ \ldots,\ \kappa_n,\ -\infty.$$

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* 'Acta Mathematica,' vol. 7, p. 326.

If n be even, the least positive and negative roots of (71) are $\kappa_{\frac{1}{2}n}$ and $\kappa_{\frac{1}{2}n+1}$, also $\kappa_{\frac{1}{2}n+1} = -\kappa_{\frac{1}{2}n}$. If the ratio $K_n(\zeta) : q_2(\zeta)$ be positive, we find that the positive roots of the period-equation (62) are situated between the following values :—

$$\infty,\ \kappa_1,\ \kappa_2,\ \ldots,\ \kappa_{\frac{1}{2}n},$$

while the negative ones which are equal and opposite to them are situated in the intervals between

$$-\infty,\ \kappa_n,\ \kappa_{n-1},\ \ldots,\ \kappa_{\frac{1}{2}n+1},$$

there being no roots between $\kappa_{\frac{1}{2}n}$ and $\kappa_{\frac{1}{2}n+1}$.

When $K_n(\zeta)$ vanishes, the roots are the n quantities

$$\kappa_1,\ \kappa_2,\ \ldots,\ \kappa_n,$$

none of which is equal to zero. If the ratio $K_n(\zeta) : q_2(\zeta)$ now become negative, the roots of (62) will at first continue to be real, being situated between the values

$$\kappa_1,\ \kappa_2,\ \ldots,\ \kappa_{\frac{1}{2}n},\ 0,\ \kappa_{\frac{1}{2}n+1},\ \ldots,\ \kappa_n.$$

This will be the case until we arrive at a value of ζ for which the period-equation has a pair of equal roots, each equal to zero. When this is so, we have

$$\frac{K_n(\zeta)}{4q_2(\zeta)} = \mathrm{Lt}_{\nu=0}\ \frac{\nu P_n(\nu)}{DP_n(\nu)} = -\frac{1}{n(n+1)},$$

whence,

$$K_n(\zeta) = -\frac{4}{n(n+1)}\,q_2(\zeta) = \frac{4}{n(n+1)}\,K_1{}^1(\zeta) \quad . \quad . \quad . \quad . \quad (72),$$

or

$$p_1(\zeta)\,q_1(\zeta) - p_n(\zeta)\,q_n(\zeta) = \frac{4}{n(n+1)}\,\{p_1(\zeta)\,q_1(\zeta) - t_1{}^1(\zeta)\,u_1{}^1(\zeta)\}.$$

When the ratio $K_n(\zeta) : -q_2(\zeta)$ or $K_n(\zeta) : K_1{}^1(\zeta)$ becomes greater than $4/\{n(n+1)\}$, two of the roots of the period-equation will become imaginary.

In *every* case there must be at least one positive root between each of the quantities

$$\kappa_1,\ \kappa_2,\ \ldots,\ \kappa_{\frac{1}{2}n}$$

and corresponding negative roots, so that under no circumstances can equation (62) have *more* than one pair of imaginary roots.

Numerical Solutions of the Period-Equations.

17. For a spheroid of given eccentricity, α, and therefore ζ, are known. Now, the functions $p_n(\zeta)$, $t_n{}^s(\zeta)$ can be expanded in finite terms of ζ in exactly the same way

as the ordinary spherical harmonics, while $q_n(\zeta)$, $u_n^s(\zeta)$ can be expressed in finite terms of ζ, $\cot^{-1}\zeta$, *i.e.*, of ζ, α; hence, the function $K_n^s(\zeta)$ can be calculated for any value of ζ.* By HORNER'S method we may then approximate to the values of the roots of the equation in κ in the simpler cases. The periods of the waves are the corresponding values of $\pi/\kappa\omega$, while ω is expressed in terms of ρ by equation (50).†

To obtain some idea of the relative frequencies of the various waves, I have tabulated the values of κ thus calculated for harmonics of the second, third, and fourth degrees for a spheroid in which $\zeta = 1$ or $\alpha = \pi/4$, the eccentricity being, therefore, $\frac{1}{2}\sqrt{2}$. The results are embodied in the accompanying Table. As already stated, the positive roots correspond to waves rotating more slowly than the liquid, or relatively in the direction opposite to that of rotation of the mass, while those having the double sign correspond to symmetrical oscillations of the liquid.

TABLES of the Values of $\kappa\left(=\dfrac{\lambda}{2\omega}\right)$ for Waves on a Spheroid whose Eccentricity $= \sin\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$.

Rank of Harmonic.	I. Harmonics of the Second Degree.				
2 (sectorial)	1·2126108,	− 0·2126108.			
1	1·280776,	− 0·780776.			
0 (zonal)	± 1·128465.				
Oscillatory waves					
	II. Harmonics of the Third Degree.				
3 (sectorial)	1·569830,	− 0·569830.			
2	1·677377,	0·423263,	− 1·100640.		
1	1·6008928,	0·8267846,	− 0·0733654,	− 1·3543120.	
0 (zonal)	± 1·5178954,	± 0·5122368.			
Oscillatory waves					
	III. Harmonics of the Fourth Degree.				
4 (sectorial)	1·852560,	− 0·852560.			
3	1·806374,	0·366650,	− 1·173024.		
2	1·921878,	0·730910,	− 0·107365,	− 1·545423.	
1	1·924662,	0·890193,	0·585670,	− 0·654623,	− 1·745902.
0 (zonal)	± 1·994751,	± 0·685895.			

I find that the period of the symmetrical or zonal harmonic oscillation of the second degree (in which the surface remains spheroidal) is, in this spheroid, 0·8599258 of the corresponding time of oscillation in a non-rotating spherical mass of liquid of the same density.

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* "On the Expression of Spherical Harmonics of the Second Kind in a Finite Form," 'Cambridge Philosophical Proceedings,' December, 1888.

† See THOMSON and TAIT'S 'Natural Philosophy,' vol. 2, § 772.

Sectorial Harmonic Waves.

18. When $s = n$, $D^s P_n(\nu)$ is numerical, and $D^{s+1} P_n(\nu)$ is zero; thus, the period-equation reduces to

$$4\kappa(\kappa - 1) = nK_n^n(\zeta)/q_2(\zeta) \quad . \; . \; . \; . \; . \; . \; . \quad (73),$$

of which the roots are given by

$$\kappa = \tfrac{1}{2}\{1 \pm \surd(1 + nK_n^n(\zeta)/q_2(\zeta))\} \quad . \; . \; . \; . \; . \; . \quad (74).$$

The condition that these roots may be real is that

$$q_2(\zeta) + nK_n^n(\zeta) > 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (75),$$

that is

$$p_1(\zeta)\, q_1(\zeta) - t_n^n(\zeta)\, u_n^n(\zeta) - \frac{1}{n}\{p_1(\zeta)\, q_1(\zeta) - t_1^1(\zeta)\, u_1^1(\zeta)\}$$

must be positive.

These results have been obtained previously by POINCARÉ in the special case in which $n = 2$, but in his investigation an extraneous factor has been introduced into the period-equation, giving a third root ($\kappa = 1$) which does not properly belong to it.

The expression for ψ in (21) is here proportional to

$$(1 - \mu'^2)^{\frac{1}{2}n}(\nu^2 - 1)^{\frac{1}{2}n}\, e^{\iota(n\phi + 2\omega\kappa t)},$$

that is, in Cartesian coordinates, to

$$(x + \iota y)^n e^{2\iota\omega\kappa t},$$

and is independent of z.

Thus, the motion of the liquid is "two dimensional," and takes place in planes parallel to the equatorial plane of the spheroid. By the laws of vortex motion the molecular rotation or spin of the actual motion of the liquid is therefore everywhere constant and equal to ω, being that due to the rotation of the liquid. In other words, the wave motion of the liquid *relative* to the rotating axes is irrotational.

Small Free Precession of the Spheroid.

19. Another case of some interest is when the harmonics determining the small periodic relative motions are of the second degree and first rank. Putting $n = 2$, $s = 1$, the period-equation (66) reduces to

$$4\kappa^2(\kappa - 1)\, q_2(\cot\alpha) - [(\kappa - 1)\sec^2\alpha + \kappa]\, K_2^1(\cot\alpha) = 0 \quad . \; . \quad (76).$$

Now, whatever the value of α may be, $\kappa = \frac{1}{2}$ is always a root of this equation. For, if we put $\kappa = \frac{1}{2}$ in the left-hand side, it becomes

$$
\begin{aligned}
&= -\tfrac{1}{2} q_2 (\cot \alpha) + \tfrac{1}{2} \tan^2 \alpha \, \mathrm{K}_2{}^1 (\cot \alpha) \\
&= \tfrac{1}{2} \{ \mathrm{K}_1{}^1 (\zeta) + \mathrm{K}_2{}^1 (\zeta)/\zeta^2 \} \\
&= \tfrac{1}{2} \{ (\zeta^2 + 1) p_1 (\zeta) \, q_1 (\zeta)/\zeta^2 - t_1{}^1 (\zeta) \, u_1{}^1 (\zeta) - t_2{}^1 (\zeta) \, u_2{}^1 (\zeta)/\zeta^2 \} \\
&= \tfrac{1}{2} (\zeta^2 + 1) \int_\zeta^\infty \left\{ \frac{1}{\zeta^2} - \frac{1}{\zeta^2 + 1} - \frac{1}{\zeta^2 (\zeta^2 + 1)} \right\} \frac{d\zeta}{\zeta^2 + 1} = 0 \quad . \quad . \quad . \quad . \quad (77),
\end{aligned}
$$

as was to be proved.

Substituting from the relation just found in (76), the period-equation becomes

$$(4\kappa^3 - 4\kappa^2 + \kappa) \tan^2 \alpha - (2\kappa - 1) \sec^2 \alpha = 0,$$

or, dividing throughout by $(2\kappa - 1) \tan^2 \alpha$, the other two roots are given by

$$2\kappa^2 - \kappa - \operatorname{cosec}^2 \alpha = 0,$$

whence

$$
\begin{aligned}
\kappa &= \tfrac{1}{4} \{ 1 \pm \sqrt{(1 + 8 \operatorname{cosec}^2 \alpha)} \} \\
&= \tfrac{1}{4} \{ 1 \pm \sqrt{(9 + 8\zeta^2)} \} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78).
\end{aligned}
$$

The expression for ψ is proportional to

$$(1 - \mu'^2)^{\frac{1}{2}} \mu' (\nu^2 - 1)^{\frac{1}{2}} \nu \sin (\phi + 2\omega\kappa t - \epsilon),$$

that is, to

$$z \{ x \sin (2\omega\kappa t - \epsilon) + y \cos (2\omega\kappa t - \epsilon) \},$$

while the height of the displacement of the surface is proportional to

$$\varpi \, z \{ x \sin (2\omega\kappa t - \epsilon) + y \cos (2\omega\kappa t - \epsilon) \}.$$

Remembering that this displacement is so small that its square may be neglected, it can be readily shown by the usual methods of analytical geometry that, if, as is here supposed, the ellipticity of the spheroid be finite, the displaced surface is a spheroid of the same form and dimensions as the original spheroid, and can be obtained by turning the latter through a small angle about the line

$$x \sin (2\omega\kappa t - \epsilon) + y \cos (2\omega\kappa t - \epsilon) = 0, \qquad z = 0.$$

This will, however, no longer be true if the ellipticity of the spheroid is a small quantity comparable with the height of the small displacement, or the surface is spherical or nearly spherical. In such cases it will be found that the displaced surface is an ellipsoid, differing in form from the original spheroid by small quantities of the first order, whose axes make finite, not small, angles with those of the spheroid.

Suppose the liquid spheroid is rotating steadily about its axis of figure with angular velocity ω, and that this axis does not quite coincide with our fixed axis of z, but is inclined to it at a small angle, while the axes of x, y rotate about the axis of z with angular velocity ω. The coordinates of the fluid particles will now no longer be constant, but will undergo small periodic changes. In the time $2\pi/\omega$ both the fluid particles and the axes will come round to their original positions; thus, the period of the *apparent* relative oscillations is $2\pi/\omega$, although the liquid is in reality rotating steadily. This accounts for the root $\kappa = \frac{1}{2}$, which occurs in the period-equation, a result which may be completely verified by rigorous analytical methods.

The movements corresponding to the other two roots are somewhat similar to precession, the axis of figure of the spheroid turning about the axis of z, to which it is inclined at a small angle.

Stability of the Spheroid.

20. We have already alluded to POINCARÉ'S investigations of the condition that the spheroid, if viscous, may be secularly stable, which requires that the energy of the system for the given angular momentum must be a minimum in the spheroidal form. The greatest eccentricity corresponds to the least value of ζ which causes any one of the coefficients $K_n^s(\zeta)$ to vanish, and this is shown to be that given by $K_2^2(\zeta) = 0$, whence, as in THOMSON and TAIT (§ 772),

$$1/\zeta = \tan\alpha = f = 1·39457.$$

If the liquid be perfectly inviscid, the criteria are very different. So long as the roots of the period-equations for the various waves and oscillations are all real, the spheroid cannot be unstable. It will, however, become unstable if for any harmonic the equation in κ has a pair of complex or imaginary roots. For, calling these roots $l \pm m\iota$, we get the possible surface displacements

$$h = C_n^s \varpi T_n^{(s)}(\mu)\, e^{\iota s\phi} e^{2\omega(\iota l + m)t},$$
$$h = C_n^s \varpi T_n^{(s)}(\mu)\, e^{-\iota s\phi} e^{2\omega(-\iota l + m)t},$$

compounding into the displacement

$$h = C_n^s \varpi T_n^{(s)}(\mu) \cos(s\phi + 2\omega lt)\, e^{2\omega mt},$$

which increases indefinitely with the time.

Let us imagine that our spheroid is subject to constraints such as to freely allow of its surface undergoing harmonic displacements of degree n and rank s, but which allow of no other displacements (such constraints are, of course, purely theoretical). The spheroid, if at all viscous, will be secularly stable or unstable according as

$$K_n^s(\zeta) > 0 \quad \text{or} \quad < 0,$$

and we have seen that the latter condition can only hold if $n - s$ be even, that is, if the displacement be one symmetrical with respect to the equatorial plane. The critical form is that in which

$$K_n^s(\zeta) = 0.$$

But since, when $K_n^s(\zeta)$ changes sign, the roots of the period-equation at first continue real, the limits of eccentricity for which the perfect spheroid is "ordinarily" stable are in every case wider than those consistent with secular stability if the liquid be viscid. The critical form is determined by the condition that the period-equation must have a pair of equal roots.

In my paper on "The Waves on a Viscous Rotating Cylinder"* I have endeavoured to further elucidate the difference between "ordinary" and "secular" stability. Assuming the displacement from relative equilibrium to be proportional to $e^{-\alpha_1 t}$, α_1 is always complex for viscous liquid, and the condition that the disturbance may not increase with the time is that the real part of α_1 must be positive. Both real and imaginary parts change sign when $\alpha_1 = 0$, the corrugations becoming relatively fixed and the liquid figure becoming a form of "bifurcation." Relative equilibrium is then critical. But, if there be no viscosity, α_1 may be purely imaginary, as in the present case, when κ is real, and the waves will neither increase nor diminish in amplitude with the time; thus, a change in the sign of one of the roots of the period-equation merely implies a change in the relative direction of the wave.

Moreover, it appears that the criteria of ordinary and secular stability will be different only if the angular velocities of the waves be different in the two opposite directions, and this can only be the case if the liquid be rotating.

Reverting to the perfect liquid spheroid, the determination of the greatest eccentricity consistent with ordinary stability involves the question, if ζ be gradually diminished, what is the harmonic displacement for which the period-equation of the waves first commences to have complex roots? It appears probable that this happens for $n = 2$, $s = 2$. With this assumption, we see, by (75), that the critical value of ζ is given by the equation

$$2\{p_1(\zeta)\,q_1(\zeta) - t_2^2(\zeta)\,u_2^2(\zeta)\} + t_1^1(\zeta)\,u_1^1(\zeta) - p_1(\zeta)\,q_1(\zeta) = 0\,;$$

this leads to

$$(3\zeta^4 + 8\zeta^2 + 1)\cot^{-1}\zeta - (3\zeta^3 + 7\zeta) = 0,$$

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* 'Cambridge Philosophical Proceedings,' 1888.

whence I find by trial and error that

$$1/\zeta = \tan\alpha = f = 3\cdot 1414567\ldots.$$

This result agrees with that found by RIEMANN, who treated the problem as a special case of the general motion of a liquid ellipsoid.

It does not, however, seem possible to justify the above assumption as to the nature of the displacements by a perfectly general rigorous proof. The condition that the period-equation should have a pair of equal roots is far too complicated to allow of this point being fully proved in the way that POINCARÉ has done for secular stability. It is certain that the spheroid will be unstable for all values of $\tan\alpha$ greater than $3\cdot 1414567$; it is probable, but not certain, that it will be stable for all values less than this limit.

Spheroids of Small Ellipticity.

21. If the eccentricity of the spheroid, and, therefore also, its angular velocity, be small, the period-equations for the waves are much modified. The value of ζ will become very great, and we shall suppose it to be so great that ζ^{-2} is a small quantity that can be neglected. Since α is small, we may put $\cos\alpha = 1$, $\sin\alpha = \alpha = 1/\zeta$.

The function $t_n^s(\zeta)$ is proportional to ζ^n since the other terms in it involve only ζ^{n-2} and lower powers of ζ. Hence, to this approximation,

$$t_n^s(\zeta)\, u_n^s(\zeta) = \zeta^{2n}\int_\zeta^\infty \frac{d\zeta}{\zeta^{2n+2}} = \frac{\zeta^{-1}}{2n+1} \quad \ldots\ldots \quad (79),$$

and

$$\mathrm{K}_n^s(\zeta) = \zeta^{-1}\left(\tfrac{1}{3} - \frac{1}{2n+1}\right) = \frac{2(n-1)}{3(2n+1)}\zeta^{-1} \quad \ldots\ldots \quad (80);$$

moreover,

$$q_2(\zeta) = -\,\mathrm{K}_1^1(\zeta) = \tfrac{2}{15}\zeta^{-3} \quad \ldots\ldots \quad (81),$$

whence, as in THOMSON and TAIT (§ 771),

$$\omega^2 = \tfrac{8}{15}\pi\rho\gamma\,\zeta^{-2} \quad \ldots\ldots \quad (82).$$

Firstly, suppose that the values of κ remain finite in the limit. Then $\nu = \kappa$ ultimately, and, since $q_2(\zeta)$ is negligible in comparison with $\mathrm{K}_n^s(\zeta)$, equation (65) gives

$$s\mathrm{D}^s\mathrm{P}_n(\kappa) + (\kappa - 1)\,\mathrm{D}^{s+1}\mathrm{P}_n(\kappa) = 0 \quad \ldots\ldots \quad (83),$$

having $n - s$ real roots between 1 and -1.

In the case of the oscillations symmetrical about the axis ($s = 0$) the equation for κ is ultimately

$$\mathrm{DP}_n(\kappa) = 0 \quad \text{or} \quad \frac{1}{\kappa}\mathrm{DP}_n(\kappa) = 0 \quad \ldots\ldots \quad (84),$$

according as n is odd or even.

The frequencies of these waves or oscillations are proportional to the angular velocity of the liquid. As the latter is diminished without limit they become relatively unimportant, and finally cease to exist, for the limiting case of a mass of liquid without rotation oscillating about the spherical form.

Secondly, suppose the periods of the waves remain finite in the limit. Put $2\omega\kappa = \lambda$, so that $2\pi/\lambda$ is the period. Since λ remains finite, κ will increase without limit as ω diminishes, and, therefore, equation (66) gives, to the first order of small quantities,

$$\frac{2\zeta^{-2}}{15}\frac{\lambda(\lambda-2\omega)}{\omega^2} - \frac{2}{3}\frac{(n-1)}{(2n+1)}s - \frac{2}{3}\frac{(n-1)}{(2n+1)}(n-s)\frac{\lambda-2\omega}{\lambda} = 0,$$

whence, by (82),

$$\lambda(\lambda-2\omega) = \frac{8n(n-1)}{3(2n+1)}\pi\rho\gamma\left\{1 - \frac{n-s}{n}\cdot\frac{2\omega}{\lambda}\right\} \quad . \quad . \quad . \quad . \quad (85).$$

If we put $\omega = 0$, we get

$$\lambda^2 = \frac{8n(n-1)}{3(2n+1)}\pi\rho\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86),$$

the well-known result for the oscillations of a liquid sphere. Denoting by Λ^2 the expression

$$\frac{8}{3}\frac{n(n-1)}{(2n+1)}\pi\rho\gamma,$$

we find

$$\lambda^2 - \Lambda^2 = \frac{2\omega}{\lambda}\left(\lambda^2 - \frac{n-s}{n}\right),$$

whence, substituting $\lambda = \pm\Lambda$ in the small terms, we get

$$\lambda = \pm\Lambda + \frac{s}{n}\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87).$$

Remembering the expressions found in § 13 for the relative and actual angular velocities of the corresponding waves, this result may be stated as follows:—The effect of communicating a small angular velocity ω to a spherical mass of gravitating liquid will be to add an angular velocity $(n-1)\,\omega/n$ to the angular velocities of all the free waves which are determined by harmonics of degree n.

The symmetrical oscillations will be unaffected by rotation to this order of approximation. If we proceed to a higher approximation by taking into account small quantities of the second order, the equations become much more complicated. But, for a spheroid similar to the Earth, the above approximation would be practically sufficient.

Forced Tides.

22. We now revert to the applications of the methods of this paper to the investigation of the tides produced on the surface of the spheroid by the influence of periodic variations of pressure over the surface of the liquid, or by the attractions of disturbing bodies in the neighbourhood of the spheroid.

In this connexion, the equations found in §§ 9, 10 will be required, viz., if at the surface

$$V_2 - p_2/\rho = \Sigma\Sigma\Sigma W^s_{(n,\kappa)} T_n^{(s)}(\mu) \sin(s\phi + 2\omega\kappa t - \epsilon_{n\kappa}{}^s) \quad . \quad . \quad . \quad (88),$$

and

$$h = \varpi\, \Sigma\Sigma\Sigma C^s_{(n,\kappa)} T_n^{(s)}(\mu) \sin(s\phi + 2\omega\kappa t - \epsilon_{n\kappa}{}^s) \quad . \quad . \quad . \quad . \quad (89),$$

then $C^s_{(n,\kappa)}$ will be given in terms of $W^s_{(n,\kappa)}$ by the relation

$$3\frac{M\gamma}{c} C^s_{(n,\kappa)} \left\{ K_n{}^s(\zeta) - \frac{4\kappa D^s P_n(\nu)}{s D^s P_n(\nu)/(\kappa - 1) + \sec^2\alpha \,.\, \nu D^{s+1} P_n(\nu)/\kappa} q_2(\zeta) \right\} = W^s_{(n,\kappa)} \quad (90),$$

where, as in (12), (15),

$$\left.\begin{aligned} \nu &= \frac{\kappa\cos\alpha}{\sqrt{(1 - \kappa^2\sin^2\alpha)}}, \\ \text{and} \qquad \zeta &= \cot\alpha \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91).$$

Also, from (39), we have

$$12\frac{M\gamma}{c} C^s_{(n,\kappa)} \frac{4\kappa D^s P_n(\nu)\, q_2(\zeta)}{s D^s P_n(\nu)/(\kappa - 1) + \sec^2\alpha \,.\, \nu D^{s+1} P_n(\nu)/\kappa} = -A^s_{(n,\kappa)} T_n{}^s(\nu) \quad . \quad (92).$$

The value of ψ at any point of the surface being

$$[\psi] = A^s_{(n,\kappa)} T_n{}^s(\nu)\, T_n^{(s)}(\mu) \sin(s\phi + 2\omega\kappa t - \epsilon_{n,\kappa}{}^s) \quad . \quad . \quad . \quad . \quad (93),$$

is determined in terms of $C^s_{(n,\kappa)}$ by the last equation (92).

23. An interesting case occurs when $\kappa = 1$. The period of the tides will then be half that of a complete revolution of the liquid, and they may therefore be called "semidiurnal" with reference to the spheroid. Except in the case when $s = 0$, equation (90) gives

$$3\frac{M\gamma}{c} C_{(n)}{}^s K_n{}^s(\zeta) = W_n{}^s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94);$$

also, from (92), $A_n{}^s T_n{}^s(\nu) = 0$, and, therefore, $[\psi] = 0$; hence, it is evident that ψ must also vanish throughout the liquid.

The height of the forced tides is therefore the same as we should get by the "equilibrium theory," *i.e.*, by neglecting the small relative motions of the fluid particles entirely. In fact, these relative motions have no effect on the height of the tides. It does not follow that they do not exist; in fact, it is evident, on the contrary, since the tides move relatively to the liquid, that they must exist. But on referring to equations (34) we see that, when $\kappa = 1$, the small relative velocity components U, V, W may be finite, even though ψ vanishes.

The zonal oscillations are, however, given by

$$3\frac{\mathrm{M}\gamma}{c}\mathrm{C}_n\left\{\mathrm{K}_n(\zeta) - \frac{8\cos^2\alpha\, q_2(\zeta)}{n(n+1)}\right\} = \mathrm{W}_n \quad . \quad . \quad . \quad . \quad . \quad . \quad (95),$$

since $\nu = 1$, and therefore $\mathrm{P}_n(\nu) = 1$, $\mathrm{DP}_n(\nu) = \frac{1}{2}n(n+1)$. For these oscillations ψ does not vanish.

24. Another interesting application is to determine the height of the permanent corrugations produced by disturbing forces which remain constant and fixed relatively to the rotating liquid. We now have to take $\kappa = 0$; therefore, $\nu = 0$ and $\nu/\kappa = \cos\alpha$.

If s is different from zero, then, whether $n - s$ be odd or even, equation (90) gives us

$$3(\mathrm{M}\gamma/c)\,\mathrm{C}_n{}^s\mathrm{K}_n{}^s(\zeta) = \mathrm{W}_n{}^s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94),$$

and (92) gives $\mathrm{A}_n{}^s\mathrm{T}_n{}^s(\nu) = 0$, whence $[\psi] = 0$; and therefore ψ is everywhere zero.

If $s = 0$ and n is odd, then $\mathrm{P}_n(0) = 0$, $\mathrm{DP}_n(0)$ is finite, and, as before, we find

$$3(\mathrm{M}\gamma/c)\,\mathrm{C}_n\mathrm{K}_n(\zeta) = \mathrm{W}_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94\text{A}),$$

and

$$\psi = 0.$$

Lastly, let $s = 0$ and let n be even. This is the case of a harmonic disturbance which is symmetrical both about the axis of the spheroid and also with respect to its equatorial plane. Then, as in § 16, $4\nu\mathrm{P}_n(\nu)/\mathrm{DP}_n(\nu)$ approaches the finite limit $-4/\{n(n+1)\}$, when ν is diminished indefinitely, and therefore

$$3\frac{\mathrm{M}\gamma}{c}\mathrm{C}_n\left\{\mathrm{K}_n(\zeta) - \frac{4q_2(\zeta)}{n(n+1)}\right\} = \mathrm{W}_n \quad . \quad . \quad . \quad . \quad . \quad . \quad (96),$$

$$[\psi] = -\frac{12}{n(n+1)}\frac{\mathrm{M}\gamma}{c}\mathrm{C}_n q_2(\zeta)\,\mathrm{P}_n(\mu) \quad . \quad . \quad . \quad . \quad . \quad . \quad (97).$$

In the first two cases the height of the corrugations is given by the "equilibrium theory," and, since $\psi = 0$, it follows from equations (34) that U, V, W are all zero. Thus, the fluid continues to rotate as if rigid in a form differing slightly from the original spheroid, as we should most naturally expect. But in the last case, since ψ is finite, there will be a finite relative motion of the liquid with respect to the moving axes. That this must be the case may be seen as follows :—The spheroid is supposed to be deformed from its original form by the action of the given conservative forces and surface pressures. The displacement does not vanish at the equator ; hence, if we consider the fluid particles at the surface, forming a circle round the equator, the displacement must necessarily increase or diminish the size of this circle. By THOMSON'S circulation theorem, the circulation in this circuit must remain the same as before, since the liquid is supposed perfect; hence, the angular velocities of the fluid particles in this circle must be altered, and they can no longer continue to rotate about the axis of the spheroid with the original angular velocity ω. Therefore the disturbance must produce permanent relative motions of the liquid, unless there be any viscosity present, in which case the mass will ultimately rotate as if rigid in the deformed figure, and the " equilibrium theory " will again become applicable.

Tides due to Action of a Satellite.

25. We shall conclude by showing how to determine the forced tides due to the presence of a small satellite of mass m revolving in any orbit about the spheroid.

If we take ϕ' to be the longitude of any point on the spheroid, measured from a plane fixed in space, with which the moving plane of (y, z) coincides at time $t = 0$, then, ϕ being the longitude measured from the latter plane, we have

$$\phi' = \phi + \omega t \quad \dots\dots\dots\dots \quad (98).$$

Let $(\mu_1, \zeta_1, \phi'_1)$ be the spheroidal coordinates of the mass m at time t, ϕ'_1 being measured from the fixed initial plane. Then, at any point (μ, ζ, ϕ') whose distance from the mass is R, we have

$$V_2 = m\gamma/R \quad \dots\dots\dots\dots \quad (99).$$

Since $\zeta_1 > \zeta$, $1/R$ can be expanded in spheroidal harmonics by the formula

$$\begin{aligned} 1/R = 1/c \, \Sigma_{n=0}^{n=\infty} (2n+1) [&P_n(\mu) \, p_n(\zeta) \, P_n(\mu_1) \, q_n(\zeta_1) \\ &+ 2 \, \Sigma_{s=1}^{s=n} \{(n-s)!/(n+s)!\} \, T_n^{(s)}(\mu) \, t_n^s(\zeta) \, T_n^{(s)}(\mu_1) \, u_n^s(\zeta_1) \cos s(\phi' - \phi'_1)] \quad (100). \end{aligned}$$

Since the motion of the satellite is supposed known, $(\mu_1, \zeta_1, \phi'_1)$ are known functions of t. In order to complete the solution we must suppose the quantities $P_n(\mu_1) \, q_n(\zeta_1)$, $T_n^{(s)}(\mu_1) \, u_n^s(\zeta_1) \cos s\phi'_1$, and $T_n^{(s)}(\mu_1) \, u_n^s(\zeta_1) \sin s\phi'_1$ expanded by FOURIER'S theorem in simple harmonic functions of the time. If the period of the satellite in its orbit be $2\pi/L$, the expansion will only involve circular functions of multiples of Lt. We then write $\phi + \omega t$ for ϕ', and resolve all *products* of sines and cosines involving ϕ or t in terms of circular functions of sums and differences. The expression V_2 will now be a triple series of the same form as in equation (88), and the heights of the tides due to each term of the series may be determined separately by the application of equations (89), (90).

26. Suppose that the attracting mass rotates round the spheroid in the equatorial plane in a circle of radius $c\sqrt{(Z^2+1)}$ with angular velocity $(1-2l)\,\omega$ (referred to fixed axes).

We then have $\mu_1 = 0$, $\zeta_1 = Z$, $\phi'_1 = (1-2l)\,\omega t$, $\phi' - \phi'_1 = \phi + 2l\omega t$. Therefore

$$V_2 = m\gamma/c\, \Sigma_{n=1}^{n=\infty}(2n+1)\left[P_n(\mu)\,p_n(\zeta)\,P_n(0)\,q_n(Z)\right.$$
$$\left. + 2\Sigma_{s=1}^{s=n}\{(n-s)!/(n+s)!\}\,T_n^{(s)}(\mu)\,t_n^s(\zeta)\,T_n^{(s)}(0)\,u_n^s(Z)\cos s(\phi + 2l\omega t)\right] \quad (101).$$

We have

$$P_n(0) = 0, \ (n \text{ odd}),$$
$$P_n(0) = \frac{(-1)^{\frac{1}{2}n}.n!}{2^n(\frac{1}{2}n!)^2}, \ (n \text{ even}),$$
$$T_n^{(s)}(0) = 0, \ (n - s \text{ odd}),$$
$$T_n^{(s)}(0) = \frac{(-1)^{\frac{1}{2}(n+s)}(n+s)!}{2^n.\frac{1}{2}(n+s)!.\frac{1}{2}(n-s)!}, \ (n - s \text{ even}),$$

so that the expansion only involves harmonics which are symmetrical with respect to the equatorial plane.

The first term in the above expansion is a harmonic of the first degree and rank, and determines the motion of the spheroid as a whole about the centre of mass of the spheroid and the attracting body. This motion can be taken separately.

Taking, in the usual way,

$$h = \varpi\, \Sigma_{n=2}^{n=\infty}\{C_n P_n(\mu) + \Sigma_{s=1}^{s=n} C_n^s T_n^{(s)}(\mu)\cos s(\phi + 2l\omega t)\} \quad . \quad . \quad . \quad (102),$$

where the summation includes only even values of $n - s$, we find, by the method of § 24,

$$C_n = \frac{2n+1}{3}\,\frac{m}{M}\,\frac{p_n(\zeta)\,P_n(0)\,q_n(Z)}{K_n(\zeta) - 4q_2(\zeta)/\{n(n+1)\}}$$
$$= (-1)^{\frac{1}{2}n}.\tfrac{1}{3}(2n+1)\frac{m}{M}\,\frac{n!}{2^n(\frac{1}{2}n!)^2}\cdot\frac{p_n(\zeta)\,q_n(Z)}{K_n(\zeta) - 4q_2(\zeta)/\{n(n+1)\}}, \ (n \text{ even}) \quad (103);$$

also, if $n - s$ be even,

$$C_n^s = (-1)^{\frac{1}{2}(n+s)}\tfrac{2}{3}(2n+1)\frac{m}{M}\cdot\frac{n-s!}{2^n.\frac{1}{2}(n+s)!\,\frac{1}{2}(n-s)!}\,t_n^s(\zeta)\,u_n^s Z$$
$$\div\left\{K_n^s(\zeta) - \frac{4ls(ls-1)\,q_2(\zeta)\,D^sP_n(\nu_s)}{sD^sP_n(\nu_s) - (ls-1)\sec^2\alpha.\nu_s D^{s+1}P_n(\nu_s)/(ls)}\right\} \quad . \quad (104),$$

where, in this case,

$$\nu_s = ls \cos \alpha \,(1 - l^2 s^2 \sin^2 \epsilon)^{-\frac{1}{2}} \quad . \quad . \quad \quad . \quad . \quad . \quad . \quad . \quad (105),$$

the corresponding value of κ being ls.

Equations (102), (103), (104) fully determine the height of the forced tides and deformations on the surface of the spheroid due to the attracting body.

If, instead of one attracting mass m, we suppose two equal masses $\frac{1}{2}m$ on opposite sides of the body, the expression for V_2, and therefore for h, will contain only zonal harmonics, and harmonics of even rank, but these will be the same as before.

Harmonic Tides of the Second Order.

27. If the body be very distant from the spheroid, Z will be great, and the series (101) will converge very rapidly, so that the harmonics of the second degree are the most important.

Taking $n = 2$, $s = 2$, we get

$$C_2^2 = \frac{5}{12} \frac{m}{M} \cdot \frac{t_2^2(\zeta)\, u_2^2(Z)}{K_2^2(\zeta) - 4l(2l-1)\, q_2(\zeta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (106),$$

and the corresponding term in the height of the forced tide is

$$h_2^2 = \frac{15}{4} \frac{m}{M} \frac{(\zeta^2 + 1)\, u_2^2(Z)}{K_2^2(\zeta) - 4l(2l-1)\, q_2(\zeta)} (1 - \mu^2)\, \varpi \cos 2 (\phi + 2m\, \omega t) \quad . \quad (107).$$

If the attracting body be rotating in the same direction as the liquid, but with less angular velocity (as in all cases of astronomical interest), $2l$ lies between 0 and 1. If, in addition, $K_2^2(\zeta)$ is positive, the denominator in the expression for h_2^2 cannot vanish, and the tide produced by this term cannot therefore become very large. In fact, the angular velocities of both the free harmonic tides will lie beyond the above-mentioned limits, and will neither of them coincide with that of the attracting body. If, however, $K_2^2(\zeta) = 0$, and if, in addition, the attracting body be fixed in space, so that $2l = 1$, this forced tide will increase indefinitely. The spheroid will now have a semi-diurnal free tide, which will be fixed in space, and will coincide with the forced tide due to the attracting body. The equilibrium in the spheroidal form will therefore be completely broken up.

If the attracting body be rotating very slowly about the spheroid, the same thing will happen if $K_2^2(\zeta)$ has a certain corresponding small negative value.

Since the spheroid is secularly stable or unstable according as $K_2^2(\zeta)$ is positive or negative, the mode in which its relative equilibrium will be destroyed if $K_2^2(\zeta)$ becomes negative will depend on circumstances. If the liquid possess but little viscosity, the changes due to secular instability will take place very slowly, but the effect of the tide generating force due to an attracting body when the angular velocity of one of the free tides coincides with that of the body will become very great. If, however, the viscosity be considerable, it will prevent the forced tides from becoming large, and will cause the liquid to rapidly assume the form of a JACOBIAN ellipsoid, owing to the spheroidal form being secularly unstable.

Conclusion.

28. The results of the present paper suggest several considerations, which may possibly throw further light on the past history of the Solar system. The criteria of stability applicable to the two cases where the spheroid is formed of perfect and of viscous liquid respectively have been already discussed in § 20. In the last paragraph I have alluded to the possibility that equilibrium in the spheroidal form may be broken up by an attracting body which causes the harmonic tides of the second order to increase indefinitely, in accordance with Professor DARWIN'S hypothesis.

The present analysis, however, suggests that the same thing may happen in the case of harmonic tides of higher order than the second, and, moreover, the results arrived at concerning the number and situation of the roots of the frequency-equation render this hypothesis quite admissible. Except for harmonics of the second order, many of the free tides will rotate in the same direction as the liquid, but with less angular velocity, even though the spheroid be secularly stable; and, if an attracting body should be rotating about the spheroid with the same angular velocity as one of these tides, they would certainly rise to an enormous height, and the liquid might, perhaps, ultimately be broken up into two or more detached masses.

Take, for example, the sectorial harmonic waves of order n. If one of these be fixed in space, we must, from § 13, have the corresponding value of $\kappa = \frac{1}{2} n$, and, by (73), this leads to the condition

$$\mathrm{K}_n^n(\zeta) = (n - 2)\, q_2(\zeta).$$

If n is greater than 2, this will be satisfied for some secularly stable form of the spheroid. Under these circumstances, the presence of a fixed attracting body near the spheroid would cause the sectorial harmonic waves of the n^{th} order to increase indefinitely. The only obstacle in the way of the present supposition is that when the distance of the attracting body is at all considerable, the harmonic components of tide generating force of the higher orders become very small in comparison.

Another question of astronomical interest is whether a rotating spheroid can be broken up into one or more *rings* of rotating liquid. This can only happen if one of the zonal harmonic oscillations increase indefinitely or become unstable.

Now, as long as the spheroid continues stable for such zonal harmonic displacements when the liquid is supposed perfect, the frequencies of these oscillations will none of them vanish ; hence, their amplitude cannot be increased indefinitely by the attractions of bodies remaining in the equatorial plane of the spheroid ; and the only way in which this can take place is under the influence of the tide generating force due to a satellite whose orbit is inclined at a considerable angle to the equatorial plane of the spheroid; the effect being greatest if their planes be perpendicular. There is still the possibility that, contrary to our hypothesis in § 20, a perfect liquid spheroid may first become unstable for some displacement symmetrical about the axis ; and, unless this question be fully decided, we are not in a position to say that such is not the case, and that LAPLACE's hypothesis is wholly unfounded.

VII. *On the Magnetisation of Iron and other Magnetic Metals in very Strong Fields.*

By J. A. Ewing, *B.Sc., F.R.S., Professor of Engineering in University College, Dundee, and* William Low.

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§ 1. Early in 1887 we communicated to the Royal Society a short account of experiments made to examine the magnetic behaviour of iron when subjected to strong magnetic force by what we called the "isthmus" method of magnetisation.* Since then the experiments have been continued and extended by applying stronger magnetic forces, and by testing samples of nickel, cobalt, and various steels, as well as wrought iron and cast iron.† It may be well to preface an account of the more recent experiments by a short summary of the results stated in our earlier paper.

§ 2. The method of experiment consisted in turning the piece of metal whose magnetisation was to be examined to the form of a bobbin with a narrow neck or isthmus, and placing that between the pole-pieces of a powerful electromagnet. The sample was furnished with a spreading cone at each end, to facilitate the convergence of the lines of magnetic induction upon the central neck. The magnetisation was measured by means of an induction coil of fine wire wound in a single layer, or, in some cases, in two layers, upon the metal of the neck. Outside of this coil, and at a small definite distance from it, a second induction coil was wound in order to measure the magnetic field in the space between the two coils. This served a double purpose: it enabled a proper deduction to be made from the values of the induction measured within the inner coil, to allow for the space between the surface of the iron neck and the centre of the thickness of the coil; and it gave values of the magnetic force in the space immediately surrounding the iron, from which an inference might be drawn as to the value of the force within the neck itself. As there was no free magnetism on the circumference of the neck, in the medial plane, the force within the metal was continuous there with the force outside, and it will be shown later that when a suitable slope was given to the conical ends of the bobbin the variation of force in the medial plane in directions at right angles to the axis was so small that the external field

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* "On the Magnetisation of Iron in Strong Fields," 'Roy. Soc. Proc.,' vol. 42, p. 200.

† Preliminary notices of some of the later results were communicated to Section A of the British Association at Manchester ('Report of the British Association for 1887,' pp. 586 and 587).

6.4.89.

must have formed (in such cases) a very approximately accurate measure of the force acting on the metal. In other cases, when the cones were more blunt, the force in the external field was somewhat greater than the mean force within the metal.

§ 3. Figs. 1 and 2, copied from our earlier paper, show the forms of bobbin originally used, and the pole-pieces of the magnet by which they were magnetised. With bobbins of the type of fig. 1, the magnetic induction in the neck and the field in the surrounding air were measured by suddenly turning the bobbin round, end for end; in bobbins of the type of fig. 2, the measurements were made by suddenly withdrawing the bobbin from its place between the pole-pieces. In the latter case, the induction measured was the excess of the whole induction ($\mathfrak{B}$) above the residual induction ($\mathfrak{B}_r$) which persisted when the bobbin was drawn out. In iron bobbins the residual magnetisation was found to be sensibly constant from the lowest to the highest value of the inducing field employed in these experiments, but the form of the bobbin made the amount of this residue small. It was measured in bobbins of the type of fig. 1, by comparing the result of turning the bobbin round with the effect of drawing the bobbin out; and, in the first instance, its value in iron bobbins of the type of fig. 2, was estimated to be about the same as in bobbins of the type of fig. 1. In later experiments, when other more retentive metals were being examined, and the residual magnetism consequently formed a more important part of the whole, its value was directly determined by using built-up bobbins which allowed one conical end to be withdrawn; the residual magnetism was then determined after the bobbin had been removed from the field by slipping off (in one operation) one of the conical ends, along with an induction coil which had been wound for this purpose upon a loose ring over the central neck.

Fig. 1.

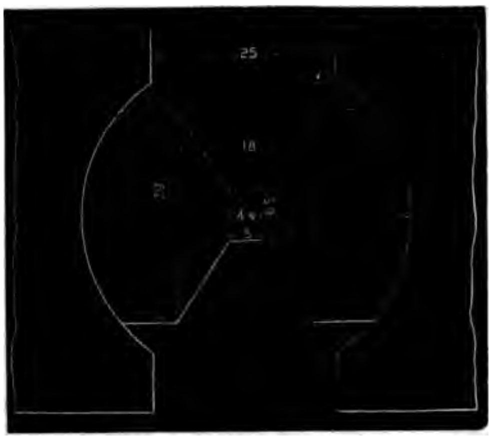
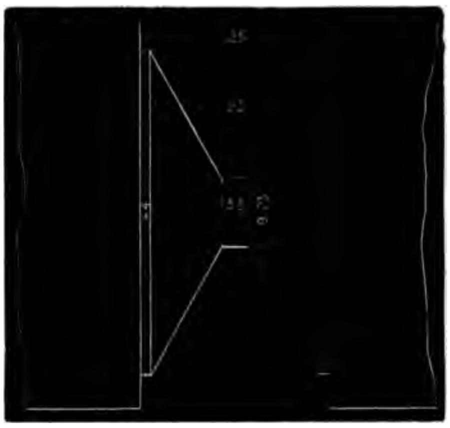


Fig. 2.



Wrought Iron.

§ 4. In the early experiments solid bobbins of the form and dimensions shown in fig. 1, were tested, one of Lowmoor, and another of Swedish wrought iron, with necks 6·5 mm. in diameter and 5 mm. long. The magnetic force was measured in an annular space between the inner and outer induction coils, about 1·3 mm. wide and closely contiguous to the iron neck : for brevity we shall call the magnetic force thus measured in the surrounding air space the "outside field." Tables I. and II. below, which are copied for convenience of reference from our earlier paper, give observed values of the induction $\mathfrak{B}$, and of the outside field for various strengths of current in the coils of the field magnets. They also give values of the quantity ($\mathfrak{B}$ — outside field)/4π, which would be a measure of the intensity of magnetisation $\mathfrak{I}$ if the outside field were fairly representative of the mean magnetic force within the metal of the neck itself (since $\mathfrak{B} = 4\pi\mathfrak{I} + \mathfrak{H}$) ; and also of the quantity $\mathfrak{B}$/outside field, which on the same proviso would measure the permeability μ. The residual induction $\mathfrak{B}_r$ was 510 in the Lowmoor and 500 in the Swedish sample. The magnetic quantities are stated in c.g.s. units.

TABLE I.—Lowmoor Wrought Iron.

Current in field magnets, ampères.	Outside field.	$\mathfrak{B}$.	$\frac{\mathfrak{B} - \text{outside field}}{4\pi}$.	$\frac{\mathfrak{B}}{\text{outside field}}$.
1·98	3,630	24,700	1680	6·80
4·04	6,680	27,610	1670	4·13
5·81	7,800	28,870	1680	3·70
7·60	8,810	29,350	1630	3·33
11·0	9,500	30,200	1650	3·18
13·5	9,780	30,680	1660	3·14
16·2	10,360	30,830	1630	2·98
21·6	10,840	31,370	1630	2·89
26·8	11,180	31,560	1620	2·82

TABLE II.—Swedish Wrought Iron.

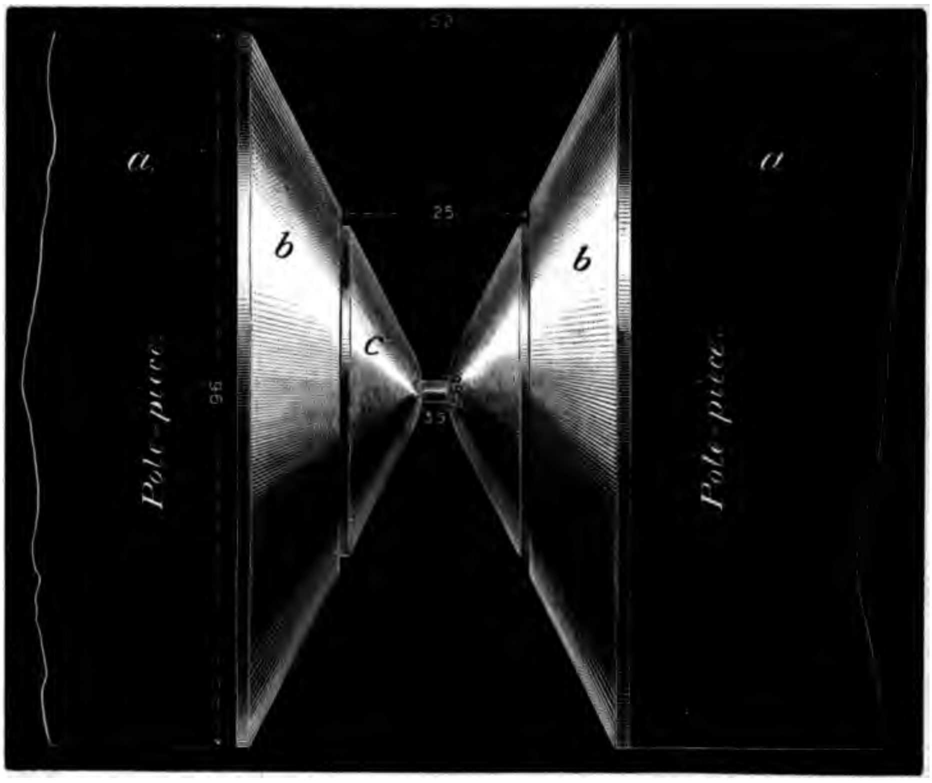
Current in field magnets, ampères.	Outside field.	$\mathfrak{B}$.	$\frac{\mathfrak{B} - \text{outside field}}{4\pi}$.	$\frac{\mathfrak{B}}{\text{outside field}}$.
4·08	6,690	27,960	1700	4·18
7·77	8,900	29,730	1660	3·34
10·9	9,510	30,820	1700	3·24
14·2	10,000	31,210	1690	3·12
16·5	10,360	31,630	1700	3·05
18·9	10,810	31,720	1670	2·94
22·9	10,880	32,060	1690	2·95
26·5	11,200	32,360	1690	2·90

Results closely accordant with these were also obtained with samples of the form shown in fig. 2; and, as bobbins with flat ends were most convenient, especially in very strong fields, the subsequent experiments were all made with the flat-ended type.

§ 5. The pole-pieces of the magnet used in the early experiments were only 5·25 cm. square, and it was clear that with a larger magnet a greater concentration of the induction might be produced. Professor TAIT was kind enough to lend us the large magnet of the Edinburgh University Laboratory, and all the subsequent work has been done with it.

The Edinburgh magnet has a pair of vertical limbs about 50 cm. long and 10·7 cm. in diameter. These are united by a horizontal yoke at the bottom, and are furnished on the top with movable pole-pieces in the form of rectangular blocks of soft wrought iron, the cross-section of which is 9·6 cm. square. The limbs are wound along a length of 49 cm. with a number of coils which are grouped in series, making about 1600 turns in all. The currents employed by us ranged up to 40 ampères, and the greatest value of the line-integral of the magnetic force was consequently about 80,000. To allow the old bobbin, of the form of fig. 1, to be effectively used, we furnished the magnet poles with a pair of intermediate conical pieces of soft wrought iron, which virtually formed an extension of the bobbin's ends. Fig. 3 is a full-size sketch of the poles a, a, with the intermediate pieces b, b, and the bobbin c, in place. The figure shows the size to which the neck of the bobbin was finally turned down in experiments which are described below. The dimensions are given in millimetres.

Fig. 3.



§ 6. In the first instance, however, a Lowmoor bobbin of the dimensions shown in fig. 1, which had been used in the earlier observations, was tested in the Edinburgh magnet without being turned down. The following are the particulars of this experiment and the results :—

Diameter of central neck, 9·23 mm. Length of neck, 3·5 mm.

Diameter to middle of inner induction coil, 9·48 mm.

Diameter to middle of outer induction coil, 10·99 mm.

Inner induction coil, a single layer of twelve turns of silk-covered wire, 0·25 mm. in diameter over the silk.

Outer induction coil, a single layer of seven turns of the same wire.

TABLE III.—Lowmoor Wrought Iron.

Current in field magnets, ampères.	Outside field.	$\mathfrak{B}$.	$\frac{\mathfrak{B} - \text{outside field}}{4\pi}$.	$\frac{\mathfrak{B}}{\text{outside field}}$.
15·0	15,990	36,460	1630	2·28
18·5	18,410	36,970	1480	2·01
28·5	18,380	37,320	1510	2·04
33·0	18,570	37,610	1520	2·03
38·5	18,900	37,990	1520	2·01

A Swedish iron sample of the same shape gave an induction of 37,620 in a field of about the same force, with a current of 38 ampères in the field magnet coils.

§ 7. To push the induction in the Lowmoor sample to a still higher value, the neck of the bobbin was turned down to a diameter of 3·97 mm., the slope of the conical ends being approximately maintained. The inner coil was re-wound with a mean diameter of 4·22 mm., and the outer coil with a mean diameter of 5·7 mm. The following are the results for a magnetising current of about 38 ampères, the mean of several measurements being taken :—

Outside field.	$\mathfrak{B}$.	$\frac{\mathfrak{B} - \text{outside field}}{4\pi}$.	$\frac{\mathfrak{B}}{\text{outside field}}$.
25,620	43,500	1430	1·7

§ 8. A final effort to raise the induction was then made by again turning down the neck of the sample to the size shown in fig. 3. The diameter of the neck was reduced to 2·66 mm., but its length was not reduced in the same proportion : to leave room for a sufficiently long induction coil, a little of the metal of the cones close to the neck was cleared away in the manner shown by the sketch. The section of the neck was now less than $\frac{1}{1500}$ that of the pole-pieces. The bobbin was annealed after being turned down. The inner induction coil was wound in a single layer of ten turns with a mean diameter of 2·93 mm. The outer coil was a single layer of eight turns with a mean diameter of 4·36 mm. With these conditions the induction was forced to the enormous value of 45,350 c.g.s. units, though the outside field between the two coils had a somewhat smaller value than before. This anomaly does not necessarily imply that the measurements were in error, for, as will appear from what follows, the relation of the outside field to the force within the metal is materially affected by the form of the conical ends, and that form had been altered, as has just been said, in the region close to the neck. The excessive smallness of the neck in this case, however, made it more difficult than before to measure the outside field with precision. The following are mean results for the strongest magnetising currents :—

Outside field.	$\mathfrak{B}$.	$\dfrac{\mathfrak{B} - \text{outside field}}{4\pi}$.	$\dfrac{\mathfrak{B}}{\text{outside field}}$.
24,500	45,350	1660	1·85

§ 9. With regard to the quantity ($\mathfrak{B}$ − outside field)/4π, it will be noticed that, if we exclude the last (somewhat doubtful) case, there is a progressive decrease as the induction rises, within the range covered by these experiments. With a field of 5000 or 6000, the value of this quantity was 1700 in the Swedish sample and 1680 in the Lowmoor sample, and it fell to 1430 as the field was raised to 25,000. This gives great interest to the question, whether the field as measured in the outside space has the same, or nearly the same, value as the magnetic face within the metal; for in that case we should have evidence that the intensity of magnetisation $\mathfrak{I}$ passes a maximum and begins to decrease under the action of very strong fields, and this is a result which WEBER's molecular-current theory of diamagnetism, extended as MAXWELL has extended it to a paramagnetic substance, would lead us to expect.* After a careful examination of this important point, we have concluded, for reasons given below, that the apparent decrease of $\mathfrak{I}$ in the experiments described above is in all probability wholly due to the outside field being greater than the field within the metal, and that, if there is any variation in the real value of $\mathfrak{I}$ in strong fields, it is smaller than our method of experiment can detect.

§ 10. An attempt to investigate the uniformity of the field (in a medial plane along lines radiating from the axis) was made by building up a bobbin over the neck of which four induction coils were wound, one above another, with small annular spaces between. The lowest coil was wound on the iron neck, and the other three on thin brass tubes turned to slip one on another, with flanges at the ends to preserve a definite clearance between them and to keep them concentric. The innermost coil had a mean diameter of 3·2 mm., and the outermost a mean diameter of [illegible] mm. The annular space between them, 2·3 mm. thick, was in this way divided into three parts in each of which the field was measured. It was found that in every case the field decreased progressively with increase of distance from the axis. Thus, in one instance, the field fell from [illegible] in the innermost ring to [illegible] in the outermost ring. It is unnecessary to describe at length these experiments, which were very laborious and which did not throw any light on the important question of how closely the force within the metal approximated to the force in the air close to the surface of the neck. Moreover, the form of the [illegible] [illegible] used in this case was [illegible] afterwards recognised, such as to give a much more uniform field than the [illegible] formerly used.

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* See MAXWELL's 'Treatise on Electricity and Magnetism,' vol. 2, chap. 22:—"If it should ever be experimentally proved that the temporary magnetisation of any substance first increases and then diminishes as the magnetising force is continually increased, the evidence of the existence of these molecular currents would, I think, be raised almost to the rank of a demonstration."

Concentration of Magnetic Force by Conical Pole-faces.

§ 11. The magnetic force in the space between the pole-pieces is made up of two parts: (1) the electromagnetic force directly produced by the current in the field-magnet coils; and (2) the force due to free magnetism distributed [illegible] [illegible] over the pole-faces. The first of these was a comparatively small part, and that [illegible] of the whole and its value must have been sensibly uniform at such small distances from the axis as those with which we are now concerned. In considering the uniformity of the field we need only deal with the force produced by free magnetism distributed over the opposing surfaces of the poles.

Fig.

§ 12. The free magnetism on each pole-face may be treated as made up of a series of co-axial circular rings [illegible] [illegible] [illegible] the axis of magnetisation. [illegible] [illegible] whole free magnetism [illegible] [illegible] at a point in the axis [illegible] [illegible] [illegible] [illegible] [illegible]. The [illegible] [illegible] which gives

$$x = \frac{r}{\sqrt{2}}; \quad \tan\theta = \sqrt{2}; \quad \theta = 54^\circ\, 44'.$$

Hence, a series of co-axial rings will be most advantageously disposed for producing force at a point on the axis if they lie on a cone having its vertex at that point, with a semi-vertical angle of 54° 44′. This conclusion is independent of the distribution of density from ring to ring.

§ 13. The greatest force will be produced when the pole-pieces are themselves saturated, so that $\mathfrak{J}$ reaches its limiting value in all parts of the metal. In that case the distribution of density from ring to ring is uniform. The surface density of free magnetism at any point of a sloping pole-face is $\mathfrak{J} \sin\theta$, where θ is the slope of the face to the axis of magnetisation. The whole quantity in each ring is $\mathfrak{J}$ multiplied by the area of the ring projected upon a plane normal to the axis—a quantity which is independent of the slope of the cone. We have, therefore, the same series of attracting rings to deal with, whatever be the slope of the convergent forces, and whether that slope be uniform or not. Given, then, a certain diameter for the neck of the bobbin to be magnetised, the greatest magnetic force will be produced at the middle of the axis of the neck if we make the expanding ends and pole faces in the form of cones, with a semi-angle of 54° 44′, and with their vertices at the middle of the neck.*

§ 14. This cone of maximum concentrative power is not, however, the form best suited for producing a uniform field. At any point in the axis $d\mathrm{F}/dy$ and $d\mathrm{F}/dz$ are zero, axes of y and z being taken in a plane parallel to the rings, and the condition for a uniform field (uniform, namely, in the neighbourhood of the axis, over a transverse plane) is that $d^2\mathrm{F}/dy^2$ and $d^2\mathrm{F}/dz^2$ shall also be zero. Consider again the attraction of a ring at any point O in the axis. Taking LAPLACE'S equation—

$$\frac{d^2\mathrm{V}}{dx^2} + \frac{d^2\mathrm{V}}{dy^2} + \frac{d^2\mathrm{V}}{dz^2} = 0,$$

and differentiating with respect to x, we have

$$\frac{d^2}{dx^2}\frac{d\mathrm{V}}{dx} + \frac{d^2}{dy^2}\frac{d\mathrm{V}}{dx} + \frac{d^2}{dz^2}\frac{d\mathrm{V}}{dx} = 0.$$

By symmetry of the field about the axis of x the second and third terms are equal; hence, writing F for $d\mathrm{V}/dx$,

$$\frac{d^2\mathrm{F}}{dx^2} + \frac{2d^2\mathrm{F}}{dy^2} = 0.$$

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* The corresponding proposition for truncated cones, with an air space between them, has lately been stated by Professor STEFAN ('Wien, Akad. Sitzber.,' Feb. 9, 1888; or 'Phil. Mag.,' vol. 25, p. 322).

The condition for a uniform field will therefore be satisfied when $d^2\mathrm{F}/dx^2 = 0$; that is, when

$$-\frac{9x}{l^5} + \frac{15x^3}{l^7} = 0,$$

which gives $x = r\sqrt{\tfrac{3}{2}}$; $\tan\theta = \sqrt{\tfrac{2}{3}}$; $\theta = 39^\circ\ 14'$. Thus, the condition is satisfied when the pole-faces are cones converging as before upon the middle of the neck, but with a semi-vertical angle of 39° 14'.*

§ 15. With a cone of any semi-angle θ, the surface density of free magnetism being $\mathfrak{J}\sin\theta$, the force at the vertex due to a ring at an axial distance x, of radius r, and of length dl, measured along the slope, is

$$2\pi r\,dl.\ \mathfrak{J}\sin\theta.\ x/l^3, \quad \text{or} \quad 2\pi\mathfrak{J}\sin^2\theta\cos\theta\,dr/r.$$

The whole force at the vertex is

$$2\pi\sin^2\theta\cos\theta\int_a^b\frac{\mathfrak{J}dr}{r},$$

a being the radius of the neck on which the cone converges, and b the radius of the base to which it spreads.

Hence (treating $\mathfrak{J}$ as uniform), with a pair of truncated cones, joined by a neck at the middle of which they have their common vertex, the whole force there is

$$\mathrm{F} = 4\pi\mathfrak{J}\sin^2\theta\cos\theta\log_e\frac{b}{a},$$

which, for convenience of calculation, we shall write

$$\mathrm{F} = 28{\cdot}935\,\mathfrak{J}\sin^2\theta\cos\theta\log_{10}\frac{b}{a}\cdot$$

§ 16. Applying this to the cones of maximum concentrative power (§ 13), in which $\sin\theta = \sqrt{\tfrac{2}{3}}$ and $\cos\theta = \frac{1}{\sqrt{3}}$,

$$\mathrm{F}_{max.} = 11{\cdot}137\,\mathfrak{J}\log_{10}\frac{b}{a},$$

and the greatest value of the force will be obtained when $\mathfrak{J}$ has the saturation value (of say 1700 c.g.s. units for soft wrought iron), in which case

$$\mathrm{F}_{max.} = 18930\log_{10}\frac{b}{a},$$

an expression which measures the greatest possible force which the "isthmus" method of magnetisation can apply at a point in the axis of the bobbin (over and above the small force which is directly produced by the magnet coils). It is impracticable to produce quite so great a force as this on account of the difficulty of saturating the magnet poles.

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* We are indebted to Mr. A. TANAKADATE for suggesting this calculation of the form of poles suited to give a uniform field.

§ 17. With the cones which give the most uniform field, for which $\sin\theta = \sqrt{\frac{2}{5}}$ and $\cos\theta = \sqrt{\frac{3}{5}}$, the value of F is only

$$8{\cdot}965\ \mathfrak{I}\log_{10}\frac{b}{a},$$

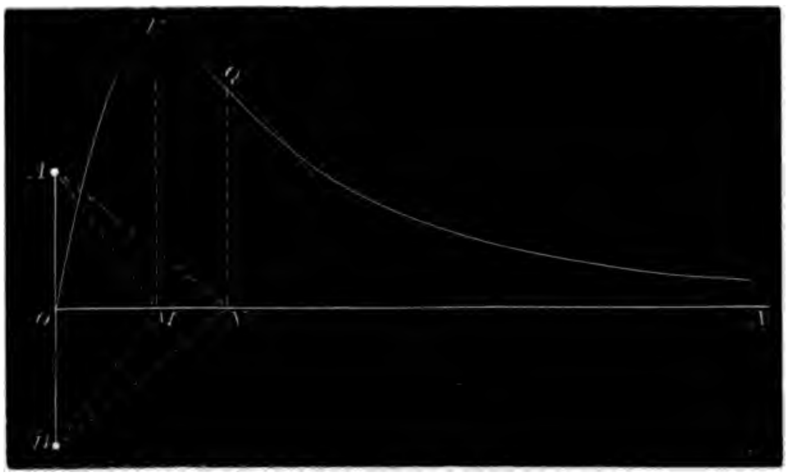
which becomes

$$15240\log_{10}\frac{b}{a}$$

in the event of the pole-pieces being of soft wrought iron and saturated.

§ 18. The curve, fig. 5, has been drawn to show how the force at the vertex varies when the angle of the cone is altered.

Fig. 5.



The base of the cone being represented by AB, any ordinate PM gives the force when the vertex is at M. In the figure, AMB represents the cone of maximum concentrative power, and ANB represents the cone giving a uniform field in the neighbourhood of the axis, Q being the point of inflection in the curve.

§ 19. In figs. 6 and 7 the same two cases are further illustrated by curves which show the sum of the forces due to two equal and opposite rings (situated on cones with a common vertex) at points along the axis.

By summing up the effects of such pairs, for the whole cone, we may judge how nearly the force is uniform from end to end of the neck of the magnetised bobbin. In a bobbin with cones of semi-angle 39° 14′ the field is sensibly uniform from end to end of the neck, except close to the ends, where it is slightly reduced, and (§ 14) this longitudinal uniformity implies transverse uniformity.

§ 20. When the semi-vertical angle of the cones is greater than 39° 14′, the force at points on the axis has a maximum at the common vertex, and, since $d^2\mathrm{F}/dx^2$ and $d^2\mathrm{F}/dy^2$ have opposite signs (§ 14), the field is stronger at places near the axis than on the axis itself. In a bobbin with a narrow neck this may have the effect of making the field in the closely surrounding air space greater than the mean field within the neck.

§ 21. We may now apply the above conclusions to elucidate the experiments which have been described. The form of bobbin used in them had been chosen, without reference to theory, as one likely to give a strong concentration of magnetic induction, and it chanced to come very near the best form for this purpose. The cones had a semi-angle of 60°, and their vertices were nearly coincident (overlapping very slightly, see fig. 2).

Applying the formula of § 15, we have, for $\theta = 60°$,

$$F = 10{\cdot}85\ \mathfrak{J} \log_{10} \frac{b}{a},$$

which is only two and a half per cent. short of the force attainable by using cones of maximum concentrative power. Moreover, it must be borne in mind that in actual use of the isthmus method the strongest induction will be reached when the semi-angle is rather *greater* than 54° 44′, for $\mathfrak{J}$ is itself a function of the angle, decreasing when the angle is decreased, on account of the augmented "resistance" of the whole magnetic circuit. For this reason we probably obtained as much concentration with cones of 60° as we should have obtained with cones of 54° 44′. Further, in the last experiment (§ 8), the neck of the bobbin had been turned down to the smallest size we found it practicable to work with. It is clear, therefore, that no materially higher value of $\mathfrak{B}$ than the value already obtained was possible with the apparatus at our disposal.

Fig. 6.

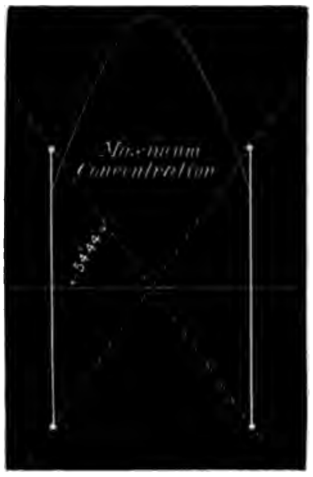
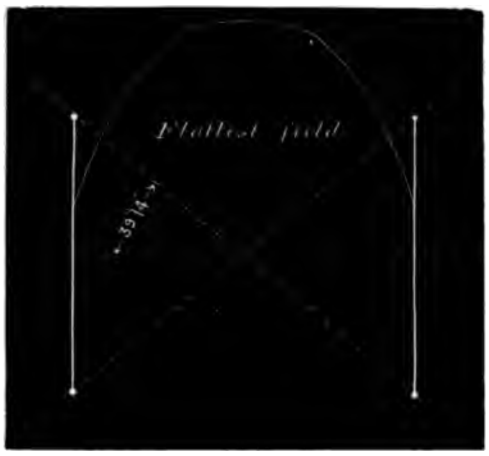


Fig. 7.



§ 22. From the relation of the line integral of magnetic force to the length of iron and mean length of air in the magnetic circuit, we estimate that the mean value of $\mathfrak{J}$

in the magnet cores and pole-pieces cannot have exceeded 1400 when the magnetising current was at its strongest. The distribution of this over the conical pole-faces was not uniform. Close to the neck it reached the saturation value (of, say, 1680 or 1700), being gathered there at the expense of outlying portions. This want of uniformity of $\mathfrak{I}$ in the pole-faces increases the magnetic force in the neck, but when a distribution of $\mathfrak{I}$ is assigned it is easily taken account of in applying the formula of § 15.

§ 23. Taking the experimental case (§ 7), in which the diameter of the neck was $\frac{1}{25}$ of the diameter to which the cones spread, we calculate that the magnetic force at the middle of the axis was probably about 22,500, and at other points of the axis it was less.

Now, the measured value of the field in the air, close to the bobbin's neck, was in this instance 25,620. To produce this force at the axis would require that the value of $\mathfrak{I}$ in the pole-pieces should have been nothing less than 1690 all over, that is to say, it would require that the poles should have been saturated from axis to circumference—a quite impossible supposition. It is clear that the force in the air close to the neck was in this case distinctly greater than the mean force within the neck.

The measured induction $\mathfrak{B}$ within the neck was 43,500. If we accept 22,500 as the mean value of $\mathfrak{H}$ within the neck (remembering that while $\mathfrak{H}$ increases from the axis to the circumference it diminishes from the middle towards the ends), the value of $\mathfrak{I}$ in the neck would be $(43{,}500 - 22{,}500)/4\pi = 1670$, which is, as nearly as may be judged, the same value as was produced by the application of magnetising forces of moderate strength.

§ 24. These considerations establish a very strong presumption that the apparent decrease of $\mathfrak{I}$ in the experiments, that is to say, the observed decrease in the quantity $(\mathfrak{B} - \text{outside field})/4\pi$ under very strong forces is to be explained by the fact that the outside field was stronger than the field within the neck; and that the true value of $\mathfrak{I}$ is sensibly constant throughout the range of magnetic forces examined, namely, from about 4000 to 24,000 c.g.s. units.

Further Experiments on Wrought Iron.

§ 25. To put this matter further to the proof, we continued the experiments with another bobbin, also of Lowmoor wrought iron, the conical ends of which were shaped so as to produce a much more uniform field. The shape which would give the most uniform field was not chosen, for that would have imposed so low an upper limit on the strength of the field that the test would have been rather inconclusive. By way of compromise, a bobbin was turned of the shape and dimensions shown in fig. 8, with cones of semi-angle 45°, as a form which combined high concentrative power with a fair approximation to uniformity of field. The advantage, in respect of uniformity of field, which this bobbin had over the one formerly used may be judged from figs. 9 and 10, which show the longitudinal variation of force due to a pair of rings in the two cases. The length and diameter of the neck were 3·42 mm. The outside field and the induction were measured as before, and it was found that they were decidedly less than in the former instance, chiefly, of course, because of the greater mean thickness of air space now present between the magnet poles, which reduced the mean value of $\mathfrak{I}$ in them. But what is important to our present purpose is to note that now, owing to the greater uniformity of the field, the quantity ($\mathfrak{B}$ — outside field)/4π undergoes no progressive diminution as the force rises. Table IV. gives the results. They confirm the conclusion which was provisionally stated in § 24. Here we may accept the strength of the outside field as closely approximating to the mean force within the neck, so that the first column in the table might have been styled $\mathfrak{H}$, the second last column $\mathfrak{I}$, and the last column μ.

Fig 8.

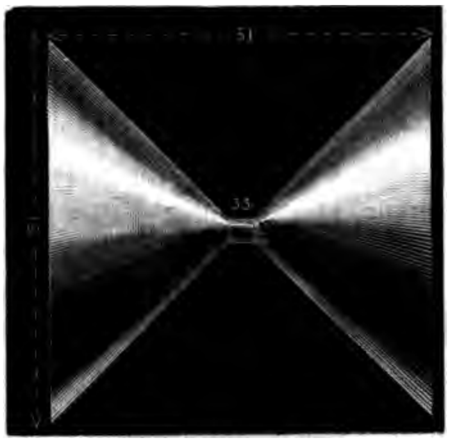


Fig. 9.

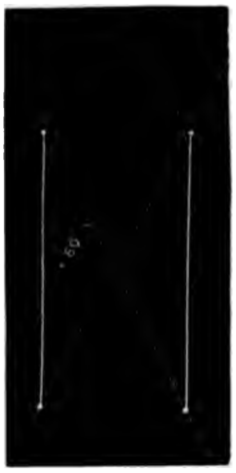


Fig. 10.

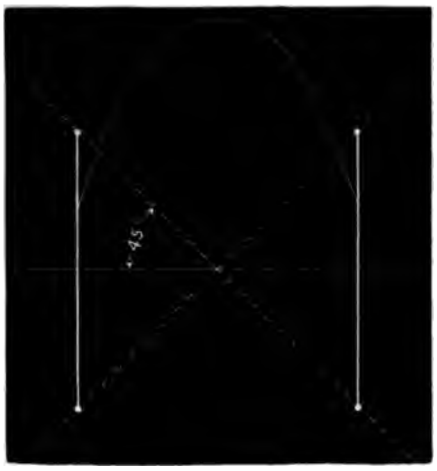


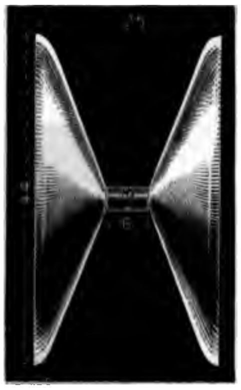
TABLE IV.—Lowmoor Wrought Iron.

Outside field.	$\mathfrak{B}$.	$\frac{\mathfrak{B}\ -\ \text{outside field}}{4\pi}$.	$\frac{\mathfrak{B}}{\text{outside field}}$.
3,080	24,130	1680	7·83
6,450	28,300	1740	4·39
10,450	32,250	1730	3·09
13,600	35,200	1720	2·59
16,390	36,810	1630	2·25
18,760	39,900	1680	2·13
18,980	40,730	1730	2·15

§ 26. It remains to describe the results of tests of other samples of wrought iron and of cast iron, of various steels, manganese steel, nickel, and cobalt. Messrs JOWITT and SONS, of Sheffield, were kind enough to supply specimens of pure Swedish iron and of steel, with which a number of experiments were made.

The following results were obtained with a bobbin of Swedish iron, described by Messrs. JOWITT as of the " L^{s}Lancash." brand, a good Swedish iron made by the Lancashire hearth process. The form and size of the bobbin are shown in fig. 11.

Fig. 11.



The cones were blunt, and their vertices were at some distance from one another, the general effect being similar to that of sharper cones with a common vertex, giving a fairly uniform, but not excessively strong, field.

TABLE V.—Swedish Iron, " L^{s}Lancash." Brand.

Outside field.	$\mathfrak{B}$.	$\frac{\mathfrak{B}\ -\ \text{outside field}}{4\pi}$.	$\frac{\mathfrak{B}}{\text{outside field}}$.
1,490	22,650	1680	15·20
3,600	24,650	1680	6·85
6,070	27,130	1680	4·47
8,600	30,270	1720	3·52
18,310	38,960	1640	2·13
19,450	40,820	1700	2·10
19,880	41,140	1700	2·07

It will be noticed that the quantity in the third column, which no doubt approximates very closely to the intensity of magnetisation $\mathfrak{I}$, is practically constant, except for errors of observation. Its mean value is 1685.

§ 27. Another bobbin, described by Messrs. Jowitt as Swedish wrought iron of the celebrated Ⓛ brand, the purest and most expensive iron used in commerce, made by the Walloon process, was also turned to the shape shown in fig. 11, and tested as follows :—

Table VI.—Fine Swedish Iron, Ⓛ Brand.

Outside field.	$\mathfrak{B}$.	$\frac{\mathfrak{B} - \text{outside field}}{4\pi}$.	$\frac{\mathfrak{B}}{\text{outside field}}$.
5,310	25,670	1620	4·83
17,680	38,080	1620	2·15
19,240	39,540	1620	2·06

It would seem that the saturation value of $\mathfrak{I}$ is specifically less in this iron than in previous samples.

Cast Iron.

§ 28. Table VII., which is copied from our former paper, gives the results of tests made with a bobbin of cast iron of a form similar to that shown in fig. 1, the magnetisation being measured by reversing the bobbin in the field. The residual induction was also measured by withdrawing the bobbin, and was found to have a nearly constant value of about 400 c.g.s. units.

Table VII.—Cast Iron.

Outside field.	$\mathfrak{B}$.	$\frac{\mathfrak{B} - \text{outside field}}{4\pi}$.	$\frac{\mathfrak{B}}{\text{outside field}}$.
3,900	19,660	1250	5·04
6,400	21,930	1240	3·42
7,710	22,830	1200	2·96
8,080	23,520	1230	2·91
9,210	24,580	1220	2·67
9,700	24,900	1210	2·57
10,610	25,600	1190	2·46

§ 29. Table VIII. relates to a later test, made with Professor Tait's magnet, in which a central spindle of cast iron was used to form the neck of the bobbin, but the conical ends were of wrought iron shrunk on to the ends of the spindle. Fig. 12 shows a section of this bobbin. A similar construction has been adopted in many other cases; the use of the more permeable metal—wrought iron—for the cones has, of course, the advantage of strengthening the induction in the neck. Here the residual magnetism was measured, after the observations were otherwise complete, by slackening one of the cones, so that it might be slipped off the spindle. A suitable induction coil, wound on a ring, was then slipped on; the whole bobbin was magnetised and removed from the field, the loose end and the coil were then slipped off together, and the ballistic effect of this was observed. In these measurements the bobbin was demagnetised by the method of reversals, to get rid of the effect of previous stronger magnetisations. A similar procedure was followed in finding the residual magnetism of steel samples. The residual magnetism is allowed for in the values of $\mathfrak{B}$ given below.

Fig. 12.

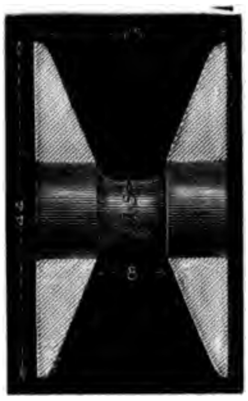


TABLE VIII.—Cast Iron.

Current in field magnets, ampères.	Outside field.	$\mathfrak{B}$.	$\frac{\mathfrak{B} - \text{outside field}}{4\pi}$.	$\frac{\mathfrak{B}}{\text{outside field}}$.
1·57	4,560	20,070	1230	4·40
3·62	9,120	24,630	1230	2·70
5·95	11,770	27,680	1270	2·35
8·1	13,460	28,710	1210	2·13
14·2	14,690	30,160	1230	2·05
23·0	16,200	30,920	1170	1·91
40·0	16,900	31,760	1180	1·88

These two experiments agree in assigning about 1240 as the saturation value of $\mathfrak{I}$ in this cast iron; and the apparent diminution in fields of the greatest strength is, of course, to be set down to the cause which has been fully explained in connection with wrought iron—an excess of the "outside field" over the mean force within the metal, owing to the cones being too blunt to give a very uniform field.

Steel.

§ 30. Of the following experiments, Nos. 1 to 5 were made with samples of steel supplied by Messrs. Jowitt, containing various percentages of carbon. The sample was built, in each case, into a bobbin of the form shown in fig. 13, with wrought iron cones. No. 6 was made with a specimen of Whitworth's fluid compressed steel, built with wrought iron cones into the bobbin of fig. 14. Observations were made in the strongest fields only.

Fig. 13.

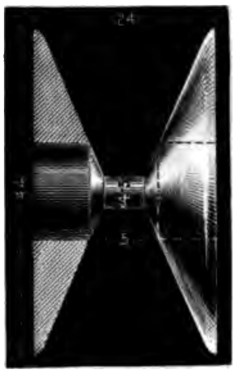


Fig. 14.

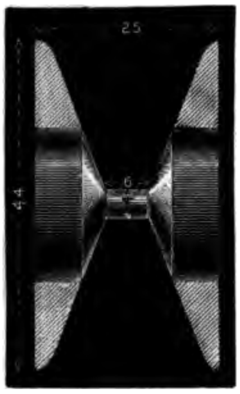


Table IX.—Steel of Various Qualities.

Description of steel.	Outside field.	$\mathfrak{B}$.	$\frac{\mathfrak{B} - \text{outside field}}{4\pi}$.	$\frac{\mathfrak{B}}{\text{outside field}}$.
1. **Bessemer** steel, containing about 0·4 per cent. of carbon	17,610	39,880	1770	2·27
2. **Siemens-Martin** steel, containing about 0·5 per cent. of carbon	18,000	38,860	1660	2·16
3. Crucible steel for making chisels, containing about 0·6 per cent. of carbon	19,470	38,010	1480	1·95
4. Finer quality of crucible steel for chisels, containing about 0·8 per cent. of carbon	18,330	38,190	1580	2·08
5. Crucible steel, containing about 1 per cent. of carbon	19,620	37,690	1440	1·92
6. **Whitworth** fluid compressed steel	18,700	38,710	1590	2·07

§ 31.—The following tests were made with a piece of Vickers' tool steel, built with wrought iron cones into the bobbin shown in fig. 15. In this case the magnetising field must have been sufficiently uniform to make the first column in the table represent $\mathfrak{H}$, the last μ, and the second last column $\mathfrak{I}$ very nearly. This steel had great coercive force; the residual magnetic induction (entered in the table under $\mathfrak{B}_r$) was scarcely constant, even in fields of over 10,000, and $\mathfrak{I}$ appeared to be still increasing in the strongest field to which the experiment extended.

Fig. 15.

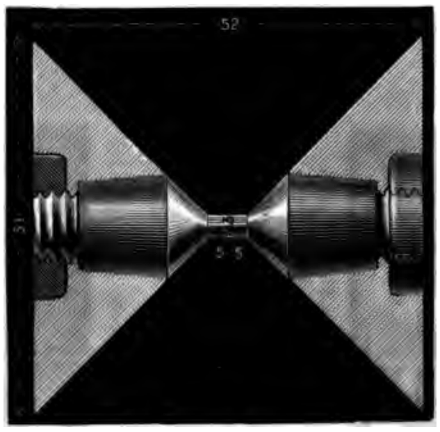


TABLE X.—VICKERS' Tool Steel.

Outside field. ($\mathfrak{H}$.)	$\mathfrak{B}_r$.	$\mathfrak{B}$.	$\frac{\mathfrak{B} - \text{outside field}}{4\pi}$. ($\mathfrak{I}$.)	$\frac{\mathfrak{B}}{\text{outside field}}$. ($\mu$.)
6,210	7350	25,480	1530	4·10
9,970	7670	29,650	1570	2·97
12,170	8000	31,620	1550	2·60
14,660	8030	34,550	1580	2·36
15,530	8030	35,820	1610	2·31

Taken together, the experiments on steel render it probable that there are specific differences in the saturation values of $\mathfrak{I}$ for different steels, smaller values being found in high- than in low-carbon steels. This is to be expected, in view of the decidedly low saturation value of $\mathfrak{I}$ found in cast iron.

Manganese Steel.

§ 32.—At the suggestion of Dr. J. HOPKINSON, we have examined the action of strong magnetic forces upon the remarkable alloy of iron and manganese lately introduced by Mr. R. A. HADFIELD, of Sheffield, which has many peculiar mechanical and electrical properties.* The experiments of HOPKINSON,† BOTTOMLEY,‡ and BARRETT § have shown that this steel is almost wholly destitute of magnetic susceptibility. Hopkinson found that a force $\mathfrak{H}$ of 244 produced an induction $\mathfrak{B}$ of 310, which makes the permeability only 1·27. Mr. Hadfield was kind enough to supply us with a bar which contained about 12 per cent. of manganese and 0·8 per cent. of carbon. The metal is excessively hard, but, by raising the bar to a bright red heat and quenching it in water, it was softened sufficiently to allow pieces to be turned, with considerable difficulty, into forms suitable for these experiments.

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* See Mr. HADFIELD's paper on Manganese Steel, 'Inst. Civ. Engin. Proc.,' February 28, 1888.

† "Magnetisation of Iron," 'Phil. Trans.,' 1885, p. 462.

‡ 'Report of the British Association for 1885,' p. 903.

§ 'Roy. Dublin Soc. Proc.,' vol. 5, 1886, p. 360.

One piece of the bar was turned into a solid bobbin, of the size and shape shown in fig. 16, and with that the following observations were made :—

Fig. 16.

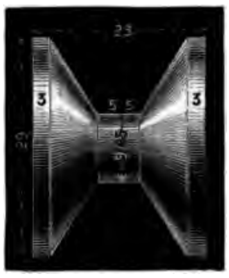
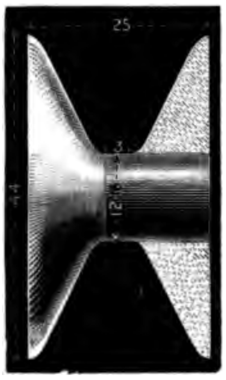


Table XI.—Hadfield's Manganese Steel.

Outside field.	$\mathfrak{B}$.	$\frac{\mathfrak{B} - \text{outside field}}{4\pi}$.	$\frac{\mathfrak{B}}{\text{outside field}}$.
2000	2770	61	1·38
3250	4560	104	1·40
3720	5090	109	1·37
4100	6010	152	1·47
5200	7320	185	1·41

Fig. 17.



§ 33. To push the induction to higher values, another bobbin was built up (fig. 17), with a central spindle cut from the bar of manganese steel, and with cones of wrought iron. The following measurements were made with it :—

TABLE XII.—HADFIELD'S Manganese Steel.

Outside field.	$\mathfrak{B}$.	$\frac{\mathfrak{B} - \text{outside field}}{4\pi}$.	$\frac{\mathfrak{B}}{\text{outside field}}$.
1930	2,620	55	1·36
2380	3,430	84	1·44
3350	4,400	84	1·31
5920	7,310	111	1·24
6620	8,970	187	1·35
7890	10,290	191	1·30
8390	11,690	263	1·39
9810	14,790	396	1·51

§ 34. The figures in the last two columns of Tables XI. and XII. show as much regularity as can be expected, when it is borne in mind that they depend upon the small differences between two large quantities which had to be separately measured. The two sets of results agree well. They show that the permeability of manganese steel is, as nearly as may be judged, constant from fields of 2000 to 10,000 units, with a value approximating to 1·4 in this sample. It follows from this that, notwithstanding the excessive resistance which this material opposes to being magnetised, a respectably high intensity of magnetisation will be produced by the application of a sufficiently strong force. In the second experiment $\mathfrak{I}$ was raised to nearly 400. It is very remarkable that scarcely any of this magnetisation remains when the force is withdrawn. One might have expected that a material which resists magnetisation so strongly would possess much coercive force. In fact, however, the residual magnetism (which was determined in the second sample in the usual way, by slipping off one of the iron cones along with an induction coil) was so small that it scarcely admitted of measurement by the apparatus which served to measure the induced magnetism. After applying the strongest field the value of the residual induction $\mathfrak{B}_r$ was only about 30. It is well known that with ordinary iron and steel the magnetisation wholly, or almost wholly, disappears when the magnetising force is withdrawn, provided the force is less than a certain amount. This elastic stage in the process of magnetisation, the limits of which are exceedingly narrow in soft wrought iron, but somewhat wider in hard iron, common steels, and nickel, seems to extend, in manganese steel, up to the strongest force we have been able to apply.

Nickel.

§ 35. Two specimens of nickel, supplied by Messrs. JOHNSON and MATTHEY, have been tested. The first was cut from a bar previously used in examining the permeability of nickel when in a state of compression under the action of ordinarily weak magnetising forces.* It contained about 0·75 per cent. of iron. The bar was annealed and was built into a bobbin with wrought iron cones, and the neck was turned down to a diameter of 4 mm. (fig. 18).

Fig. 18.

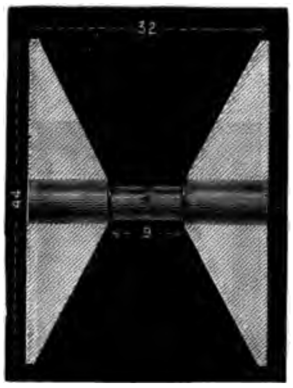


Table XIII.—Annealed Nickel (0·75 per cent. of Iron).

Outside field.	$\mathfrak{B}$.	$\frac{\mathfrak{B} - \text{outside field}}{4\pi}$.	$\frac{\mathfrak{B}}{\text{outside field}}$.
3,450	9,850	510	2·86
6,420	12,860	510	2·00
8,630	15,260	530	1·77
11,220	17,200	480	1·53
12,780	19,310	520	1·51
13,020	19,800	540	1·52

Here $\mathfrak{I}$ is sensibly constant, with a mean value of 515, of which about 160 was residual.

§ 36. The second sample was a hard-drawn wire, not annealed before testing, which contained less iron than the other (0·56 per cent.). Perhaps for this reason, the value of $\mathfrak{I}$ in it was less. The nickel formed the central spindle of a bobbin with wrought iron cones, and with a neck 5·7 mm. in diameter.

Table XIV.—Hard-drawn Nickel (0·56 per cent. of Iron)

Outside field.	$\mathfrak{B}$.	$\frac{\mathfrak{B} - \text{outside field}}{4\pi}$.	$\frac{\mathfrak{B}}{\text{outside field}}$.
2,220	7,100	390	3·20
4,440	9,210	380	2·09
7,940	12,970	400	1·63
14,660	19,640	400	1·34
16,000	21,070	400	1·32

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* Ewing, "Magnetic Qualities of Nickel, Supplementary Paper," 'Phil. Trans.,' A, 1888.

Cobalt.

§ 37. Lastly, a piece of cobalt was tested which was cut from a cast bar supplied by Messrs. JOHNSON and MATTHEY, and turned to form the centre of a bobbin with wrought iron cones (fig. 19), and with a neck 4·48 mm. in diameter. It was found to contain 1·66 per cent. of iron.

Fig. 19.

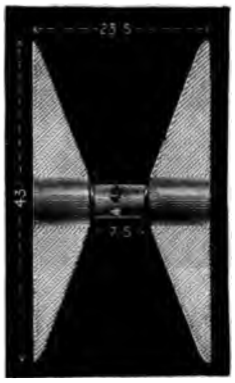

TABLE XV.—Cobalt.

Outside field.	$\mathfrak{B}$.	$\frac{\mathfrak{B} - \text{outside field}}{4\pi}$.	$\frac{\mathfrak{B}}{\text{outside field}}$.
1,350	16,000	1260	12·73
4,040	18,870	1280	4·98
8,930	23,890	1290	2·82
14,990	30,210	1310	2·10

It appears from this that the saturation value of $\mathfrak{I}$ for this specimen of cobalt is about 1300, or a little greater than the value we have found for cast iron. In the second, third, and fourth of these observations the residual magnetism was sensibly constant ($\mathfrak{B}_r = 1260$); in the first it was a little less.

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§ 38. We may conclude that under sufficiently strong magnetising forces the intensity of magnetisation ($\mathfrak{I}$) reaches a constant, or very nearly constant, value in wrought iron, cast iron, most steels, nickel, and cobalt. The magnetic force at which $\mathfrak{I}$ may be said to become practically constant is less than 2000 c.g.s. units for wrought iron and nickel, and less than 4000 for cast iron and cobalt. In stronger fields, the relation of magnetic induction to magnetic force may be expressed by the formula

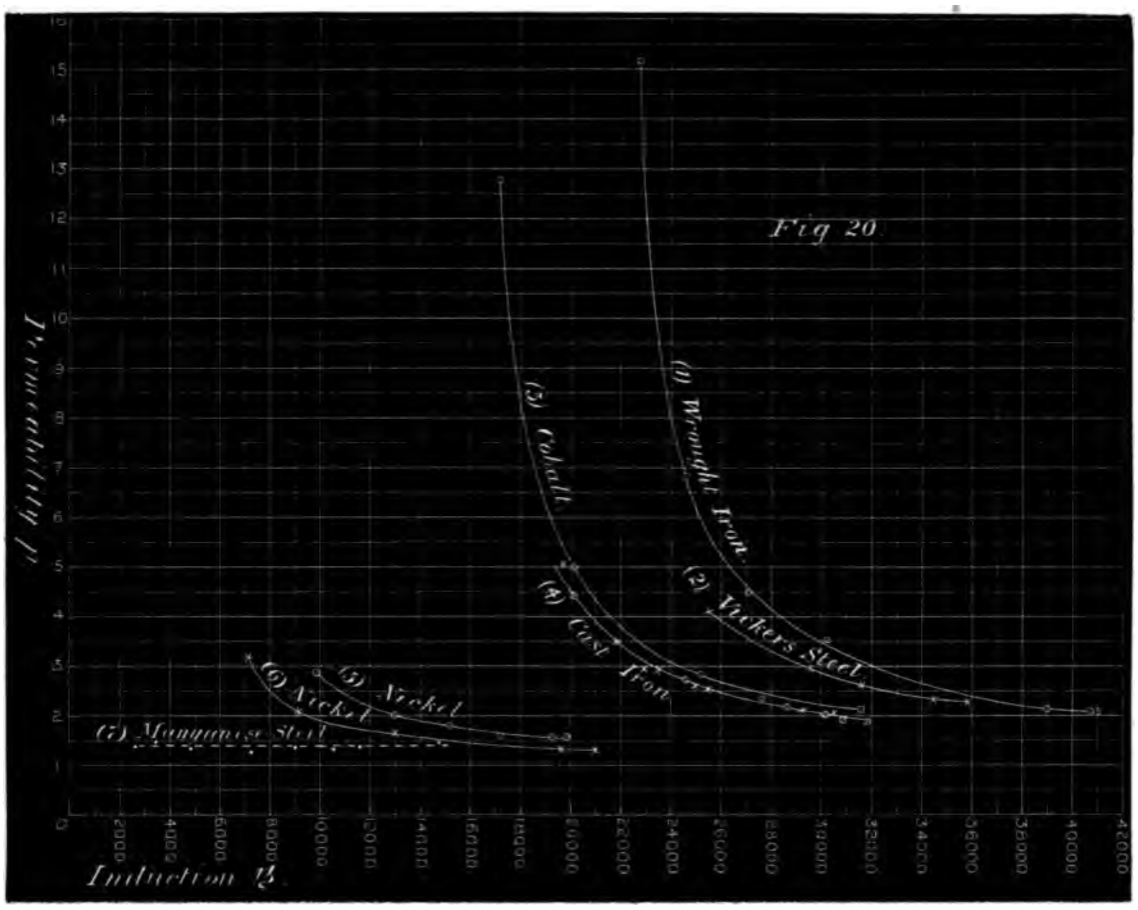
$$\mathfrak{B} = \mathfrak{H} + \text{constant}.$$

For the particular specimens we have tested, the value of this constant ($4\pi\mathfrak{I}$) is about

21,360 in wrought iron, 15,580 in cast iron, 5030 and 6470 in nickel, and 16,300 in cobalt.

The experiments give a definite meaning to the term "saturation," as applied to magnetic state. When magnetisation is measured by the induction 𝔅, the term saturation is inapplicable; there is apparently no limit to the value to which the induction may be raised. But, when we measure magnetisation by the intensity of magnetism 𝔍, we are confronted with a definite limit—a true saturation value, which is reached or closely approached by the application of a comparatively moderate magnetic force. There is nothing to show that the approach to this limit is not asymptotic; but in wrought iron it is practically reached before the magnetic force rises to 2000 c.g.s., and after that a ten-fold increase in the force produces no material change in the intensity of magnetism.

Fig. 20.



§ 39.—The results are further summarised in fig. 20, in which Rowland's curve, showing the relation of the permeability μ to the induction 𝔅, is drawn from the data supplied by the experiments on—

(1) Swedish wrought iron (Table V.).
(2) Vickers' tool steel (Table X.).
(3) Cobalt (Table XV.).
(4) Cast iron (Tables VII. and VIII. The points taken from Table VII. are marked thus, ×, and those from Table VIII. thus, ⦶.).
(5) Annealed nickel with 0·75 per cent. of iron (Table XIII.).
(6) Hard-drawn nickel with 0·56 per cent. of iron (Table XIV.).
(7) Hadfield's manganese steel (Table XII.).

If the magnetic force $\mathfrak{H}$, instead of the induction $\mathfrak{B}$, had been taken as abscissa, the curves (with the exception of those relating to Vickers' steel and manganese steel) would have sensibly lain upon rectangular hyperbolas with $\mathfrak{H} = 0$ and $\mu = 1$ for asymptotes.

VIII. *Some Observations on the Amount of Light Reflected and Transmitted by Certain Kinds of Glass.*

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[PLATE 8.]

INTRODUCTION.

ALTHOUGH for both theoretical and practical purposes it is important to know the amount of light reflected from the surface of glass, and the loss which light suffers in passing through glass, but few accurate experiments appear to have been made on this subject.

Dr. ROBINSON, in the report on the Melbourne telescope ('Phil. Trans.,' 1869, p. 127), gives an account of experiments made by Lord ROSSE, Mr. GRUBB, and himself to determine the amount of light transmitted by telescopic object glasses, and through various kinds of flint and crown glass; he states that Lord ROSSE'S and Mr. GRUBB'S experiments were made with a BUNSEN'S photometer, and his own with a ZÖLLNER'S photometer. Dr. ROBINSON assumed the truth of FRESNEL'S formulæ, and then calculated the values of the extinction coefficients ϵ^{-nt} from the expression $n = (\log \rho^2 - \log \mathrm{I})/(t \times \text{modulus})$, where I is the intensity of the emergent light, t the thickness of the glass, and ρ^2 the coefficient giving the amount which escapes reflection at the two surfaces.

Mr. ROOD published in the 'American Journal of Science' for 1870 (vol. 50, p. 1) an account of some observations he had made, with a modification of BUNSEN'S photometer, of the loss which light suffers in passing through crown glass. He states that the formulæ for reflection were originally given by YOUNG, and obtained subsequently by POISSON and FRESNEL, but believes that no accurate experiments have ever been made to test their truth. He used two plates of glass, ·15 mm. and 1·677 mm. thick, and assumed that with such thin plates the loss of light was practically entirely due to reflection. He found that with one plate, of which the index was 1·5236, and which therefore should have transmitted 91·736 per cent. of the incident light, 91·440 per cent. actually passed through; and with the other, of which the index was 1·5225, 91·155 was transmitted, instead of 91·763.

VOGEL published in the 'Berliner Monatsberichte' for 1877 (p. 138) some determinations of the absorption of crown, and of two kinds of flint, glass for different parts of the spectrum, but states that, as the three glasses were not equally well polished, no conclusion can be drawn as to their relative merits.

M. ALLARD ("L'Intensité des Phares," 'Annales des Ponts et Chaussées,' 1876, p. 31) states that the absorption of light by the glass which it passes through is given "par une formule exponentielle, mais on peut sans grande erreur la supposer proportionelle à l'épaisseur et l'évaluer à raison de 0·03 par centimètre de verre traversé," but does not give an account of any experiments made to determine the amount absorbed.

Since the experiments described in this paper were commenced Lord RAYLEIGH has published ('Roy. Soc. Proc.,' vol. 41, p. 275) an account of some determinations he has made of the intensity of the light reflected from glass at a nearly perpendicular incidence; he came to the conclusion that "recently polished glass surfaces have a reflecting power differing not more than 1 or 2 per cent. from that given by FRESNEL's formula; but that after some months or years the reflection may fall off from 10 to 30 per cent., and that without any apparent tarnish."

The experiments of which I have the honour of presenting an account to the Royal Society were commenced in order to determine, if possible, the amount of light lost by transmission through glass, without assuming the truth of the formulæ for reflection, and also to determine experimentally the amount reflected from the surface of the glass.

It was thought that one way in which this might be effected would be by taking plates of the same kind of glass, but of different thicknesses, and observing the amount of light which passed through; the reflection from the first surface would be the same in all cases, whilst that from the second would only differ slightly in amount, as, owing to the increased absorption of the thicker pieces, less light would reach the second surface; but, the absorption being small, and photometric methods not very exact, it was thought that this would hardly produce any sensible difference in the results.

It was also hoped that it might be found possible to determine the reflected light directly, since, as Lord RAYLEIGH has pointed out (*loc. cit.*) with reference to Professor ROOD's experiments, any error in the measured amount of the light transmitted would give rise to a very much greater error in the estimated amount of the reflected light. This was subsequently accomplished by measuring the relative intensities of the illumination produced by two Argand gas flames, when the light from both fell directly on the photometric surfaces, and when the light from one fell directly, whilst that from the other reached the photometer after reflection from the surface of a piece of glass.

Experiments were also made to ascertain whether repolishing altered in any way the reflective power of the glass; and the polarising angles before, and after, repolishing were also determined.

The observations were made with five plates of Messrs. CHANCE'S "lighthouse" glass, for which the author is indebted to Mr. KENWARD, the plates being 6·5 mm., 11·5 mm., 15 mm., 18·5 mm., and 24·3 mm. thick respectively, and with different thicknesses of Messrs. FIELD'S "ordinary dense flint." A block of this glass was procured from Mr. HILGER, by whom a slice 7 mm. thick was cut off and polished, and then the remainder of the piece formed into a rectangular block measuring 91·3 mm. × 69·5 mm. × 49 mm., and the six faces carefully polished.

Some measurements were also made with a piece of ordinary plate glass 6 mm. thick.

The ordinary plate glass was green when seen edgewise, Messrs. CHANCE'S glass was slightly green when viewed in the same manner, and the flint glass was distinctly yellow by daylight.

Mr. KENWARD states that "the lighthouse glass is of a special mixture, varying slightly from time to time; it is of the nature of hard crown glass." The average refractive index for the sodium line of Messrs. CHANCE'S hard crown is stated by them to be 1·5172, and of their soft crown 1·5146; and in a letter which accompanied the glass its index was said to be "about 1·51 or 1·52."

The refractive indices for the sodium line of the crown glass, the flint glass, and the plate glass used in these experiments were determined in the ordinary way with small prisms made of each kind of glass by Mr. HILGER; the values found were:—

Crown glass	1·5145,
Flint glass	1·6330,
Plate glass	1·5274.

The indices of the crown and plate glass were also determined with the wedges of these two kinds of glass used for the reflection experiments; that for the crown was found to be 1·5137, and of the flint 1·6385; the refracting angles of these two wedges being only 9° 39′ 45″ and 9° 51′ 19″, whilst those of the small prisms were 59° 45′ 19″ and 59° 45′ 17″, it seemed probable that the values obtained with the latter were the most accurate, and they were therefore taken as the true values of the indices.

The account of the experiments is given in Part I., and the results deduced from them in Part II., the account of the experiments being divided into six sections:— 1. Amount of light transmitted. 2. Amount of light reflected at a nearly perpendicular incidence. 3. Amount of light reflected at a nearly perpendicular incidence after repolishing. 4. Amount of light reflected at various incidences between 0° and 90° by the crown glass before and after repolishing. 5. Amount of light transmitted after repolishing. 6. Values of the polarising angles before and after repolishing.

PART I.

Section I.—Determination of the Amount of Light Transmitted.

A photometric arrangement similar to the one described in PICKERING'S 'Physical Manipulations,' vol. 1, p. 132, was used for the greater number of these determinations (see Plate 8, fig. 1). Two similar pieces of looking glass 125 mm. × 125 mm. were fixed vertically at the ends of a horizontal board rather more than 2 metres long and 27 cm. wide, the mirrors being 2 metres apart; a small Argand gas burner, giving a flame 15 mm. in diameter, was fixed opposite the middle point of the line joining the centres of the mirrors, and at a horizontal distance of 20 cm. from the line; the mirrors were so adjusted that they reflected the light of the lamp towards each other.

A block of wood, resting on four small metal rollers and guided by two strips of wood fixed 7·8 cm. apart on the upper surface of the horizontal board, carried the photometer; an index was fixed to the wood block, and a scale, divided into millimetres, to the board, the zero being at one of the mirrors.

The photometer of which a description was given in 'Roy. Soc. Proc.,' vol. 35, p. 27, and 'Phil. Mag.,' series 5, vol. 15, p. 423, was used (Plate 8, fig. 1A); it consisted essentially of two pieces of white paper so placed that, whilst each was illuminated by one only of the two lights to be compared, both were visible to the observer. Three pieces of wood were screwed to the block, and between these the photometer was placed; this arrangement, whilst permitting the photometer to be reversed, so that the light from each of the two mirrors could be made to fall first on the one and then on the other paper, ensured its always being replaced in the same position.

The whole arrangement was optically equivalent to two exactly similar sources of light which always retained the same relative intensity, and were at a distance apart of a little more than twice the distance between the two mirrors.

A wooden screen was placed between the lamp and the observer, and, in order to cut off stray light reflected from the horizontal board, screens, with apertures in them rather larger than the apertures in the photometer box, were placed on either side of it, and between it and the mirrors; the edges of the apertures in the screens being formed of metal filed to a feather edge, and then, together with all the wood work, painted a dead black.

Assuming that the two sides of the Argand flame were equally bright, that the reflective powers of the two looking glasses were equal, and that there was no stray light, or at least that there was an equal amount from either side, then at a point midway between the two mirrors the intensities of the light reflected from the mirrors would necessarily have been equal, and, in order to determine the amount of light which passed through any transparent substance, it would merely have been necessary to interpose the substance in question between one of the mirrors and the photometer, and determine, by means of the photometer, the new position of equality.

But, as such assumptions might be erroneous, the experiments were actually made in the following manner :—First, the glass plate was fixed on one side of the photometer, whose position was altered until both papers appeared equally bright; six readings having been thus made, the glass was then placed on the other side, and six more readings made. The glass was then replaced in its original position, and six more readings made, and so on. After thirty-six readings the photometer was reversed, and another set of thirty-six readings made in the same manner. In making the readings, the photometer was first placed too much to the right, and then moved to the left till the point at which the illumination was equal was reached; it was then pushed to the left, and gradually moved back to the right, till the papers appeared equally bright, being thus brought to the position of equality alternately from one side and from the other.

In order to prevent the possibility of any light reflected from the edges of the plates reaching the photometer, the plates were placed either close to it or close to one of the screens. For two sets of readings the glass was placed against the opening in the box enclosing the photometer, and for one set against the opening in the screen, except in the case of the thick piece of flint glass, which was too large and heavy to be carried by the block on which the photometer rested, and which was therefore always placed against one of the screens. The position of the glass made no difference in the measurements.

The percentage transmitted of light falling upon the glass was calculated in the following manner:—Half the difference between the means of each set of six observations with the plate on either side of the photometer was taken; half the distance between the two mirrors, plus or minus this quantity, together with the distance between the lamp and the mirrors, gave the two distances from the lamp at which there was equality of illumination. The mirrors being 200 cm. apart and the lamp at a horizontal distance of 20 cm. from the middle of the line which joined their centres, the half difference was added to or subtracted from 202.*

Calling these distances a and b, a being the lesser, a correction x had to be made in the value of a for the optical shortening of the path of the light due to its passage through the glass; this was calculated by the ordinary formula $x = e(1 - 1/n)$, where e is the thickness of the plate, and n its refractive index, the distance of the lamp from the glass being sufficiently great to allow the formula for perpendicular incidence to be used without introducing any sensible error.

The percentage amount of light transmitted was given by the expression $100(a - x)^2/b^2$.

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* Strictly speaking, the geometrical, and not the arithmetical, mean of the readings should have been taken, as the intensity of light varies inversely as the square of the distance. Calling the two sources of light, to which the lamp and mirrors were equivalent, m and n, and the distance between them x, the

The surfaces of the glass plates were always cleaned with a wash-leather immediately before the plates were used, but, in order to ascertain whether this was sufficient, a number of readings were made with a piece of plate glass treated in this way, and then the plate was cleaned with strong nitric acid, washed with water, with alcohol, and again with water, dried with a clean cloth, and finally rubbed with a wash-leather. The means of the twelve readings made immediately before, and immediately after, the plate had been so treated were identical.

Table I. gives the readings made with the 24·3 mm. plate of crown glass. The observations made with the other pieces of glass were about as concordant.

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two positions of the photometer in which there is equality x_1 and x_2, and the coefficient of transparency for the particular piece of glass k, then

$$\frac{mk}{x_1^2} = \frac{n}{(x - x_1)^2}, \quad \text{and} \quad \frac{nk}{(x - x_2)^2} = \frac{m}{x_2^2},$$

then

$$k = \frac{nx_1^2}{m(x - x_1)^2}, \quad \text{and } k = \frac{m(x - x_2)^2}{nx_2^2},$$

$$k^2 = \frac{x_1^2(x - x_2)^2}{x_2^2(x - x_1)^2},$$

$$k = \frac{x_1(x - x_2)}{x_2(x - x_1)}.$$

The difference, however, between the values of k obtained by taking the quotient of the squares of the geometrical means of the readings, and by taking the quotient of the squares of the arithmetical means,

$$\left(\frac{\frac{x_1 + (x - x_2)}{2}}{\frac{x_2 + (x - x_1)}{2}}\right)^2,$$

was very small, and, as the readings themselves could not be made with any very great degree of accuracy, the simpler process was used for calculating out the results.

TABLE I.

A.					
cm.	cm.	cm.	cm.	cm.	cm.
97·1	110·8	96·1	110·4	96·1	109·6
97·3	111·1	96·8	110·2	96·8	110·0
96·0	110·4	96·7	109·6	96·5	109·3
96·8	111·2	96·4	109·8	96·5	110·6
96·3	109·8	95·9	109·3	95·8	109·5
96·7	110·7	96·5	110·0	95·8	109·7
Mean 96·6	110·7	96·4	109·9	96·2	109·8
B.					
94·3	106·3	93·5	107·2	94·6	107·1
94·2	107·3	94·2	106·4	94·5	106·7
93·6	106·5	93·7	106·5	94·0	107·1
94·2	106·8	93·6	106·9	94·3	107·5
94·0	106·8	94·1	106·8	94·4	106·8
94·4	106·9	94·8	106·1	94·4	107·1
Mean 94·1	106·7	94·0	106·3	94·4	107·0

Table II. gives the mean readings (1st column), their half differences (2nd), the values of $a - x$ and b (3rd and 4th), and the percentage amount of light transmitted (5th), A and B being the readings made in the two positions of the photometer.

TABLE II.

Crown Glass. 6·5 mm. plate. $x = 2$ mm.

Readings.	Half differences.	$a - x$.	b.	Percentage of light transmitted.	Percentage of light transmitted. Mean of A and B
A.					
cm.	cm.				
97·7–106·1	4·2	197·6	206·2	91·84	
97·1–106·3	4·6	197·2	206·6	91·11	
96·7–105·4	4·3	197·5	206·3	91·65	
				91·53	
B.					
94·7–103·4	4·3	197·5	206·3	91·65	
95·1–104·0	4·4	197·4	206·4	91·47	
95·1–104·1	4·5	197·3	206·5	91·29	
				91·47	91·50

Crown Glass. 11·5 mm. plate. $x = 4$ mm.

Readings.	Half differences.	$a - x$.	b.	Percentage of light transmitted.	Percentage of light transmitted. Mean of A and B.
A.					
cm.	cm.				
96·2–106·3	5·0	196·6	207·0	90·20	
96·3–106·4	5·0	196·6	207·0	90·20	
96·3–106·4	5·0	196·6	207·0	90·20	
96·4–106·5	5·0	196·6	207·0	90·20	
				90·20	
B.					
94·1–104·9	5·4	196·2	207·4	89·49	
94·4–104·6	5·1	196·5	207·1	90·02	
94·4–105·1	5·3	196·3	207·3	89·67	
95·3–104·9	4·8	196·8	206·8	90·56	
				89·93	90·07

Crown Glass. 15 mm. plate. $x = 5$ mm.

Readings.	Half differences.	$a - x$.	b.	Percentage of light transmitted.	Percentage of light transmitted. Mean of A and B.
A.					
cm.	cm.				
95·5–107·1	5·8	195·7	207·8	88·69	
95·8–106·9	5·5	196·0	207·5	89·22	
95·6–106·6	5·5	196·0	207·5	89·22	
				89·04	
B.					
94·3–104·9	5·3	196·2	207·3	89·58	
94·0–105·2	5·6	195·9	207·6	89·04	
94·2–105·4	5·6	195·9	207·6	89·04	
				89·22	89·13

Crown Glass. 18·5 mm. plate. $x = 6$ mm.

Readings.	Half differences.	$a - x$.	b.	Percentage of light transmitted.	Percentage of light transmitted. Mean of A and B.
A.					
cm.	cm.				
97·1–108·7	5·8	195·6	207·8	88·60	
96·7–109·0	6·1	195·3	208·1	88·08	
96·2–108·1	5·9	195·5	207·9	88·43	
				88·37	
B.					
94·7–105·8	5·5	195·9	207·5	89·13	
94·6–106·4	5·9	195·5	207·9	88·43	
94·6–106·4	5·9	195·5	207·9	88·43	
				88·66	88·51

Crown Glass. 24·3 mm. plate. $x = 8$ mm.

Readings.	Half differences.	$a - x$.	b.	Percentage of light transmitted.	Percentage of light transmitted. Mean of A and B.
A.					
cm.	cm.				
96·6–110·7	7·0	194·2	209·0	86·34	
96·4–109·9	6·7	194·5	208·7	86·86	
96·2–109·8	6·8	194·4	208·8	86·68	
				86·63	
B.					
94·1–106·7	6·3	194·9	208·3	87·55	
94·0–106·3	6·1	195·1	208·1	87·90	
94·4–107·0	6·3	194·9	208·3	87·55	
				87·70	87·16

Flint Glass. 7 mm. plate. $x = 3$ mm.

Readings.	Half differences.	$a - x$.	b.	Percentage of light transmitted.	Percentage of light transmitted. Mean of A and B.
A.					
cm.	cm.				
95·8–108·4	6·3	195·4	208·3	88·00	
96·2–108·1	6·0	195·7	208·0	88·52	
96·1–107·1	5·5	196·2	207·5	89·40	
96·5–108·1	5·8	195·9	207·8	88·87	
				88·70	
B.					
93·5–105·0	5·7	196·0	207·7	89·05	
94·0–105·4	5·7	196·0	207·7	89·05	
94·2–105·5	5·7	196·0	207·7	89·05	
94·3–106·1	5·9	195·8	207·9	88·70	
				88·94	88·83

Flint Glass. 49 mm. thick. $x = 19$ mm.

Readings.	Half differences.	$a - x$.	b.	Percentage of light transmitted.	Percentage of light transmitted. Mean of A and B.
A.					
cm.	cm.				
94·5–109·2	7·3	192·8	209·3	84·85	
94·5–109·0	7·2	192·9	209·2	85·02	
94·9–109·0	7·0	193·1	209·0	85·36	
95·3–109·2	6·9	193·2	208·9	85·53	
				85·19	
B.					
93·1–106·8	6·8	193·3	208·8	85·70	
93·3–107·4	7·0	193·1	209·0	85·36	
93·1–107·0	6·9	193·2	208·9	85·53	
93·9–107·3	6·7	193·4	208·7	85·88	
				85·62	85·40

Flint Glass. 69·5 mm. thick. $x = 27$ mm.

Readings.	Half differences.	$a - x$.	b.	Percentage of light transmitted.	Percentage of light transmitted. Mean of A and B.
A.					
cm.	cm.				
93·1–110·3	8·6	190·7	210·6	81·99	
93·5–110·3	8·4	190·9	210·4	82·32	
93·6–110·4	8·4	190·9	210·4	82·32	
				82·21	
B.					
92·5–108·9	8·2	191·1	210·2	82·65	
92·9–108·9	8·0	191·3	210·0	82·98	
93·1–109·0	7·9	191·4	209·9	83·15	
				82·93	82·57

Flint Glass. 91·3 mm. thick. $x = 35$ mm.

Readings.	Half differences.	$a - x$.	b.	Percentage of light transmitted.	Percentage of light transmitted. Mean of A and B.
A.					
cm.	cm.				
92·7–110·6	8·9	189·6	210·9	80·82	
92·9–110·9	9·0	189·5	211·0	80·66	
93·1–110·7	8·8	189·7	210·8	80·98	
				80·82	
B.					
91·6–110·1	9·2	189·3	211·2	80·34	
92·0–110·2	9·1	189·4	211·1	80·50	
92·3–109·8	8·7	189·8	210·7	81·15	
				80·66	80·74

Three plates of 6 mm. Crown Glass. Cemented together. $x = 6$ mm.

Readings.	Half differences.	$a - x$.	b.	Percentage of light transmitted.
A.				
cm.	cm.			
96·2–108·6	6·2	195·2	208·2	87·90
96·3–108·4	6·0	195·4	208·0	88·25
96·3–108·6	6·1	195·3	208·1	88·08
				88·08

The light reflected by the right-angled prism was examined with a double-image prism and a plate of selenite, and was found to be completely unpolarised.

The observations were made by first determining the position of the analyser in which the field appeared equally bright throughout; then the plate of glass to be examined was placed between the right-angled prism and the white surface, and the new position of the analyser, in which there was equality of illumination, observed. As the intensity of the light which traversed the two Nicols varied as the square of the cosine of the angle between their principal sections, the percentage amount transmitted by the glass was given by 100 × $\cos^2 \alpha'/\cos^2 \alpha$, where α is the angle between the principal sections of the Nicol, when the field was uniformly bright without the glass, and α' when it was interposed.

The analysing Nicol was first rotated "clockwise," and readings made in each of the four quadrants of the position in which the two halves of the field appeared equally illuminated; the Nicol was then rotated "counter-clockwise," and four similar readings made, the mean of the eight readings being taken as the true position. The glass was then interposed between the diaphragm and the reflecting prism, and eight readings of the new position of the analyser, in which there was equality, made in the same way.

To determine the light transmitted by each piece of glass, four sets of eight observations were made without the glass, and four sets with it. Table III. gives the first set of each for the 6·5 mm. plate of crown glass; the other sets were about as concordant.

TABLE III.

Without Glass.

Readings of analyser.		Mean.*	α.
42° 25′	42° 45′	42° 35′	47° 25′
137 0	139 30	138 15	48 15
222 25	223 15	222 50	47 10
317 35	318 30	318 0	48 0
		Mean . . .	47 42

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* As the intensity of the light varies as the square of the cosine, the geometrical, and not the arithmetical, is the true mean; but the observations did not appear to be sufficiently concordant to make it worth while to employ the longer process.

With 6·5 mm. plate of Crown Glass.

Readings of analyser.		Mean.*	α.
40° 15′	38° 0′	39° 07′	50° 53′
138 30	139 05	138 47	48 47
222 0	223 45	222 52	47 08
319 30	320 20	319 55	49 55
		Mean . . .	49 11

Table IV. gives the values of α determined in this way for the 6·5 mm., 11·5 mm., and 15 mm. plates of crown glass, and the percentage amount of light transmitted, as calculated from these numbers.

TABLE IV.

Crown glass. 6·5 mm. plate.

α.		Percentage amount of light transmitted.
Without glass.	With glass.	
47° 42′	49° 11′	94·32
45 44	49 36	86·22
46 16	48 37	91·45
46 54	48 37	93·61
		Mean 91·40

11·5 mm. plate.

α.		Percentage amount of light transmitted.
Without glass.	With glass.	
42° 03′	44° 53′	91·05
40 14	43 45	89·54
41 16	44 37	89·68
40 09	42 54	91·85
		Mean 90·53

15 mm. plate.

α.		Percentage amount of light transmitted.
Without glass.	With glass.	
41° 54′	45° 29′	88·73
40 23	43 42	90·08
40 17	43 44	89·82
		Mean 89·54

This method is clearly incapable of giving very accurate results. It is difficult to judge of the equality of the illumination in the two halves of the field, and also the angle through which the Nicol has to be turned to make the comparison is small, and, therefore, a slight error in the determination of the value of α makes a very considerable one in the result.

The measurements with the polarising photometer not being entirely satisfactory, another form of photometer was devised (Plate 8, figs. 3 and 3A). It consisted essentially of two white surfaces illuminated by the same lamp, the light falling very nearly perpendicularly on both. One surface was at a constant distance from the lamp, whilst the other could be brought nearer to, and moved further from it; one surface was seen directly, and the other through the glass to be examined, and then the distance of the movable surface from the lamp altered till both surfaces appeared equally bright.

One surface was fixed at a distance of 78 cm. from the lamp and in the same horizontal plane, whilst the second was fixed to a vertical screen, which could be moved backwards and forwards by means of a pulley and catgut band along a board with a divided scale. The surfaces consisted of a double thickness of white paper, as it was found by taking a double thickness the apparent illumination was increased, a portion of the light which passed through the first paper being reflected back by the second. In front of the lamp two right-angled prisms were placed. They were held in position by two pieces of wood, through which a screw was passed, one prism being slightly in advance of the other, and overlapping it to a small extent. To adjust the prisms they were placed on a smooth table, the one resting directly on the table, and the other on a thin piece of card, and, after being adjusted, fixed in position by means of the screw. When so placed, the line dividing the two fields of view was much narrower than when the front surfaces of the prisms were in the same vertical plane and their edges in contact. A screen was placed between the prisms and the lamp, which were only 8 cm. apart.

The light of the lamp fell nearly perpendicularly on the white paper, and the direction of that reflected back to the prism was also nearly normal to the paper, so that the distance of the paper from the lamp could be altered without altering to any considerable extent the angle at which the light fell upon it, or the angle under which it was seen.

The distance of the moveable surface from the lamp, when its reflection in the prism and that of the fixed surface appeared equally bright, was determined (1) without any glass being interposed, (2) with a plate of glass between the fixed surface and the prisms, and (3) between the moveable surface and the prisms, six readings being made of each of these positions, and then the prisms reversed so that the surface which had been seen by reflection in the one was seen by reflection in the other, and six more readings made.

Calling the apparent brightness of the fixed surface C, the distance of the moveable surface from the lamp without the glass x, with the glass between the fixed surface and the prisms x', and with the glass between the moveable surface and the prisms x'', and the coefficient of transparency of the particular piece of glass k, then the two surfaces will appear equally bright when

$$\mathrm{C} = \frac{1}{x^2}; \qquad \mathrm{C}k = \frac{1}{(x')^2}; \qquad \text{and } \mathrm{C} = k\,\frac{1}{(x'')^2};$$

whence

$$k = \left(\frac{x}{x'}\right)^2, \quad \text{or} \quad \left(\frac{x''}{x}\right)^2, \quad \text{or more simply} \quad \left(\frac{x''}{x'}\right).$$

The percentage amount of light transmitted by the 6·5 mm. and the 24·3 mm. plates of crown glass was determined in this way. Table V. gives the results.

TABLE V.

6·5 mm. plate.

Readings.		Percentage of light transmitted.
x''.	x'.	
cm.	cm.	
76·0	64·4	92·42
77·8	64·9	91·64
		Mean 92·03

24·3 mm. plate.

Readings.		Percentage of light transmitted.
x''.	x'.	
cm.	cm.	
78·0	87·0	89·65
74·1	85·9	86·26
		Mean 87·95

In order to have obtained really accurate results with this method it would have been necessary to have made a large number of observations, and taken their mean; but, as the results obtained with it, and with the polarising photometer, agreed fairly well with one another and with the far larger number of observations made with the first method, it was thought unnecessary to continue the observations, the agreement between those obtained by all three methods being sufficient to show that there was but slight, if any, probability of a constant error due to the photometer itself used in the first series.

Table VI. gives the results of the three methods and the probable error of each determination calculated by the ordinary formula, $0·674 \sqrt{(\Sigma e^2)}.\,/n$.

TABLE VI.—Percentage Amount of Light Transmitted.

	First method.	Second method.	Third method.
Crown glass—			
6·5 mm. plate	91·50 ± ·06	91·40 ± 1·07	92·03 ± ·18
11·5 „	90·07 „ ·07	90·53 „ ·32	
15·0 „	89·13 „ ·07	89·54 „ ·22	
18·5 „	88·51 „ ·08		
24·3 „	87·16 „ ·15	. .	87·95 „ ·81
Flint glass—			
7·0 mm. thick	88·83 „ ·09		
49·0 „	85·40 „ ·07		
69·5 „	82·57 „ ·11		
91·3 „	80·74 „ ·07		
Common plate glass—			
6 mm. thick	87·68 „ ·10		
3 plates of 6 mm. crown glass .	88·08 „ ·05		

attached to the goniometer was made as small as possible and blackened, and the goniometer itself was covered with a black cloth whilst the observations were being made.

In order to measure the intensity of the light reflected by the glass both when incident normally and at various angles, that is, to compare the intensity of light which under certain circumstances would be partially polarised with the intensity of unpolarised light, it was necessary that the photometric surface should be normal both to the incident light and to the line of sight; hence, the photometer which had been used in the transmission experiments clearly could not be used, nor indeed could a Bunsen's disk or any of its modifications. A new form of photometer was, therefore, devised. Two wooden screens were fixed to the sides of a block 10 cm. across, similar to the one which had carried the photometer in the first set of experiments; in these rectangular apertures were cut, 3 cm. by 2 cm., and "parchment" paper fastened over them; the two right-angled glass prisms which had been used in the third method for determining the amount of the transmitted light (Plate 8, fig. 3A, p. 260) were placed between the screens, and in a line with the apertures. The two papers, each illuminated by the light of one of the lamps, were seen by reflection in the prisms, and by moving the block, to which an index was fixed, along the photometer board a position could be found in which the two images appeared equally bright.

Glass only reflecting from 4 to 5 per cent. of the light incident normally upon its surface, and the photometer scale being only 2 metres long, it was impossible to compare the intensities of the direct and reflected light when the two lamps had the same illuminating power. During the first set of experiments, those marked A, the necessary difference was obtained by keeping the flame of the comparison lamp turned down rather low. Subsequently, in the determinations marked B, the same result was obtained by different sized apertures in the metal chimneys; that in the chimney of the goniometer lamp was 10 mm. by 18 mm., whilst the one in the chimney of the comparison or fixed lamp was only 10 mm. by 6 mm.; the gas flames were so regulated that the apertures appeared completely filled with a uniformly bright flame. When the gas pressure changes slightly, the size of a flame, and not its intrinsic brilliancy, is mainly what alters; and, therefore, by limiting the visible portion of the flame in this way a greater constancy in the ratio between the illumination produced by the two lamps was obtained; but, as the tables show, the measurements made when the whole flame, and those in which the central portions only were used, agree satisfactorily.

This method for determining the reflective power possesses the obvious defect that, owing to the necessary alteration in the course of the light, the direct and reflected light cannot be interchanged, and, therefore, a constant source of error may easily exist. It seems probable that, to a small extent, such was the case, and that the measured amounts of the reflected light were slightly too high.

When the lamp attached to the goniometer was so placed that the light fell directly on the photometer. The whole beam of light which passed through the aperture in the chimney fell directly on the screen at the end of the photometer bench and no light diffused from the lamp reached the photometer. When, however, the lamp was so placed that the light was reflected from the glass, part of the light fell on the wall of the room, and, although this was at a distance of about [illegible] meters and painted black, some light must have been diffused from it towards the photometer. The metal frame, also in which the glass was held, although made as small as possible and blackened, certainly reflected some light. In order to obtain some idea of the amount of light which reached the photometer from these sources the [illegible] was fixed in the proper [illegible] meter without any glass. The left half of the field was not [illegible] dark, but the amount of illumination was so slight as to be inappreciable. It [illegible] almost imperceptible when the light from the comparison lamp illuminated the other half of the field. Hence the error from this cause can only be small.

In order to eliminate as far as possible any error due to a change in the relative amount of illumination produced by the lamps, four readings were [illegible] of the position of the photometer when the two halves of the field were equally bright, the light of both lamps falling directly on it. Four readings of the position when the light was reflected from the glass were then made, and then four more with the direct light. The mean of the eight readings being taken as the true position of the photometer when the light reached it directly.

Calling the two sources of light [illegible] the distances between them [illegible] the positions of the photometer [illegible] which there is equality of illumination [illegible] and k the coefficient of reflection of the particular piece of glass, then

[illegible]

therefore

[illegible]

The [illegible]

[illegible]

the axes of the flames, which were about 15 mm. in diameter; in all the others, for which portions only of the flames were used, the distances were measured from the apertures in the chimneys.

The same glass was used for these experiments that had been employed in the transmission experiments already described. In order to prevent the light reflected from the second surface of the glass reaching the photometer the measurements were made with prisms cut from the 18·5 mm. plate of crown glass, and from the flint glass block; one face of each of these prisms was formed by one of the original surfaces of the glass, and these faces, which had not been repolished, were used for the reflection experiments.

Table VII. gives some of the actual measurements made with the prism of crown glass. The first four columns contain measurements made with the light from the whole surface of the flames; the others with light from the central portions only. In the first set the light was incident upon the glass at an angle of 6° 17′, and in the second at 6° 47′.

TABLE VII.

A.				B.			
Direct light.	Mean with direct light.	Reflected light.	Mean with reflected light.	Direct light.	Mean with direct light.	Reflected light.	Mean with reflected light.
cm.	cm.	cm.	cm.	cm.	cm.	cm.	cm.
128·8				117·0			
129·8				118·9			
127·8				118·1			
130·1		28·8		119·3		24·0	
	128·8	30·1	28·7		118·9	24·2	24·1
127·8		28·0		118·6		23·8	
129·4		28·0		120·1		24·3	
127·6				119·3			
129·1		29·2		120·2		23·7	
	127·9	28·7	28·8		119·4	24·2	24·2
126·3		27·7		118·3		24·1	
128·0		29·6		120·3		24·7	
126·7				119·2			
128·1		28·5		119·6		24·1	
	127·2	28·7	28·4		118·8	25·0	24·4
126·1		28·0		118·3		24·5	
128·0		28·6		118·1		24·2	
126·3				118·2			
127·9		27·8		118·6		23·5	
	127·0	28·9	28·2		118·3	24·3	24·1
126·9		27·6		118·1		24·2	
128·0		28·4		117·5		24·3	
125·8				118·5			
127·3				119·6			

From the mean results the values of x_1, x_2, $(x - x_1)$ and $(x - x_2)$ were obtained; and from these the percentage amount of light reflected by the glass calculated by the formula $K = \left\{\frac{x_2(x - x_1)}{x_1(x - x_2)}\right\}^2$. These values, and others obtained from similar sets of measurements, are contained in Table VIII.

Table VIII.

Crown Glass.

Angle of incidence.	x_1.	$(x - x_1)$.	x_2.	$(x - x_2)$.	Per cent. of incident light reflected.
A. 6° 17′	168·7	88·1	68·6	188·2	3·62
	167·8	89·0	68·7	188·1	3·75
	167·1	89·7	68·3	188·5	3·78
	166·9	89·9	68·1	188·7	3·78
6° 17′	171·5	85·3	71·7	185·1	3·71
	173·6	83·2	74·5	182·3	3·84
	174·3	82·5	74·0	182·8	3·67
	175·6	81·2	76·4	180·4	3·83
B. 6° 47′	156·1	95·2	61·3	190·0	3·87
	156·6	94·7	61·4	189·9	3·82
	156·0	95·3	61·6	189·7	3·93
	155·5	95·8	61·3	190·0	3·95
7° 30′	158·4	92·5	62·2	188·7	3·70
	157·8	93·1	61·7	189·2	3·70
	157·8	93·1	62·4	188·5	3·81
	157·7	93·2	62·3	188·6	3·81
	157·5	93·4	61·8	189·1	3·76
				Mean	3·78

Similar measurements were made with the flint glass wedge. The results are given in Table IX.

Table IX.

Flint Glass.

Angle of incidence.	x_1.	$(x - x_1)$.	x_2.	$(x - x_2)$.	Per cent. of incident light reflected.
10°	156·2	94·7	68·2	182·7	5·12
	155·9	95·0	67·8	183·1	5·09
	155·1	95·8	68·7	182·2	5·42
	155·6	95·3	69·1	181·8	5·42
10°	157·6	93·3	68·9	182·0	5·02
	157·6	93·3	69·4	181·5	5·12
	157·2	93·7	69·4	181·5	5·20
				Mean	5·20

Section III.—Amount of Light Reflected at a nearly Perpendicular Incidence after Repolishing.

Lord **Rayleigh** found ('Roy. Soc. Proc.,' vol. 41, p. 389) that, although the glass surfaces he examined were free from any apparent tarnish, the amount of light they reflected was largely increased by repolishing. The wedge of crown glass was, therefore, repolished on December 21, 1887, by means of a disk of wood charged with putty powder and mounted in a lathe (the same method that Lord **Rayleigh** had used), and its reflective power redetermined immediately; it was found to reflect 4·29, instead of 3·78, per cent. The glass was again examined on January 5, 1888, and it then reflected 4·20 per cent.

Two days later the glass was repolished a second time with fine rouge, and again examined; it reflected 4·22 per cent. of the incident light.

After an interval of five months, on June 13, this piece of glass, the surface of which had become considerably tarnished, was rubbed with wash-leather until the moisture deposited on the glass by breathing gently on it evaporated quite uniformly; it was then examined, and found to reflect 4·42 per cent. of the incident light. The next day it was repolished for the third time, and examined immediately; it reflected 4·30 per cent.

An attempt was made to repolish the flint glass wedge on February 28, 1888, with both putty and rouge, but a surface free from scratches could not be obtained. Its reflective power, however, was increased, and it reflected 6·20 per cent., instead of 5·20, after this imperfect polishing.

The surface not being satisfactory, the glass was sent to Mr. **Hilger** to be repolished. It was received back on the evening of March 2, and examined on the 3rd; it reflected 6·06 per cent.

After three months the glass was again examined, the film which had formed on its surface having been previously removed by rubbing with a wash-leather; it only reflected 5·71 per cent., although the surface appeared perfectly polished. On June 11 it was repolished with very fine washed rouge, and was found to reflect 6·25 per cent. of the incident light. Two days later, on June 13, it only reflected 5·73 per cent.

The polishing with putty powder was effected by means of a soft wood disk, mounted in a lathe, the disk being kept moist. For the rouge a polisher was formed by cementing a piece of silk to a sheet of plate glass fastened to a table, and charging this with carefully washed rouge: the glass was held against the rapidly rotating disk in the one case, and rubbed over the fixed surface in the other. The relative velocities of glass and polisher were very different in the two cases, and with the disk the friction was sufficiently great for the glass to become sensibly warm.

Tables X. and XI. give the details of these experiments; with the crown glass the angle of incidence was 7° 30′, and with the flint glass 10°.

TABLE X.

Crown Glass.

	x_1.	$(x - x_1.)$	x_2.	$(x - x_2.)$	Per cent. of incident light reflected.
December 21, 1887.—Repolished with putty	157·8	93·1	65·0	185·9	4·25
	157·0	93·9	64·8	186·1	4·34
				Mean	4·29
January 5, 1888	157·8	93·1	64·3	186·6	4·13
	156·6	94·3	63·9	187·0	4·23
	156·0	94·9	63·5	187·4	4·25
				Mean	4·20
January 7, 1888.—Repolished with rouge	155·7	95·2	63·0	187·9	4·20
	154·9	96·0	62·6	188·3	4·24
	155·5	95·4	63·3	187·6	4·28
	156·2	94·7	63·2	187·7	4·17
				Mean	4·22
June 13	159·6	91·3	67·5	183·4	4·43
	159·6	91·3	67·3	183·6	4·40
	159·8	91·1	67·7	183·2	4·44
				Mean	4·42
June 14.—Repolished with rouge	160·8	90·1	68·1	182·8	4·36
	160·9	90·0	67·6	183·3	4·25
	160·9	90·0	67·8	183·1	4·29
				Mean	4·30

TABLE XI.

Flint Glass.

	x_1.	$(x - x_1.)$	x_2.	$(x - x_2.)$	Per cent. of incident light reflected.
February 28, 1888.—Repolished with rouge and putty	154·6	96·3	71·2	179·7	6·09
	153·4	97·5	70·8	180·1	6·24
	152·9	98·0	70·5	180·4	6·27
				Mean	6·20
March 3, 1888, 5 P.M. — Repolished by Mr. HILGER.	156·7	94·2	73·6	177·3	6·23
	157·4	93·5	73·0	177·9	5·94
	157·4	93·5	73·8	177·1	6·13
	157·1	93·8	73·6	177·3	6·14
				Mean	6·11
March 3, 1888, 9 P.M.	156·1	94·8	71·6	179·3	5·88
	155·4	95·5	71·5	179·4	6·00
	155·4	95·5	71·6	179·3	6·02
	155·2	95·7	72·1	178·8	6·17
				Mean	6·02
June 5, 1888	159·9	91·0	74·1	176·8	5·69
	159·9	91·0	74·7	176·2	5·82
	160·2	90·7	74·9	176·0	5·82
	160·4	90·5	73·6	177·3	5·51
				Mean	5·71
June 7, 1888	160·2	90·7	75·7	175·2	5·98
	160·8	90·1	74·4	176·5	5·58
	160·6	90·3	75·3	175·6	5·81
	159·7	91·2	75·3	175·6	6·00
				Mean	5·84
June 11, 1888	160·5	90·4	74·0	176·9	5·55
	160·5	90·4	74·3	176·6	5·61
	160·5	90·4	74·6	176·3	5·68
	161·0	89·9	74·6	176·3	5·58
				Mean	5·60
June 11, 1888.—Repolished with rouge	166·0	84·9	81·5	169·4	6·05
	165·7	85·2	82·5	168·4	6·34
	165·8	85·1	82·6	168·3	6·35
				Mean	6·25
June 13, 1888	160·1	90·8	74·5	176·4	5·74
	160·4	90·5	74·4	176·5	5·66
	160·5	90·4	75·1	175·8	5·79
				Mean	5·73

Section IV.—Amount of Light Reflected at Various Incidences between 0° *and* 90° *by the Crown Glass before and after Repolishing.*

In the experiments already described the light incident on the glass consisted of a divergent beam, or rather of a complex of rays, the mean incidence of which, if such a term may be used, was as nearly normal as the construction of the apparatus permitted.

Measurements were also made with the crown glass both before and after it was repolished, when the light was incident upon its surface at various angles, the angle between the axis of the beam of light and the normal to the surface being considered as the angle of incidence. Table XII. gives these results.

TABLE XII.—Crown Glass.

Angle of incidence.	Before repolishing.					After repolishing.				
	x_1.	$(x-x_1)$.	x_2.	$(x-x_2)$.	Per cent. of light reflected.	x_1.	$(x-x_1)$.	x_2.	$(x-x_2)$.	Per cent. of light reflected.
10° B.	156·5	94·6	61·4	189·7	3·83	155·5	95·4	63·1	187·8	4·25
	156·7	94·4	61·5	189·6	3·82	150·0	95·9	63·1	187·6	4·36
	156·8	94·3	61·2	189·9	3·76	154·4	96·5	63·5	187·4	4·48
						154·0	96·9	63·6	187·3	4·56
				Mean	3·80				Mean	4·41
20° B.	157·2	93·9	61·1	190·0	3·69	154·9	96·0	66·1	184·8	4·91
	156·2	94·9	60·9	190·2	3·78	154·7	96·2	65·3	185·6	4·79
	155·9	95·2	60·7	190·4	3·79	154·7	96·2	65·1	185·8	4·75
	155·7	95·4	60·7	190·4	3·81	154·4	96·5	65·2	185·7	4·81
				Mean	3·77				Mean	4·81
30° A.	171·8	85·0	73·8	183·0	3·98					
	170·4	86·4	74·1	182·7	4·23					
	170·6	86·2	73·1	183·7	4·04					
	170·6	86·2	73·1	183·7	4·04					
B.	155·0	96·1	59·3	191·8	3·67	153·6	97·3	63·0	187·9	4·51
	154·4	96·7	59·9	191·2	3·86	152·9	98·0	62·8	188·1	4·58
	153·8	97·3	59·7	191·4	3·89	153·0	97·9	62·9	188·0	4·58
	154·2	96·9	58·7	192·7	3·68	152·8	98·1	62·5	188·4	4·53
				Mean	3·92				Mean	4·55
40° B.	156·4	94·7	64·7	186·4	4·42	158·3	92·6	72·0	178·9	5·54
	156·5	94·6	65·6	185·5	4·57	158·1	92·8	71·3	179·6	5·43
	156·3	94·8	65·8	185·3	4·64	158·2	92·7	70·4	180·5	5·22
	155·7	95·4	64·4	186·7	4·47	157·4	93·5	69·3	181·6	5·14
						156·6	94·3	68·9	182·0	5·19
				Mean	4·52				Mean	5·30

TABLE XII.—continued.

Angle of incidence.	Before repolishing.					After repolishing.				
	x_1.	$(x-x_1)$.	x_2.	$(x-x_2)$.	Per cent. of light reflected.	x_1.	$(x-x_1)$.	x_2.	$(x-x_2)$.	Per cent. of light reflected.
50° A.	162·2	94·6	72·5	184·3	5·26					
	161·3	95·5	71·9	184·9	5·30					
	161·2	95·6	72·2	184·6	5·38					
	161·2	95·6	72·4	184·4	5·42					
B.	154·3	96·8	68·6	182·5	5·56	158·6	92·3	75·6	175·3	6·19
	152·8	98·3	67·8	183·3	5·66	158·3	92·6	76·2	175·3	6·29
	152·4	98·7	68·3	182·8	5·85	158·3	92·6	74·7	176·2	6·46
	151·9	99·2	67·7	183·4	5·81	157·9	93·0	74·5	176·4	6·15
				Mean	5·53				Mean	6·27
56° 34′ B.	156·4	94·7	77·3	173·8	7·25	155·3	95·6	79·7	171·2	8·21
	158·7	92·4	78·4	172·7	6·99	155·5	95·4	79·9	171·0	8·22
	159·5	91·6	79·2	171·9	7·00	155·2	95·7	79·9	171·0	8·30
	157·7	93·4	79·9	171·2	7·64	154·9	96·0	79·4	171·5	8·23
				Mean	7·22				Mean	8·24
60° A.	163·1	93·7	85·6	171·2	8·25					
	163·2	93·6	84·9	171·9	8·02					
	161·9	94·9	85·3	171·5	8·50					
	161·8	95·0	84·1	172·7	8·17					
B.	154·4	96·7	80·2	170·9	8·64	155·1	95·8	84·5	166·4	9·84
	153·4	97·7	80·7	170·4	9·10	154·5	96·4	83·5	167·4	9·68
	152·2	98·9	79·5	171·6	8·88	153·9	97·0	83·7	167·2	9·95
	151·9	99·2	78·3	172·8	8·76	153·7	97·2	83·8	167·1	10·06
				Mean	8·54				Mean	9·88
65° A.	166·8	90·0	97·0	159·8	10·73					
	166·8	90·0	97·5	159·3	10·90					
	167·4	89·4	98·6	158·2	11·08					
	168·2	88·6	97·8	159·0	10·50					
B.	154·1	97·0	87·1	164·0	11·18	156·4	94·5	91·9	159·0	12·20
	153·3	97·8	87·1	164·0	11·48	155·8	95·1	91·5	159·4	12·28
	152·1	99·0	86·6	164·5	11·74	155·8	95·1	91·6	159·3	12·32
	151·5	99·6	85·9	165·2	11·66	155·4	95·5	91·2	159·7	12·31
				Mean	11·16				Mean	12·28
70° A.	167·9	88·9	109·8	147·0	15·64					
	167·6	89·2	109·8	147·0	15·80					
	168·0	88·8	110·0	146·8	15·69					
	169·5	87·3	111·6	145·2	15·67					

TABLE XII.—continued.

Angle of incidence.	Before repolishing.					After repolishing.				
	x_1.	$(x-x_1)$.	x_2.	$(x-x_2)$	Per cent. of light reflected.	x_1.	$(x-x_1)$.	x_2.	$(x-x_2)$.	Per cent. of light reflected.
70° A.	169·2	87·6	109·1	147·7	14·62	158·7	92·2	105·8	145·1	17·94
	167·3	89·5	109·0	147·8	15·57	157·7	93·2	105·2	145·7	18·21
	166·0	90·8	106·8	150·0	15·17	157·0	93·9	104·4	146·5	18·16
	165·3	91·5	107·4	149·4	15·83	157·0	93·9	105·5	145·4	18·83
									Mean	18·28
B.	156·9	94·2	99·3	151·8	15·42					
				Mean	15·49					
75° B.						160·1	90·8	119·4	131·5	26·52
						159·5	91·4	118·9	132·0	26·64
						158·7	92·2	116·7	134·2	25·52
						157·8	93·1	117·3	133·6	26·83
									Mean	26·33

Section V.—Amount of Light Transmitted after Repolishing.

The effect of repolishing on the amount of light transmitted by the glass was examined. The first observations, those with the 24·3 mm. plate of crown glass (except the last) were made with the photometer used in the reflection experiments, the arm of the goniometer carrying the lamp being clamped in the prolongation of the line joining the fixed lamp and the axis of the goniometer. The small slot in front of the flame of the fixed lamp was subsequently replaced by one of the same size as that in front of the lamp carried by the goniometer, and most of the observations made with this arrangement, in which there were two lamps of nearly equal illuminating power. Finally, a photometer with a single lamp, two mirrors, and inclined paper surfaces, like that used for the original transmission experiments, was fitted up, and this was used for all measurements made in August and September, 1888, that is, those with the flint glass and the four last determinations of the light transmitted by the crown glass.

The measurements were made as described in Section I., but only four readings were taken in each position of the glass, and the results were calculated out by the expression $k = \frac{x_1 (x - x_2)}{x_2 (x - x_1)}$ (see p. 250), where k is the coefficient of transparency, x the distance between the two lights, and x_1 and x_2 the two positions of the photometer in which there is equality of illumination, the optical shortening of the path of the light due to its passage through the glass being of course allowed for.

The results are contained in Tables XIII. and XIV.

TABLE XIII.

Crown Glass.

	x_1.	$(x - x_1)$.	x_2.	$(x - x_2)$.	Per cent. of incident light transmitted.
6·5 mm. plate. July 6, 1888.—Cleaned, but not repolished	127·2	135·1	133·1	129·2	91·39
	126·9	135·4	132·7	129·6	91·53
	127·7	134·6	133·4	128·9	91·67
				Mean	91·53
July 13.—"Ground grey and repolished" by Mr. HILGER	125·9	136·4	132·3	130·0	90·70
	127·2	135·1	133·3	129·0	91·11
	126·0	136·3	133·1	129·2	89·73
				Mean	90·31
July 17.—Re-examined	125·8	136·5	132·2	130·1	90·70
	126·0	136·3	132·0	130·3	91·25
	125·9	136·4	132·1	130·2	90·98
				Mean	90·90
11·5 mm. plate. August 13.—Cleaned, but not repolished	187·8	196·8	198·0	186·6	89·93
	187·9	196·7	197·5	187·1	90·50
	187·8	196·8	197·5	187·1	90·40
				Mean	90·28
11·5 mm. plate *bis*. July 13.—"Ground grey and repolished" by Mr. HILGER	126·1	136·0	132·5	129·6	90·69
	126·1	136·0	133·6	128·5	89·18
	125·7	136·4	132·4	129·7	90·28
				Mean	90·05
July 16.—Re-examined	125·6	136·5	132·8	129·3	89·59
	125·6	136·5	133·0	129·1	89·32
	125·5	136·6	132·9	129·5	89·52
				Mean	89·48
July 17.—Re-examined	124·9	137·2	131·7	130·4	90·14
	124·5	137·6	132·0	130·1	89·18
	124·8	137·3	131·7	130·4	90·00
				Mean	89·77

[illegible]—continued

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]					
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
				Mean	[illegible]
[illegible]					
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
				Mean	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
				Mean	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
				Mean	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
				Mean	[illegible]
[illegible]					
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
				Mean	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
				Mean	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
				Mean	[illegible]

TABLE XIII.—continued.

	x_1.	$(x - x_1)$.	x_2.	$(x - x_2)$.	Per cent. of incident light transmitted.
24·3 mm. plate. June 15.—Cleaned, but not repolished . .	84·7	165·4	92·4	157·7	87·40
	84·8	165·3	92·5	157·6	87·41
	84·9	165·2	93·9	156·2	85·49
				Mean	86·90
June 16.—Re-examined	86·7	163·4	95·6	154·5	85·75
	87·7	162·4	95·3	154·8	87·72
	88·0	162·1	95·9	154·2	87·29
				Mean	86·92
June 16.—Polished with rouge on silk . .	84·8	165·3	92·9	157·2	86·81
	85·5	164·6	93·3	156·8	87·30
	85·6	164·5	93·8	156·3	86·71
				Mean	86·93
June 23.—Repolished by Mr. HILGER; the polish was defective	84·6	165·5	94·6	155·5	84·03
	84·4	165·7	94·3	155·8	84·15
	84·2	165·9	94·3	155·8	83·85
	84·1	166·0	94·2	155·9	83·85
				Mean	83·97
June 30.—Repolished a second time by Mr. HILGER	85·5	164·6	92·9	157·2	87·90
	85·6	164·5	92·6	157·5	88·50
	84·6	165·5	92·5	157·6	87·09
				Mean	87·84
June 30.—Re-examined	86·4	163·3	94·0	156·1	87·86
	86·3	163·8	93·7	156·4	87·94
	86·7	163·4	94·3	155·8	87·67
				Mean	87·82
July 17.—Re-examined	123·0	138·7	132·0	129·7	87·14
	123·7	138·0	132·4	129·8	87·88
	123·3	138·4	131·7	130·0	87·94
				Mean	87·65

TABLE XIV.

Flint Glass.

	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
				Mean	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
				Mean	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
				Mean	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
				Mean	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
				Mean	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
				[illegible]	[illegible]

Section VI.—Values of the Polarising Angles before and after Repolishing.

In order to determine the angles of polarisation, a form of apparatus essentially similar to that employed by SEEBECK ('POGGENDORFF, Annalen,' vol. 20, 1830, p. 27) was used. It consisted of a goniometer with a horizontal circle reading to 20″; the slit and lens of the collimator were removed and the observing telescope replaced by a tube to one end of which a vertical divided circle was fixed. A Nicol was contained in an inner tube, and by means of a vernier the position of its principal section could be read on the vertical circle to 5′. The stage of the goniometer and the arm carrying the Nicol were geared together by means of toothed wheels working into a double pinion, the number of teeth in the wheels and pinion being such that on moving the arm of the goniometer carrying the Nicol the angular velocity of the tube was twice that of the stage.

A small gas flame was placed close to the end of the collimator tube, the flame being surrounded by a blackened metal chimney with a small aperture in it, and the glass surface whose angle of polarisation was to be observed clamped to the vertical stage with its surface in the prolongation of the vertical axis of the goniometer, the stage turned till the image of the flame was seen through the Nicol, and then, by means of the pinion (the axis of which was fixed to a sliding piece), the stage and Nicol geared together.

The Nicol having been clamped with its short diagonal horizontal, the arm was moved till the light reflected by the glass was reduced to a minimum.

The end of the collimator tube nearest the lamp was bisected by a vertical thread; a pair of cross threads were placed in the inner end, and a diaphragm with a small aperture at the eye end, of the Nicol tube; and in making the observations care was taken that the image of the vertical thread should coincide with the point of intersection of the two cross threads as seen through the diaphragm.

The observations were made by moving the Nicol tube alternately towards the right and the left. As the image of the flame always remained in the field of view, and the room was completely dark, the angle at which the light was reduced to a minimum could be observed with a fair amount of accuracy.

The amount of light reflected by glass at a perpendicular incidence being small, and a satisfactory diagonal eyepiece not being available, the position of the stage in which the light was incident perpendicularly on the surface of the glass was determined by an indirect method.

The axis of the Nicol tube was first, by means of the diaphragm and cross threads, brought into the same line as that of the collimator tube, and its position read on the divided circle of the goniometer; the tube was then turned through 90°, the glass surface attached to the vertical stage and adjusted, and the stage rotated until the image of the single thread again coincided with the cross threads of the Nicol tube.

The stage reading gave the position in which the light was incident upon the surface at an angle of 45°. To verify the adjustment, the Nicol tube was clamped, first, at an angle differing by + 90°, and then at one differing by − 90°, from its original position.

The reading of the Nicol, when light polarised in the plane of incidence was cut off, had been carefully determined some years previously, and was again verified. To reduce, as far as possible, the errors due to the Nicol (which was of the ordinary construction, with its terminal faces not perpendicular to its geometrical axis), eight readings were made with the prism in one position, and eight more after it had been turned round through 180°.

The actual readings made with the 6·5 mm. plate of crown glass, which had been repolished by Mr. **Hilger**, were—

°	′	°	′	°	′	°	′
107	06	107	25	106	59	106	55
107	03	107	08	107	02	107	04
107	20	107	09	107	08	106	56
107	27	107	09	107	04	106	55.

The readings made with the other glass surfaces were about as concordant.

Table XV. gives the angles of polarisation as deduced from the means of these readings.

Table XV.

Angles of Polarisation.

	°	′	″
Crown glass—			
6·5 mm. plate, repolished by Mr. Hilger	56	22	30
11·5 ,, not repolished	56	14	30
11·5 ,, No. 2, repolished by Mr. Hilger and again with rouge	56	28	30
11·5 ,, ,, ,, with putty powder	56	49	0
15·0 ,, ,, by Mr. Hilger	56	42	30
15·0 ,, ,, with putty powder	56	49	30
18·5 ,, not repolished	56	21	30
18·5 ,, not repolished	56	25	30
24·3 ,, repolished by Mr. Hilger	56	37	30
Wedge used for reflection experiments repolished with putty powder, and subsequently with rouge	56	52	0
Flint glass—			
Face of block—not repolished	58	14	0
,, ,, repolished with putty powder and examined immediately	58	48	0
,, ,, repolished with putty powder and examined after two days	58	44	0

PART II.

When light passes through a transparent plate it is diminished by reflection at the two surfaces, and by "obstruction" within the plate, the cause of obstruction being that a part of the light which has entered the plate is absorbed and, unless the plate be absolutely homogeneous, a part scattered.

If r be the ratio of the light reflected by the first surface, and r' by the second surface, to the light incident upon them, α the coefficient of transmission, and t the thickness of the plate, then the intensity of the transmitted beam is given by the expression $i = \mathrm{I}\rho\rho'\alpha^t$, where $\rho = (1 - r)$ and $\rho' = (1 - r')$.

I, i, and t being known, by eliminating $\rho\rho'$, α can be readily calculated. The value of i depends in the case of coloured media on the refrangibility of the light, but in the case of the two kinds of glass used in these experiments it may be taken to be the same for light of all wave-lengths.

Table XVI. contains the values of α for a thickness of one millimetre of crown and flint glass, obtained by combining in pairs the five values of i for crown glass, and the four values for flint glass, contained in Table VI.

TABLE XVI.

Values of α.

Crown glass.	Flint glass.
·99685	·99906
·99690	·99884
·99744	·99887
·99729	·99893
·99692	·99837
·99752	·99897
·99750	
·99809	Mean ·99884
·99763	
·99733	
Mean ·99735	

Dr. ROBINSON, in the paper already mentioned ('Phil. Trans.,' 1869, p. 160), gives the values of n in the expression $i = \mathrm{I}\rho^2 e^{-nt}$, ρ^2 being calculated from FRESNEL'S formula, and t being the thickness in inches. From the values given by Dr. ROBINSON for a cylinder of crown, and a prism of flint, glass, both of Messrs. CHANCE'S manufacture, the values of the coefficient α were calculated for a thickness of one millimetre.

PART II.

When light passes through a transparent plate it is diminished by reflection at the two surfaces, and by "obstruction" within the plate, the cause of obstruction being that a part of the light which has entered the plate is absorbed and, unless the plate be absolutely homogeneous, a part scattered.

If r be the ratio of the light reflected by the first surface, and r' by the second surface, to the light incident upon them, α the coefficient of transmission, and t the thickness of the plate, then the intensity of the transmitted beam is given by the expression $i = I\rho\rho'\alpha^t$, where $\rho = (1 - r)$ and $\rho' = (1 - r')$.

I, i, and t being known, by eliminating $\rho\rho'$, α can be readily calculated. The value of i depends in the case of coloured media on the refrangibility of the light, but in the case of the two kinds of glass used in these experiments it may be taken to be the same for light of all wave-lengths.

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	Index for line E.	n.	a.
Crown glass	1·5200	0·0272	0·99893
Flint glass	1·6216	0·0218	0·99914

From the values of a it would appear that these two specimens of glass absorbed somewhat less light than those used in the experiments of which an account is contained in this paper, but Dr. ROBINSON'S results depend on the value of ρ being "calculated *accurately* from FRESNEL'S formula," and if, as seems probable, glass usually reflects less than the theoretical amount of light, the amount absorbed would necessarily appear less than it really was.

From the mean values of a given in Table XVI. the values of ρ (on the assumption that $\rho = \rho'$) were obtained by calculating the values of a' for the different thicknesses of the two kinds of glass used in these experiments and then introducing these values into the equation $i = I\rho\rho'a'$.

TABLE XVII.

Values of ρ.

Crown glass.		Flint glass.	
6·5 mm. plate.	·9648	7·0 mm. plate.	·9463
11·5 „	·9636	49·0 „	·9508
15·0 „	·9629	69·5 „	·9461
18·5 „	·9648	91·3 „	·9475
24·3 „	·9642	Mean . .	·9477
Mean . .	·9641		

The value of r for the crown glass is therefore ·0359, and for the flint glass ·0523.

The amount of light which according to theory should have been reflected by the glass was calculated by the expression $\left(\frac{n-1}{n+1}\right)^2$, where n is the index of refraction. These values, and also the amount of the reflected light, as determined directly (see Tables VIII. and IX.), are given in Table XVIII.

TABLE XVIII.

Percentage amount of Light Reflected.

	Observed.		Calculated.
	By transmission.	By reflexion.	
Crown glass	3·59	3·78	4·187
Flint glass	5·23	5·20	5·780

PART II.

When light passes through a transparent plate it is diminished by reflection at the two surfaces, and by "obstruction" within the plate, the cause of obstruction being that a part of the light which has entered the plate is absorbed and, unless the plate be absolutely homogeneous, a part scattered.

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I, i, and t being known, by eliminating $\rho\rho'$, α can be readily calculated. The value of i depends in the case of coloured media on the refrangibility of the light, but in the case of the two kinds of glass used in these experiments it may be taken to be the same for light of all wave-lengths.

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Dr. ROBINSON, in the paper already mentioned ('Phil. Trans.,' 1869, p. 160), gives the values of n in the expression $i = \mathrm{I}\rho^2 e^{-nt}$, ρ^2 being calculated from FRESNEL'S formula, and t being the thickness in inches. From the values given by Dr. ROBINSON for a cylinder of crown, and a prism of flint, glass, both of Messrs. CHANCE'S manufacture, the values of the coefficient α were calculated for a thickness of one millimetre.

PART II.

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·99752	·99897
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Mean ·99735	

Dr. ROBINSON, in the paper already mentioned ('Phil. Trans.,' 1869, p. 160), gives the values of n in the expression $i = \mathrm{I}\rho^2 e^{-nt}$, ρ^2 being calculated from FRESNEL's formula, and t being the thickness in inches. From the values given by Dr. ROBINSON for a cylinder of crown, and a prism of flint, glass, both of Messrs. CHANCE's manufacture, the values of the coefficient α were calculated for a thickness of one millimetre.

The observed values of the light reflected by the crown glass do not agree quite so well as those for the flint glass; this may be due to the fact that all the measurements with the flint glass were made with one thick block, whilst several pieces of crown glass were used for the transmission experiments, and these plates may have differed slightly both in their composition and in the polish of their surfaces. Making due allowance for this, and for the approximate character of all photometric measurements, the agreement between the results obtained by two entirely distinct methods is, probably, quite as close as could have been anticipated.

The calculated value for both kinds of glass considerably exceeds the observed. As has already been mentioned, the refractive indices, as determined with the large prisms used for the reflection experiments, and with the small prisms, differed slightly; the differences, however, — ·0008 and + ·0055, are not sufficiently great to affect the result to any considerable extent. Thus, at a perpendicular incidence the two theoretical values for the reflected light are 4·187 and 4·176 for the crown glass, and 5·780 and 5·856 for the flint glass, or a difference of 0·011 and 0·076 per cent. of the incident light, a quantity which is, of course, quite inappreciable photometrically.

Both the crown glass and the flint glass had been recently polished, the former by Messrs. CHANCE and the latter by Mr. HILGER, both kinds of glass having been ground with emery and polished with rouge; the crown glass was partially machine-polished but finished by hand, the flint glass entirely hand-polished. The glass surfaces were always well cleaned with wash-leather immediately before being used, as after being left in the laboratory for some days they were usually somewhat tarnished; the films, however, were easily removed, and in no case could any deterioration of the surface be detected.

The effect of repolishing the glass was to increase its reflective power, but Tables X. and XI. show that the two kinds of glass behaved somewhat differently. Immediately after repolishing both reflected more than the theoretical amount of light; but, whilst the crown continued to do so, the reflective power of the flint decreased, and after an interval it reflected the theoretical amount.

TABLE XIX.

		Crown glass.	Flint glass.
Percentage of light reflected before repolishing		3·78	5·20
" " " after repolishing	Immediately	4·27	6·14
	After an interval	4·31	5·72
" " " calculated from the observed index of refraction		4·19	5·78

A number of measurements were made of the light reflected by the crown glass at various angles before and after repolishing; the means of the results contained in

Table XII., and also those for crown glass from Table XIX., are given in the second and third columns of Table XX. The amount of light which, according to FRESNEL'S theory, should have been reflected by the glass was calculated for the various incidences by means of the formula

$$J_r^2 = \tfrac{1}{2}\left\{\frac{\sin^2(i-r)}{\sin^2(i+r)} + \frac{\tan^2(i-r)}{\tan^2(i+r)}\right\},$$

the values of r being determined from the observed value of the refracting index of the glass; the fourth column contains the results.

By assuming the truth of the theory, the value of the refractive index could, of course, be readily deduced from the amount of light reflected at a nearly normal incidence, this being equal to $\left(\frac{n-1}{n+1}\right)^2$. Before repolishing, the crown glass reflected 3·78 per cent. of the incident light; hence, the value of n would be 1·4842. Assuming that such was the case, the amount of light reflected by the glass at various angles was calculated, and these numbers are given in the fifth column.

TABLE XX.

Percentage amount of Light Reflected by Crown Glass.

Angle of incidence.	Observed.		Calculated.	
	Before repolishing.	After repolishing.	From observed value of *n*.	From calculated value of *n*.
6° 17′	3·78	4·29	4·19	
6 47				
7 30				
10 0	3·80	4·41	4·19	3·81
20 0	3·77	4·81	4·21	3·81
30 0	3·92	4·55	4·34	3·93
40 0	4·52	5·30	4·77	4·34
50 0	5·53	6·27	5·98	5·52
56 34	7·22	8·24	7·62	7·23
60 0	8·54	9·88	9·16	8·63
65 0	11·16	12·28	12·31	11·75
70 0	15·49	18·28	17·37	16·78
75 0	..	26·33	25·58	

The percentage amount of light reflected before and after repolishing and the amount calculated from the observed index of the glass are represented by the curves on Plate 8, fig. 4, where the abscissæ are the angles of incidence, and the ordinates the percentages. The curve for the values deduced from the index calculated from the amount of light reflected normally is not given, as it coincides so closely with the curve for the glass in its original state that, in order to render the differences visible, it would have been necessary to draw the diagram on a much larger scale.

More observations were made with the glass before repolishing than after, which accounts for the one curve being so much smoother than the other.

These results show (1) that repolishing increased the amount of the reflected light; (2) that before repolishing the amount of light reflected was less than the theoretical amount calculated from the observed index of refraction of the glass, but that in the case of the crown glass, by assuming a value for the index in accordance with the amount of light reflected at a perpendicular incidence, the amount reflected at other angles by the glass before repolishing was given correctly by the formula; (3) that after repolishing the observed amount of light reflected exceeded the calculated amount; (4) that in the case of the flint glass what may be described as the "polishing-effect" passed off in the course of a day or two, and then the theoretical and actual intensities of the reflected light agreed, but that with the crown glass this did not appear to be the case.

The effect of repolishing being to increase the amount of light reflected by the glass, it seemed desirable to ascertain whether repolishing would produce any change, and, if so, whether increase or diminution, in the amount transmitted. If the increase in the reflected light were due to a more perfect surface being obtained, and, therefore, to less light being irregularly reflected or diffused, the intensity of the transmitted beam would certainly not be weakened by repolishing; if, however, it were due to an increase in the refractive index of the surface-layer of the glass, then the intensity of the transmitted beam would be decreased.

Table XXI. contains the means of the values set forth in Tables XIII. and XIV., and in the second column the values for the transmitted light obtained with the same samples of glass two years previously (see Table VI.).

TABLE XXI.

Percentage amount of Light Transmitted by Crown Glass.

	Original determinations.	Not repolished.	Repolished with rouge.	Repolished with putty.
6·5 mm. plate . .	91·50	91·53	90·60	
11·5 " . .	90·07	90·28	89·77	89·75
15·0 " . .	89·13	89·31	88·61	88·72
18·5 " . .	88·51	88·55		
24·3 " . .	87·16	86·91	87·77	

Percentage amount of Light Transmitted by Flint Glass.

	Original determinations.	Not repolished.	Repolished with putty and examined	
			Immediately.	After an interval.
7 mm. thick . .	88·83	92·66	84·55	85·03
49 " . .	85·40	..		
69·5 " . .	82·57	84·35		
91·3 " . .	80·74			

These numbers show that, except with the 24·3 mm. plate of crown glass, the effect of repolishing was to decrease the amount of light transmitted by both kinds of glass; they also show that, whilst the amount transmitted by the crown glass was the same as when it was first examined, the amount transmitted by the flint glass had increased considerably, although both kinds of glass had been kept during the interval wrapped in soft paper and in the same room.

The 24·3 mm. plate of crown glass behaved differently from the others. As is stated in Table XIII., it was first cleaned, and the amount of light it transmitted determined; each surface was then polished for 20 minutes with fine rouge on a silk polisher. This produced no change, and the plate was, therefore, sent to an optician, who returned it with the statement that it had been "polished with rouge on pitch, almost dry, to get a high polish." The moment the glass was placed in the photometer the polish was seen to be defective, the surface being apparently grained, and, as the table shows, its transmissive power was greatly decreased; it was returned to the optician to be again repolished, and then it let through more light than when first tested.

The values of ρ, calculated with the value of α previously obtained (p. 280), are given in Table XXII.

TABLE XXII.

Values of ρ for the Crown Glass.

	Original determinations.	Not repolished.	Repolished with rouge.	Repolished with putty.
6·5 mm. plate . .	·9648	·9650	·9601	
11·5 " . .	·9636	·9647	·9617	·9619
15·0 " . .	·9629	·9640	·9601	·9608
18·5 " . .	·9648	·9644		
24·3 " . .	·9642	·9642	[·9672]	
Mean	·9641	·9645	·9609	

Values of ρ for the Flint Glass.

	Original determinations.	Not repolished.	Repolished with putty and examined	
			Immediately.	After an interval.
7 mm. thick . .	·9463	·9665		
49 " . .	·9508	. .	·9460	·9482
69·5 " . .	·9461	·9562		
91·3 " . .	·9475			
Mean	·9477			

These results agree with those obtained by the direct measurement of the reflected light, and show that the effect of the repolishing is to increase the amount of light reflected and to decrease the amount transmitted, and this latter effect must be due to some cause other than a more perfect surface having been obtained.

The polish of the glass plates was examined by holding them close to the aperture in one of the screens of the photometer, and allowing the beam of light from the lamp to pass through the glass, the cross-section of the beam being smaller than the surface of the glass, and all other light being carefully excluded. If the surfaces had been perfectly polished, they would have been quite invisible, but such was not the case, the shadow cast by the edge of the aperture being just visible in all cases.

Examined in this way, there did not seem to be much, if any, difference between the various plates of crown glass, of which two were in their original state and four had been repolished, nor between them and the four faces of the flint block, of which two had been repolished.

The surfaces of the crown and flint glass wedges used for the reflection experiments did not appear to be quite so good; the difference, however, was very slight. The 7 mm. plate of flint glass was much inferior to all the other pieces, the boundaries of the beam of light which passed through it being distinctly visible. When examined in a strong light there appeared to be a slight film on the surface; on continued rubbing with a wash-leather this diminished, and then the surface of the plate, when placed in a beam of light in the dark room, was less visible than before. The inferiority of the surfaces of this plate appeared to be due to the films which had formed on them, and not to any roughness due to imperfect polishing.

The truth of BREWSTER's law, that the tangent of the polarising angle of a substance is numerically equal to its index of refraction, being generally admitted, it seemed desirable to ascertain the values of the polarising angles for the different plates before and after repolishing.

Table XXIII. gives the mean results collected from Table XV. The means show that the effect of repolishing was in all cases to increase the polarising angle, a result which is in accordance with those previously obtained.

TABLE XXIII.—Polarising Angles.

Crown Glass.

	Not repolished.	Repolished with rouge.	Repolished with putty powder
	° ′ ″	° ′ ″	° ′ ″
	56 14 30	56 22 30	56 49 0
	56 21 30	56 28 30	56 49 30
	56 25 30	56 42 30	56 52 0
		56 37 30	
Mean	56 20 30	56 32 45	56 50 10

Flint Glass.

	Not repolished.	Repolished with putty powder.
	° ′ ″	° ′ ″
	58 14 0	58 48 0
		58 44 0
Mean	58 14 0	58 46 0

Table XXIV. gives the values of the refractive indices of the two kinds of glass as determined directly, and as deduced from the amounts of the reflected and transmitted light, and from the values of the polarising angles, both before and after repolishing. The values obtained in these ways do not agree well together, those deduced from the amount of the transmitted light being considerably the lowest; they show, however, that repolishing increased the theoretical value of the index as determined by three independent methods.

TABLE XXIV.

Values of the Refractive Indices.

		Crown glass.	Flint glass.
Observed		1·5145	1·6330
Calculated from amount of light reflected, determined directly.	Before repolishing	1·4830	1·6280
	After repolishing	1·5220	1·6590* 1·6290†
Calculated from amount of light reflected, deduced from amount transmitted.	Before repolishing	1·4676	1·5930
	After repolishing	1·4928	1·6055
Calculated from polarising angle.	Before repolishing	1·5017	1·6149
	After repolishing with rouge	1·5136	
	" " putty	1·5293	1·6490

* Immediately after repolishing.

† After an interval.

Sir DAVID BREWSTER stated, many years ago ('Phil. Trans.,' 1815, p. 126), that glass "acquires an incrustation or experiences a decomposition by exposure to the air which alters its polarising angle without altering its general refractive power," and added that by the action of heat alone he had produced a variation of 9° in the polarising angle of flint glass.

SEEBECK ('POGGENDORFF, Annalen,' vol. 20, 1830, p. 27) made a number of determinations of the polarising angles of different kinds of glass, and found that there was considerable difference between the observed values and those calculated from the refractive index, except in the case of surfaces which he himself had ground and polished (with emery and colcothar).

With one specimen of flint glass the difference was originally $-38'$; he then polished it himself, and found that the difference was only $+3'$; after being polished by an optician, the difference became $+28'$.

SEEBECK was of opinion that these differences were due to the treatment which the glass had received whilst being polished and cleaned, and that lapse of time made no change. The only experiments he appears to have made on this latter point were with crown glass, the surface of which, as the experiments here recorded show, does not alter, or at least only alters very slowly.

Lord RAYLEIGH found ('Roy. Soc. Proc.,' vol. 41, p. 275) that repolishing prisms of crown glass caused a considerable increase in the amount of light they reflected; but his experiments do not show that when the prisms were first examined by him they reflected less light than when they were originally polished.

CONCLUSION.

It seems probable that the amount of light reflected by freshly polished glass varies with the way in which it has been polished, and that, if a perfect surface could be obtained without altering the refractive index of the surface-layer, then the amount would be accurately given by FRESNEL'S formula, but that usually the amount differs from that given by the formula, being sometimes greater and sometimes less.

The formation of a film of lower refractive index on the glass would account for the defect in the reflected light; but, to account for the excess, it seems necessary to assume that the polishing has increased the optical density of the surface-layer, and the changes produced in the amount of light transmitted and in the angle of polarisation support this view.

After being polished, the surface of flint glass seems to alter somewhat readily, the amount of the reflected light decreasing, and the amount of the transmitted increasing, whilst with crown glass the change, if any, proceeds very slowly.

There is no evidence to show to what particular cause these changes are due.

The values of the transmission coefficients for light of mean refrangibility for the two particular kinds of glass are given, and show that for 1 centimetre the loss by obstruction amounts to 2·62 per cent. with the crown glass, and 1·15 per cent. with the flint glass.

EXPLANATION OF PLATE 8.

Fig. 1. Double mirror photometer.
- A. Photometer board.
- B. Scale.
- C. Lamp.
- DD. Mirrors.
- E. Block carrying photometer.
- F. Photometer.
- G. Wood stops.
- H. Screens.
- I. Windows in screens.

Fig. 1A. Photometer.
- A. Base.
- B. Wooden prisms.
- CC. Pieces of white paper.
- DDD. Windows in casing.

Fig. 2. Polarising photometer.
- AA. Nicols.
- B. Right-angled prism.
- C. Lamp.
- DD. Pieces of white paper.
- E. Screen.

Fig. 3. Prism photometer.
- A. Photometer board.
- B. Scale.
- C. Lamp.
- DD. Pieces of white paper.
- E. Block carrying photometric surface.
- F. Screen.
- G. Prisms.

Fig. 3A. Photometer.
- A. Base.
- BB. Right-angled prisms.
- C. Screw.

Fig. 4. Curves representing the percentage amount of light reflected by crown glass at various angles of incidence.
1. Observed values with the glass in its original condition.
2. Calculated values from observed index of refraction.
3. Observed values with repolished glass.

IX. *On the Total Solar Eclipse of August* 29, 1886.

By Captain L. DARWIN, *R.E.*, ARTHUR SCHUSTER, *Ph.D.*, *F.R.S.*, *and* E. WALTER MAUNDER.

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[PLATES 9, 10.]

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I. ORIGIN OF THE EXPEDITION AND GENERAL PREPARATIONS. BY CAPTAIN DARWIN, ARTHUR SCHUSTER, AND E. W. MAUNDER.

AN expedition for observing the Eclipse of the Sun of August, 1886, was organised and sent out by the Royal Society, the necessary funds being obtained partly by a special vote from the Treasury, partly from the annual grant to the Royal Society, and partly from the Society's private funds. A Committee appointed by the Council of the Royal Society discussed the principal questions to which observers were to direct their attention, and distributed the available instruments amongst them. It was also decided that, as far as the scientific part of the work was concerned, the observers should be independent of each other, and report separately to the Society; but that they should elect one of their members as chief, to represent them in all dealings with the authorities in the West Indies. Mr. NORMAN LOCKYER was accordingly chosen to be this representative.

The present report only deals with the photographic results obtained by its authors. Mr. NORMAN LOCKYER was the only other observer who took out photographic instruments; most unfortunately, the weather proved so bad at the station he selected that he was unable to see anything of the eclipse.

Captain ABNEY was unfortunately not able to take part in the expedition, but he gave his invaluable help to the observers in their preparations, and in this way contributed most materially to whatever success the photographic part of the expedition may have obtained. The photographic plates used by Dr. SCHUSTER and Mr. MAUNDER were prepared by him, and we wish to offer him our best thanks for the assistance he has rendered us.

The expedition left England on the 29th of July, 1886, and arrived at St. George, Grenada, on the 12th of August. A letter had kindly been sent by the Colonial Office to the colony, stating what the requirements of the expedition would be. The members consequently found on their arrival that the Governor, Mr. W. J. SENDALL, had made every possible inquiry as to the best sites for the observatories, taking into consideration the weather probabilities as well as their personal comfort; and they have to thank him for the greatest courtesy and consideration during the whole of their stay in the island. The protection of the instruments having been mentioned in the Colonial Office letter, Captain I. C. MALING, the Colonial Secretary, very kindly prepared models of huts which could be cheaply and readily constructed on the spot. When the expedition arrived at Barbadoes on their outward journey, they found these models awaiting them. A telegram was despatched to Grenada approving generally of the design, and thus work was actually commenced before the arrival of the expedition in the island.

Before the observers left England, the President of the Royal Society had written to the Admiralty requesting the co-operation of any men-of-war that might be on the station. As a result of this communication, three of Her Majesty's ships—the "Fantôme," Commander R. H. ARCHER, R.N.; the "Bullfrog," Lieutenant J. MASTERMAN, R.N.; and the "Sparrowhawk," Lieutenant C. F. OLDHAM, R.N.—were found ready and prepared to render every assistance. Every member of the expedition felt grateful for the willing way in which the valuable assistance of both the officers and men of these ships was given; and the President of the Royal Society, in a letter to the Admiralty on the return of the expedition, expressed the value to science of such ready co-operation.

After the arrival of the expedition at Grenada, two or three days were occupied in selecting stations and making preparations for conveying the observers to their destinations. St. George itself was not favourably situated, and it was moreover considered advisable to scatter the observatories as much as possible, so as to avoid the chance of a single mass of cloud proving fatal to the whole expedition. The observers were therefore as possible, the clock adjustment was carried on in the position which the equatorials were to have during totality.

Owing to the unsettled state of the weather, the preparations for the eclipse were carried on under great difficulties, and the time at our disposal was found barely sufficient. We arrived at the observatory on a Tuesday, and the remainder of that week was taken up with the erection of the equatorials and the preliminary adjustments of the various instruments. During the week preceding the eclipse much time was lost owing to the frequent interruption of the work by heavy tropical showers. Tuesday and Wednesday were wet and stormy and no direct Sun light was available, although much required, to get the instruments into working order. Thursday was fine, and in the morning good progress was made; but Friday was again wet, and was followed by a rainy night. Saturday, the day before the eclipse, was cloudy in the morning, and the Sun only appeared at intervals. Our experience has thus taught us that a fortnight's time for preparation is hardly sufficient when two observers have to look after five different instruments, all requiring careful treatment. After the days had been spent in continuous work, the evenings were taken up with the preparation of photographic chemicals and occasional star observations.

We were without intelligent help except during the two days when Mr. **Lawrance** came to Prickly Point. We had taken him out, jointly with Professor **Thorpe**, as private assistant, anticipating the great difficulties we should have to encounter. His time, however, was chiefly taken up at Hog Island, where Professor **Thorpe** was observing, but our thanks are due to him for the assistance he gave us during these two days. As for unskilled help, we engaged for the whole time one negro servant, but now consider we should have done better to have had two.

The damp climate, with its steady temperature, varying day and night only between 81° and 85°, proved very exhausting, and the work could not be carried on as well and as quickly as it might have been under more favourable circumstances.

The richness of animal life proved a source of great annoyance. Mosquitos abounded in our residence. Wasps built their nests and spiders spun their webs with remarkable rapidity. The photographic room and even one of the equatorial stands had to be cleared of wasps' nests, and a dense spider's web was found an hour before totality stretched across the slit of one of the spectroscopes; the slit having been perfectly clear the night before.

It is difficult to realise at home the special difficulties of temporary observatories, but we venture to suggest that more skilled assistance should be provided at future eclipses. Although at Prickly Point we were enabled to carry out our programme with hardly any mishaps, we feel obliged to point out the difficulties under which we worked and which might easily have led to serious accidents. For example, five minutes before totality, Captain **Darwin**'s clock stopped altogether, although it is believed every possible pains had been taken in its regulation. It was only by a lucky alteration in the balance at the last moment that it was induced to go on at all.

Captain ARCHER had kindly sent us two sailors to render help during totality, viz., SAMUEL BROWETT, signalman, and HENRY STEELE, petty officer. One of these two was placed between the two huts to call out the time at intervals of half a minute while the other was to assist Captain DARWIN in the manipulation of his instrument. They both did what was required of them admirably.

During totality Captain DARWIN required no other assistance, as his programme was not heavy; but with Dr. SCHUSTER, whose whole observations were included in that time, the case was different. He gratefully accepted therefore the help offered to him by Dr. P. F. McLEOD, the officer of health in Grenada, and by Mr. MURRAY, who accompanied the expedition as naturalist.

Captain MALING had undertaken to make a drawing of the outer parts of the corona. According to a suggestion of Mr. LOCKYER'S, a disc had been prepared 6¾ inches in diameter. When placed at the proper distance the Moon and the inner parts of the corona could be screened off by means of this disc. It was placed on a wooden support at the top of an incline which ran down from the observatory to the sea shore. A post was driven into the ground some distance away from the disc. The observer was to look through a small hole in the post. This arrangement, however, wants very careful adjustment, and we had not much time to spare, as the principal objects of the expedition monopolised nearly all our attention. Whether from want of adjustment or from other causes, we cannot now decide, Captain MALING'S drawing includes the whole of the corona down to the prominences. Captain MALING'S statement is included in this report.

The following Table gives the position of the observatory and various data connected with the eclipse:—

Latitude	12°	0′	N.
Longitude	61°	45′	W.

Commencement of totality—

	h.	m.	s.
G.M.T.	23	17	12
L.M.T.	19	10	12
Duration of totality	—	3	50

Altitude of Sun	18°	45′
Parallactic angle	80°	25′
Angle between circle of declination and Sun's axis towards East	20°	37′

III. TOTALITY AT PRICKLY POINT. BY CAPTAIN DARWIN AND ARTHUR SCHUSTER.

It has been mentioned that during the week preceding the eclipse, which took place on the Sunday, the weather had been very unfavourable; but the clouds cleared away on Saturday afternoon, the sunset was fine, and experience had taught us that a fine evening was generally followed by a fine morning. Our prospects, therefore, were very good on Saturday night, but early on Sunday morning the wind rose, which was a bad sign. At five o'clock, however, when we got up, the sky was still perfectly clear; the clouds came from the East at half-past five, at first in the form of detached cumuli, but before long the whole sky towards the East was overcast. The Sun rose behind the clouds and was still hidden when the time for first contact arrived. It was only twenty minutes before totality that the crescent of the Sun appeared, and then the wind soon made a clearance all round him.

Ten minutes before totality the sky was clear. The last instructions were given, the instruments put in position, and everybody took his appointed place. Five minutes before totality, as the darkness increased visibly, the weather seemed safe. Another minute passed and danger once more threatened; a small cloud rising from the South-East was driven by the wind right towards the Sun. The Moon from above descended over the Solar disc, and it seemed a race between the Moon and the cloud which should cover it first. Totality began, and the corona became distinctly visible, but was obscured again almost instantaneously by the cloud. For about three-quarters of a minute the corona was only seen through a film of cloud as a narrow hazy ring. The corona finally appeared, and remained clear as long as totality lasted.

Lieutenant R. J. KIDD, then Private Secretary to the Governor of Grenada, took temperature observations every five minutes for one quarter of an hour before and after totality. His numbers are as follows :—

h.	m.	° Fahr.
6	55	82
7	0	82
7	5	82
7	10	81·75
7	15	81·75
7	20	81·5
7	25	82
7	30	82·5

As the commencement of totality took place at $7^h\ 10^m$ it will be seen that during totality the thermometer only fell one quarter of a degree Fahrenheit, and reached its lowest point about ten minuntes after totality. The whole change in temperature was exceptionally small.

IV. ON THE ACCURACY REQUIRED IN ADJUSTING AN EQUATORIAL FOR PHOTOGRAPHIC PURPOSES DURING TOTAL SOLAR ECLIPSES. BY ARTHUR SCHUSTER.

Observers preparing for a total Solar eclipse have in general only a very moderate time at their disposal, and it is of great importance to them to settle beforehand to what degree of accuracy the adjustment of their instruments is to be carried. The time which is spent over adjustment in any one direction must necessarily be taken away from other more important matters, as there is never any lack of work on these occasions.

If, for instance, a photographic picture of the Solar corona is to be obtained, it would be clearly waste of time to refine on the adjustments of the equatorial or the rate of the clock beyond the point at which the Sun's change in declination would produce a visible effect. We shall see that this consideration limits the time of exposure for which the full advantage of the aperture of the lens can be realised, and this again will give us a limit beyond which it would be unnecessary to adjust the equatorial. We might, indeed, take the Sun's apparent motion into account, and point the instrumental axis, not to the pole, but to some point near it, which might easily be determined by calculation. We shall see, however, that the available time of exposure of a 4-inch lens is quite sufficient for our present requirements, and that it is therefore unnecessary to make allowance for the Sun's change of declination.

When the instrument is nearly adjusted, the relation between the true declination of a star δ and the apparent declination δ' is given by—

$$\delta = \delta' - \gamma \cos(\tau - \delta),$$

where γ is the angle between the true pole and the instrumental pole, τ is the hour angle of the star, and δ the hour angle of the instrumental pole. If δ is constant the change of apparent declination from bad adjustment in a short time t is found from the above equation to be $-\gamma t \sin(\tau - \delta)$, which will, numerically, always be smaller than γt. If the time t, measured in minutes of time, is p, we can write for this maximum change—

$$44 \times 10^{-4} p\gamma.$$

We find in this way that a change of apparent declination of one second of arc per minute could be produced if

$$\gamma = 3' \, 49''.$$

Now, the change of the Sun's declination may be, and during the last eclipse was, nearly one second of arc per minute.* It will be unnecessary, therefore, to spend any time in making the angle γ smaller than that given by the above value, as a closer adjustment may in some cases make matters worse instead of better.

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* During the total phase of the late eclipse, owing to the low altitude of the Sun (18° 45'), the apparent change of altitude due to change of refraction was about $2\frac{1}{2}$ seconds of arc; but the change in declination due to refraction was small, and generally the effects of refraction may be neglected.

Special considerations may of course induce observers to adjust their instruments more accurately. If, for instance, the equatorial adjustment is to be done once for all about a fortnight or a week before the eclipse, and if the conditions of the foundation on which the pillar rests give grounds for fear that the position may slightly change, it would be wise to aim at a greater accuracy in the first instance, so as to give some play for change.

So far the adjustments do not depend on the size of the object-glass, but, if we want to make the best use we can of the aperture, an apparent shifting of the Sun's image during the time of exposure must be confined within small limits, which we have now to fix.

The central disc of a star, or of a small object at a great distance, has an angular radius Θ in the focal plane of the telescope, given by

$$\Theta = 1{\cdot}22\,\lambda/R,$$

when R is the diameter of the objective. Experience shows that small objects can, under favourable circumstances and with strong illumination, be resolved when the centre of one diffraction disc coincides with the first dark ring; but for objects having no well-defined boundaries we must give wider limits, and I therefore take the point of fair resolution to be reached when the first two dark rings touch, so that the star discs stand perfectly clear of each other. In order to find out experimentally to what extent a small shifting of the images, such as might be produced by defective adjustment of the equatorial, might influence the possibility of resolution, I drew on a piece of paper two circles touching each other, corresponding to the two star discs, and another similar set above them slightly displaced in the line joining the centres. It was found that a displacement of one-tenth of the distance between the centres of the disc could only affect the resolution to a slight degree. We may, therefore, consider a displacement through one-fifth of the radius of each disc allowable without impairing definition. There is always something arbitrary in the fixing of these limits, but it must be done in every investigation in which we want to determine the accuracy to be aimed at. As object-glasses of different makers would probably differ 10 per cent. in their observed resolving powers, the limit chosen seems a fair and reasonable one.

We may allow then an angular shifting of

$$\phi = \Theta/5 = 1{\cdot}22\,\lambda/5R = 1{\cdot}22 \times 10^{-5}/R$$

if we substitute for λ its mean value of about 5×10^{-5} cm. This gives a value for ϕ which varies inversely with the aperture, and is equal to 2·5 seconds of arc for one centimeter aperture. If the exposure takes place during p minutes, and if the change of the Sun's declination in one minute is τ seconds of arc, the displacement will be $p\tau$, and we get for the longest time of exposure compatible with full definition—

$$\mathrm{R}p\tau = 2{\cdot}5.$$

In the West Indian Eclipse R was 10 and τ was ·9, which gives about 17 seconds as time of exposure. This is sufficient for obtaining a good image; and we conclude, therefore, that an adjustment of the polar axis to about $3'{\cdot}5$ is sufficient for full definition. Photographs taken with longer exposures would chiefly serve to fix the fainter parts of the outer corona.

By the ordinary means of adjusting an equatorial it is not difficult to obtain the necessary accuracy; nevertheless, the calculation shows that we can by no means neglect to go through the full routine of adjustment, and no equatorial should be sent out for eclipse purposes without complete appliances in the way of good finders, circles which are easily read, &c.

The adjustment of the rate of the clock to the necessary accuracy is, as will appear, much more difficult.

We have seen that the angular displacement may reach, but should not exceed, $1{\cdot}22 \times 10^{-5}/\mathrm{R}$ during the time of exposure. Reduced to seconds of time, this is equal to $\cdot 17/\mathrm{R}$, or, if the exposure is to be q seconds of time, the error of the clock should not exceed $\cdot 17/q\mathrm{R}$, which for an exposure of 17 seconds and an aperture of 10 cm. is one part in a thousand.

The instrument we had at our disposal did not allow an adjustment as accurate as this. At Prickly Point it was found difficult to go beyond an accuracy of one per cent. No record of the rate was taken at Carriacou, but it was probably no better than at Prickly Point. This would show that full definition would not be possible with an exposure of more than two seconds.*

We may finally calculate by how much the efficient aperture is reduced if the errors due to adjustment are greater than those we have hitherto allowed. If the centres of two star discs are separated by a distance $2\Theta + \mathrm{K}$, a displacement equal to K would just bring them into contact. A telescope which to the eye would resolve two stationary stars at a distance 2Θ will, if during the time of photographic exposure the displacement is K, have its effective aperture reduced in the ratio of $(2\Theta + \mathrm{K}) : 2\Theta$. Hence, if R′ be the effective aperture

$$\frac{\mathrm{R} - \mathrm{R}'}{\mathrm{R}'} = \frac{\mathrm{K}}{2\Theta}.$$

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* Some of the statements made here do not agree with those made in the Preliminary Account ('Roy. Soc. Proc.' vol. 42, p. 180). The discrepancy is accounted for by the fact that I previously took wider limits for the allowable shifting due to the motion of the telescope, and that I have since then subjected the photographs to a closer investigation as regards fineness of detail actually shown.

Hitherto we have put $K = \frac{1}{5}\Theta$; but supposing the rate of the clock is such that during the exposure $K = 2\Theta$, which was very nearly the case during the late eclipse, the above equation shows that the effective aperture was reduced to half, or that the photographs could not show anything which an eye-observer with a telescope of 2-inch aperture might not have seen.

Let us now turn our attention to the resolving powers which have hitherto been actually obtained in photographs taken of the corona.

During the Eclipse of 1871 two prominences were separated in a photograph, which were at a distance of 15 seconds, and the corona itself gives no evidence of a finer structure. The aperture used was 4 inches.*

In the corona photograph of 1882 the diameter of one prominence subtends an apparent angle of about 15 seconds. As far as the photograph is concerned, we have no reason to suppose that an aperture resolving two stars at a distance of 15 seconds could not have shown everything that is seen on the photograph, for no detail of the corona gives evidence of smaller structure. This gives an aperture of 2 inches as sufficient to resolve all the detail shown in this photograph.

An examination of the drawing made by Mr. WESLEY of the present eclipse gives substantially the same result.

The 4-inch glasses used in the eclipses which I have mentioned all give, therefore, a resolution equal to that obtainable with a 2-inch aperture on a stationary object.

During the last eclipse the regularity of the clock motion was, as we have seen, not sufficient to give more perfect images; the same cause has very likely stood in the way of more perfect images during previous eclipses. We have at present no reason to suppose that the corona possesses any structural detail that could not be seen with an aperture of two inches, and it would be very desirable if eye observations could be taken at some future eclipse which would give us some idea of the detail of the structure visible with larger instruments.

But it is very remarkable that the definition obtained is just what we should have to expect, if we take the error of clock motion into account. If the motion of the clock is regular, we should by shorter exposures get in great part over the difficulty of adjustment, but in the instruments which have been at our disposal the mechanism of transmission from the clock to the telescope is not as perfect as it might be made, and the irregular vibrating motion produced by this cause would be sufficient to damage the sharpness of the image.

There does not seem to be any advantage at present in using larger apertures in future eclipses, at least if these larger apertures are accompanied by increased focal length. Most of the adjustments have to be more accurate for them; in order to get the advantage of the increased aperture, and the difficulties of mounting and adjusting are

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* Owing to an oversight, the aperture was in the Preliminary Account stated to be 2 inches. The focal length of the lens was only half that used during the later eclipses, but the aperture was the same.

V. The Photographic Camera. By Arthur Schuster.

1. *Adjustments.*

The lens used for obtaining photographs of the corona had a clear aperture of 4 inches (10·1 cm.), and a focal length of about 5 ft. 3 in. (160 cm.). It therefore gave images of the Sun of about ·3 in. in diameter.

The accurate adjustment of the camera is a matter of some difficulty, as distant objects are not sufficiently sharp to admit of delicate focusing. It was found better to focus on a sharp object at a moderate distance, and to find by calculation the correction for parallel rays. The correction to infinity δf for a distance u of the object focussed is given by

$$\delta f = f^2/u.$$

If, for instance, u were 1 kilometer, and f, as mentioned above, 160 cm., the correction would be 2·6 mm., and, as it proved impossible to adjust the focus more nearly than to about half a millimeter, an error of 20 per cent. in the estimate of the distance is allowable.

It is a matter of importance to be able to determine accurately the position of the corona in reference to the coordinates fixed in space. In Egypt and at Caroline Island this was done by means of a platinum wire stretched across the camera directly in front of the plate; the shadow of this wire appears on all photographs, and its position was determined by a succession of instantaneous photographs of the Sun's crescent taken directly before and after totality with a stationary telescope and at short intervals of time. This plan has some disadvantages. The shadow of the wire hides certain parts of the corona, and, for instance, in the photograph taken at Caroline Island the shadow is very awkwardly placed. Then, again, it is difficult to stretch the wire sufficiently tight. The camera has to be turned directly on the Sun while the crescent is being photographed, and the heat thus concentrated on the wire lengthens it and may change its position.

A different plan was therefore adopted. Two needles were placed one on each side of the camera, the line joining the points passing approximately through the centre of the plate; one of the needles was intentionally inserted somewhat obliquely, so that its shadow might always be recognised on the photographs. The image of the needles would thus completely determine the position of the plate in the camera. The direction of the line joining the needle-points can be very accurately determined with reference to the declination circle. We need only bring the image of some object, either terrestrial or celestial, just in coincidence with one needle-point, then with the other, and note the change of reading of the hour and declination circles. For safety, and as a check, photographs of the Sun's crescent might be taken after totality. The needles, in the present instance, did not reach sufficiently far into the centre of the camera. In previous eclipses the general illumination of the sky in the neighbourhood of the corona was sufficient to affect the whole plate; but it did not on this occasion, and the images of the needle-points only appear on one plate, owing to a cause to be mentioned presently. This one plate is, however, quite sufficient to fix satisfactorily the orientation of the corona. I believe that the method could be rendered perfectly safe in future eclipses, and it is certainly more accurate than that hitherto used.

2. *Arrangements during Totality.*

Dr. P. F. MACLEOD, the resident officer of health in Grenada, and Mr. MURRAY, who accompanied the expedition as a naturalist, had offered their assistance during totality. This was gratefully accepted, and my best thanks are due to them. Mr. MURRAY undertook to screen the camera while the slides were being drawn or pushed in, so as to prevent a confusion of image due to shaking when the slides are touched. Dr. MACLEOD took charge of the slides before and after exposure, handing them as they were wanted, and placing them in a bag after they had been taken out of the camera. I had originally intended to take eight photographs, but had to change all arrangements suddenly when I saw the corona obscured by a cloud at the beginning of totality. As the brighter parts of the corona were faintly visible, and as I did not know whether the cloud would clear away in time, two photographs were taken, referred to in Mr. WESLEY'S report as Plates 1 and 2. The exposure of these must have been about 15 seconds each.

It is only in a very indirect way that I could, after the eclipse, obtain an idea of the various exposures given, and not much value is to be placed, therefore, on the numbers. I knew the time I intended to expose, but I generally closed the camera sooner, as I was always afraid of a fresh cloud coming on.

The corona came out clearly while the third exposure took place. Instead of a plate, a piece of sensitized paper had then been put into the slide, but the result was not good.

Three further plates were taken with the corona clear, the last about 15 seconds before the end of totality. After the spectroscopic cameras had also been shut, I intended to take out all slides. Unfortunately I loosened the wrong hook, and drew the shutter instead of the slide, but, noticing my mistake at once, pushed it back. That moment the Sun came out, and the last plate was partly spoilt, but parts of the corona show well, and it has been used in making out the details of the corona. The accident was fortunate, in so far as the image of the needles necessary for orientation came out very clearly owing to the illumination by the reappearing Sun.

3. *Results.*

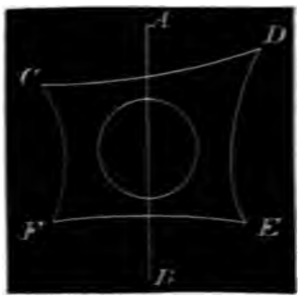
Mr. WESLEY, who has made a careful drawing of most of the recent eclipses, has described the corona as it is shown on the photographs on the present occasion, and I need not add anything to his remarks. But I should like to draw attention to one or two points which have a bearing on the speculations respecting the origin of the corona.

It is generally accepted now, I believe, that the Sun cannot have an atmosphere of anything like the extent of this luminous appendage. The reasoning generally given, and which seems conclusive, is that the pressure on the Sun's surface must, to judge from spectroscopic evidence, not exceed that on the surface of the Earth, and that an atmosphere comparable in size with the volume of the Sun would necessarily produce an enormous pressure. I do not know, however, whether attention has been drawn to the fact that, even if such a gaseous atmosphere existed, it could not be luminous to any great height unless it contained a great number of solid or liquid particles. The same laws of convective thermal equilibrium which regulate the decrease of temperature in the atmosphere act still more strongly on the surface of the Sun, and even taking the highest estimate which can reasonably be made of the Sun's temperature, the gases rising in the Solar atmosphere would quickly fall in temperature below the point at which they can be luminous. The only way in which they can be kept luminous is by containing sufficient matter to absorb the radiation from the body of the Sun, or by some independent cause, such as friction between solid and gaseous matter, or electric discharges.

If electric discharges are the cause of the luminosity, the matter through which the discharge takes place must either be supplied by the Sun or by something outside, such as streams of meteoric matter. The latter hypothesis has the great advantage that it may possibly account for the periodicity of Sun-spot phenomena, and it deserves, therefore, the attention of scientific men. Now, it occurred to me several years ago that if meteoric streams fixed in space were in any way connected with the corona we should find some evidence of them in the general shape as seen from the Earth. During eclipses which take place about the same periods of the year the Earth and the orbits of these hypothetical meteor streams would occupy the same relative positions. We might then expect some periodicity in the shape of the corona depending on the time of the year. To my surprise, I found that, as far as the evidence went, it seemed indeed to point in that direction. In the report of the Eclipse of 1875 a paragraph was inserted, which, without laying any stress on the point, called attention to the similarity between that eclipse and the one which had taken place the previous year during the same month. The eclipses which have taken place since have added to the evidence, and it seems worth while, therefore, to draw attention to it, without however in the least wishing to imply that I consider the fact as proved or even as probable.

The Solar corona often shows a rough symmetry about its axis of rotation, but deviates from complete symmetry owing to one of the halves being broader than the other. Hence, it often appears in the form of a trapezium. Fig. 1 may serve to illustrate this. AB is the axis; C, D, E, F the points of the longest rays of the corona, the distance DE being longer than the distance CF. Now, during the eclipses which have taken place early in April the eastern half of the corona is the one which is broadest, while in the eclipses observed in July and August the opposite held good and the western half was the broader. The eclipses of December and May have hitherto shown no difference between the two halves. It will no doubt be considered that the number of well ascertained coincidences is too slight to prove anything, and with this opinion I quite agree, especially as the changes in the corona which seem to depend on the Sun-spot cycle have to be taken into account. Nevertheless, it seems worth while to give the evidence here.

Fig. 1.



To begin with the eclipses of which we possess photographs, the only ones, perhaps, we can safely take into account, the following Table will illustrate my meaning :—

Date.	Year.	Corona.	Sun-spot numbers.
April 6	1875	Eastern half broader	29·5
May 17	1882	Absence of symmetry	60·0
May 6	1883	" "	63·7
July 29	1878	Western half broader	3·3
August 29	1886	" "	25·7
December 22	1870	No symmetry	135·4
December 11	1871	Symmetry, but both sides equally wide	96·[illegible]

The Sun-spot numbers represent frequency of spots as determined by Mr. R. WOLF. As far as 1878 these numbers are given in Mr. RANYARD's Eclipse volume ('Roy. Astron. Soc. Memoirs,' vol. 41) for the actual date of the eclipse. After that date the numbers are those given from time to time for the average of the whole year by Mr. R. WOLF in the 'Comptes Rendus.'

As regards the drawings, we have the following :—

Date.	Year.		Sun-spot numbers.
April 16 . .	1874	The drawings of Mr. BRIGHT and Mr. DEGERMANN show a distinctly greater width on the east side. In Miss ALICE HALL'S drawing the eastern half is less broken up than the western, but not broader. In Mr. WRIGHT's drawing the points of the streamers on the eastern side are further apart than on the west	119·1
August 7 . .	1869	Observers speak of the trapezian form of the corona, which is important, as the Sun-spots, though not at their maximum, appear in greater number at that time than in either 1882 or 1883. The drawing of Mr. MEEK, given by Mr. SCHOTT, seems the only one that is oriented relative to the Sun, and here the western side of the Sun is the broad one	77·6
August 18 . .	1868	The drawings are too uncertain to come to any conclusion, but if lines are drawn giving the ends of the extreme rays, the rays on the west are in nearly all cases further apart	42·9

We have, therefore, not one case in which the eastern side was the broadest in the autumn or the western in the spring.

If we take the time at which the greater eastern width changes to the greater western width to be about the middle of June, and middle of December, we should expect that in the eclipse which has just taken place the two sides would be about equally developed, but if there is a slight asymmetry the eastern side should be the broadest.*

VI.—THE CORONAGRAPH. BY CAPTAIN DARWIN.

1. *Description of the Instrument and its Adjustments.*

The coronagraph was designed by Dr. HUGGINS as the instrument which would give the best chance of rendering it possible to obtain photographs of the corona in Sun light. For this purpose a reflector is to be preferred to a refractor, and special precautions are taken to avoid internally reflected light. The telescope is of the Newtonian form.

The instrument which I took out is by GRUBB, of Dublin, and was mounted on an ordinary equatorial stand. The light enters a tube 4 feet long, fitted with numerous diaphragms. It then passes into a tube 5 feet 6 inches long (a little less than the focal length of the mirror), of about double the diameter of the first tube, the mirror being at the further end. The beam enters this tube at one side of the centre, and is reflected back at a small angle from the mirror at the further end to the photographic plate, which is alongside of the place where the light enters. A partition in this tube separates the incoming and reflected rays for as long a distance as possible. Below this partition there are oval diaphragms, through which both beams pass. I may here remark that this last precaution is, I rather think, a mistake; for the backs of most of these diaphragms are necessarily visible from the photographic plate, and, to whatever extent they are useful in preventing stray light from getting to the bottom of the tube, they are proportionately harmful in reflecting this light directly back on to the photographic plate, which is very near to them. I was so much afraid of the direct Sun light striking the back of these oval diaphragms that I placed a temporary diaphragm of only 2 inches in diameter at the bottom of the outer tube, thus ensuring that the beam of Sun light should pass clear through the diaphragms in the inner tube. I should gladly have dispensed with this if I had dared.

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* [*Note added July* 14, 1889.—This prediction has not been verified. In the Eclipse of January 1, 1889, although there is not much difference between the two sides, the western half is the broader. Nevertheless, the considerations in the text are of some use, as they show the importance of taking account of the relative position of Sun and Earth in discussing the shape of the corona in different eclipses.]

The two tubes are held in position in a long wooden frame. There is a half-open space between the tubes, which has to be covered up by a cloth or other temporary contrivance. This opening should be provided with a permanent cover.

I arranged an instantaneous shutter close in front of the plate, that is to say, in the position which Dr. HUGGINS considered most advantageous. It consisted of a long wooden slide, with a rectangular opening 6 inches wide, which was drawn across an opening of 2⅝ inches diameter by two pieces of elastic. It was released by cutting a thread. Both the elastics and the thread were fastened to the body of the telescope, and not to the camera or slide which takes the plate. This latter part is not too firmly fixed to the body, and I was anxious to avoid vibrations.

As Dr. SCHUSTER has fully described the adjustment of his instrument for equatorial movement, focus, &c., I will only remark that the same methods were used in the case of the coronagraph in so far as they were applicable.

2. *Observations in Sun Light and during the Partial Eclipse.*

The most important observations to be made with the coronagraph were with the view of testing the practicability of obtaining photographs of the corona during Sun light by this instrument. This could be done in two ways :—

(1.) By obtaining photographs shortly before or after the eclipse, and comparing any irregularity that might appear in the halo round the Sun with photographs of the corona taken during totality—a similarity of form indicating that the corona had been photographed.

(2.) By taking photographs during the partial eclipse. Then, if the light of the corona produced any effect on the plate, the dark limb of the Moon would be seen against it.

It was advisable in trying the first of the above experiments to take a series of photographs in each of the reversed positions of the instrument. Then any irregularity in the illumination round the Sun which was reversed on turning the instrument over would clearly be proved to be due to instrumental causes; whereas any irregularities which were visible in photographs taken in both positions, and which were not thus reversed, must be due to some outside causes, either atmospheric or coronal. In my instrument reversal was only possible near noon; and on this account, as well as on account of the Sun being at its greatest elevation, the best results would be obtained by taking one series of photographs shortly before noon in one position, and another series in the other position soon afterwards.

With regard to the length of exposure, I thought that there was no advantage to be gained for either test by giving a longer exposure than was necessary for the air glare to produce a result on the plate. I estimate the exposure that I gave with the automatic shutter at between one-fifth and one-tenth of a second.

The plates which I used for these tests consisted of gelatino-chloride dry plates on paper and on glass. These were prepared for me by two makers: by Mr. A. COWAN (for Messrs. MARION and Co.), to whom I am indebted for making for me a specially prepared batch of plates, and by Mr. WARNERKE.

Chloride plates were chosen in preference to bromide plates, because it was considered that their relatively greater sensitiveness to ultra-violet rays would be advantageous for distinguishing between the corona and the air glare. The plate to be chosen is no doubt the one which is most sensitive in that part of the spectrum where there is the maximum difference of intensity of light in the spectra of the corona and of the sky. I may here remark that Dr. SCHUSTER informs me that, according to his photograph, the maximum difference would appear to be in the less refrangible part of the spectrum, rather than in the violet; and, if that is the case, bromo-iodide plates would, I think, have been preferable.* Paper was considered better than glass, as tending to reduce the halation to a minimum; as a fact, I relied chiefly on the paper negatives.

I used various developers for these chloride plates, without observing any marked difference in the results:—

(1.) Hydrokinone, 1 grain to the ounce. To 2 oz. of the above add ½ dr. of a saturated solution of potassic carbonate, with common salt as a restrainer.

(2.) Potassic citrate, 136 grs.; potassic oxalate, 44 grs.; water, 1 oz. To 3 parts of the above add 1 part of ferrous sulphate, 140 grs.; sulphuric acid, 1 drop; water, 1 oz. Potassic bromide was used as a restrainer.

(3.) A weak ferrous sulphate developer.

Some of the plates were developed in the West Indies, and some on my return home. The results appeared to be the same, but they were, I think, much easier to develop in England. Ice was used freely in Grenada, but I do not think the gelatine became quite hardened during the time of development, and the plates appeared to be more liable to fog.

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* See p. 341.

3. *Results of First Test, or Photographs taken before the Eclipse.*

For the first test, that is, the one by means of photographs taken before or after the eclipse, I had intended to have taken several series, but fortune did not favour me. We reached Prickly Point on a Tuesday. By the following Sunday, although a great deal remained to be done in the way of final preparations and adjustments, I was able to take my first photograph. This left me six clear days before the eclipse in which to make the final arrangements, to obtain that experience so necessary when working in a new climate, and, in my case, with an instrument with which I was not very familiar, and also to obtain, if possible, one or more series of photographs for the test in question. I may here remark that, before starting, Dr. **Huggins** had given me every possible assistance and advice, without which I could hardly have undertaken the work; and that I had also spent several mornings in London in practising with the coronagraph, but from want of time I had to entrust a great deal of the troublesome preparation at home to Mr. **Lawrance**, to whose careful attention at this period I owe a great deal. As already remarked, the weather was very unfavourable, and on several days photographic work was an impossibility; on others I could only get casual photographs in the intervals between the frequent tropical showers, or in gaps in the cloudy sky. For two days after the eclipse the weather was also unfavourable. In fact it was only on the day before the eclipse that I succeeded in getting a series of photographs about noon in both positions; but, the rapidity of change in form of the corona being an unknown quantity, the shorter the interval the more valuable would the photographs be. The sky on that day was variable, but generally clear—as clear as it ever was during our stay in the West Indies, but not, I think, as clear as it often is in England.

Soon after coming home, Mr. **Wesley**, who has had great experience with regard to photographs of a similar nature, very kindly undertook to examine this series. Several of the negatives exhibited corona-like markings, but Mr. **Wesley** could not find any detail on any one of the photographs which was confirmed by the others. This was done before he had had an opportunity of seeing the photographs of the corona taken during totality. Had he succeeded in making a drawing of the supposed corona, its comparison with the true corona would have therefore afforded a valuable test as to its genuineness; under the circumstances this was, of course, impossible. Mr. **Wesley** also informs me that he thinks that "the photographs taken on the same side are as little comparable with each other as with those taken on the opposite sides," which, as far as it goes, indicates that the irregularities are due to atmospheric rather than to instrumental causes. My own opinion on the above points coincides with that of Mr. **Wesley**; but I was especially glad to obtain his assistance, because of his large experience, and also because he had not at that time seen any photograph of the corona.

4. *Results of Second Test, or Photographs taken during the Partial Eclipse.*

The second of the two tests could be applied by taking photographs either during the eclipse, before or after totality, or sufficiently near first or last contact for the Moon to be still eclipsing the corona. I succeeded in taking over twenty photographs during these periods. Many of these, when developed, showed what I may describe as a false corona, that is, an increase of density near the Sun, including the part between the cusps, and therefore in front of the Moon. In none of these can the limb of the Moon be seen against the corona as a background. Besides subsequent examination, I watched these photographs very carefully during development, without result. I mention this circumstance because they appear to have gone back considerably, so that, in several instances in which the air glare was clearly visible before fixing, it is now barely discernible.

I have also searched for any trace of a remarkably large prominence in the place where I knew it should be found, but without result. This prominence was hardly, if at all, covered by the Moon after totality, that is, during the period in which nearly all my photographs were obtained. Prominences are certainly more actinic than the corona, and we should therefore expect them to appear on the plate if the corona is obtainable by this method. However, if the air glare increases much more rapidly in intensity than the corona does as the Sun is approached, this argument is not sound, as the prominences might then be more overpowered by sky light than the outer parts of the corona would be.

5. *Photographs taken during Totality.*

It will be observed that the two experiments or tests just described were made by taking photographs before and after the eclipse, and during the partial phases. But during totality the instrument was not idle. The following was the programme which I had laid down for myself during the 3 minutes 50 seconds available :—

During the first minute a photograph was to be taken with the prismatic camera. After that, six plates were to be exposed with the coronagraph—four chloride plates with the same length of exposure as that given during Sun light, and two bromide plates with exposures of 5 and 10 seconds respectively.

This programme could not be followed exactly. Immediately after I had commenced exposing the prismatic camera, I looked up and found that a light cloud was drifting across the corona. The sky became clear again in about 50 seconds. I was anxious not to take any photographs with the coronagraph during the exposure of the prismatic camera, for fear of vibration; but, as nearly a minute had been lost, something had to be sacrificed, and I decided to take some of the photographs with the coronagraph before putting the cap on the prismatic camera. I do not think that the work has suffered in consequence, and at all events I obtained all the plates I had allowed for in my programme.

The instantaneous photographs of the corona were complete blanks, proving, I think, that the exposure had been far too short. I developed them with the same solutions, and for at least the same length of time, as when developing, immediately beforehand, some of the plates exposed during the partial eclipse; the instrument was in the same condition as before and after totality, when successful photographs were taken. These circumstances are worth mentioning, to show that I did not fall into some of the commoner traps with which the photographer is surrounded. Three of the photographs showed signs of fog, which was probably only due to the length of time which I allowed them to stay in the developer.

The long exposure photographs were not taken with any special object beyond that of obtaining a record of the corona. The plates used were bromo-gelatine dry plates prepared by Captain ABNEY, and I used the ordinary alkaline development. The extension of the corona shown on these plates is not very great, and they show signs of vibration; they have, however, I hope, been of use to Mr. WESLEY in his drawing of the inner parts of the corona. As my main object was to get instantaneous photographs, these plates had to be taken without removing the automatic shutter; the shutter had, therefore, to be worked by hand, and this probably caused the vibration. It may, however, have been caused by a puff of wind; and on another occasion I should take far greater precautions against this danger by surrounding myself with canvas screens in all exposed directions, and as high as possible.

6. *Comments and Conclusions.*

Returning again to the consideration of my observations with regard to the special uses of the coronagraph, it will be seen that my results are adverse to the possibility of obtaining photographs of the corona during Sun light with this instrument. It is, however, I consider, by no means proved that the method is impossible, for there are several reasons why this trial should not be considered conclusive.

(1.) The atmospheric conditions were very unfavourable. The air was fully charged with moisture, and on the morning of the eclipse the sky was certainly not of that dark blue which, no doubt, indicated atmospheric purity. It was slightly hazy, and not, I think, as clear as an average English blue sky. About a minute after totality I noticed a halo with prismatic colours round the Sun—an indication, I presume, of suspended matter.

(2.) The Sun was at a low elevation during the eclipse, and the station was only about a couple of hundred feet above the sea. Both these circumstances, no doubt, increased the air glare.

(3.) Professor THORPE's observations at this eclipse show that the light from the corona was not so bright as on other occasions. This also appears to be the general impression amongst other observers who had seen previous eclipses. If this was due to the corona not being so luminous, the opportunity was, no doubt, an unfavourable one independently of the state of the atmosphere. But this effect is not to be distinguished from the light being diminished by an impure or dense atmosphere. Experiments comparing the light of the corona with Sun light immediately before and after the eclipse would be necessary to settle this point.

(4.) The exposure was too short. I feel confident that at Grenada an alteration in this respect would not have materially altered the result; but, even with the atmosphere in its unfavourable condition, I think a larger exposure should have been adopted. Before leaving England I had come to the conclusion that the right length of exposure would be that sufficient, and not much more than sufficient, for the air glare to produce an effect on the plate. I thought that the corona would be more readily visible if the background of the sky was faintly developed on the negative; and in this opinion I think Dr. HUGGINS would have agreed, as his photographs seem to have been treated in that way. I found that my instantaneous shutter, when set at its slowest rate, gave the required result, and I was at the time quite contented. I have, however, very carefully reconsidered this subject, and now come to the conclusion that this was a mistake. The question to be settled is—At what photographic density do we get the greatest ratio of small changes of shade in the negative, as seen by the eye, compared with the changes in luminosity of the objects photographed? Captain ABNEY gives some results in the form of curves in his Text-book of Science on Photography, which, if modified to suit this problem, would appear to indicate that the ratio is greatest when the photographic density is about one-third or half way between white and the deepest shade due to the full development of the image of a bright object.*

If this is the right interpretation—and I believe it to be so—then it is evident, in order to have the best chance of photographing the small difference of shade between the sky and the corona, I should have given an exposure which would have allowed me to bring up the sky to what I may call a third or a half full density. In order to have done this I should have lengthened my exposure.

Acting in accordance with the advice of Dr. HUGGINS, I adopted a slow development with weak solutions. I could not have followed a higher authority, but I must confess that I have some doubts whether a longer exposure with a quicker but well-restrained development might not have produced better results. The best time for watching for faint outlines is during development, and with a well-exposed plate we have an opportunity of observing the image in every stage.

All these considerations appear to me to point to longer and more varied exposures than I gave, together with a well-varied development.

My conclusions are, therefore:—

(1.) That my results do not prove the impossibility of photographing the corona in Sun light.

(2.) But they prove that under certain circumstances the light of the corona is not sufficient to produce any effect on chloride dry plates with an exposure which is sufficient for the air glare or false corona.

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* See p. 256. The ordinates of his curves represent the shade as measured by the proportion of black to white; the scale of shade as measured by the eye is somewhat different from this. The abscissæ represent the length of exposure, which I imagine to be equivalent to differences of luminosity.

Captain ABNEY pointed out to me—and I think correctly—that the sky or background to the Sun, as shown on the photographs, is due to two causes: first, the light of the sky, and, secondly, the reflected light from the interior of the instrument. Dispersion of light from the mirror or its surroundings would shed a uniform light over the whole surface of the exposed plate; whereas the sky is brightest near the Sun, and possibly not always quite uniform or regular in its appearance. Now, on several of the photographs that I took there is a perfectly uniform light over the whole sky, with no apparent increase of brightness near the limb of the Sun. This constitutes the only hopeful sign I have seen; because, if the above views are correct, it indicates that I have not given a sufficiently long exposure to photograph the sky round the Sun, and gives hopes that the false sky that I did obtain might be diminished by instrumental alterations.

7. *Discussion on the possibility of obtaining Photographs of the Corona during Sun Light.*

But there are certain considerations which appear to me to indicate that a practical method of photographing the corona during Sun light is not likely to be obtained.

If I am right in considering that the increased density round the Sun in these photographs is a true picture of the sky, and not due to irradiation or internal reflection, then it will be seen that this sky illumination is not uniform round the Sun, and that the form of the false corona thus formed would always be difficult to distinguish from the form of the true corona, and would be liable to be mistaken for it. It is quite possible that changes in the appearance of the sky may be more visible when photographed than when seen direct, and that these changes may be as sudden as that from blue sky to cloud. At high altitudes this difficulty would no doubt be lessened, but it would, I think, always be a source of trouble. If, however, these irregularities in the false corona are not due to irregularities in the sky, I can only say that the instrumental defect which causes them is a very difficult one to cure.

In considering the utility of the photographs of the corona taken in Sun light, if obtainable, the effect of the image being superimposed on the much denser picture of the air glare should be carefully considered. In the first place, it is to be observed that the air glare or light of the sky increases rapidly as the Sun is approached, and that the light of the corona also increases in a similar manner. This will, I think, cause the corona to be distinguished with difficulty in the photographs taken in Sun light. This may not be so much the case when considering abrupt changes of shade in the corona, but, in so far as the form of the corona is distinguished by a gradual change of light, it certainly will be. In the following figures, which have no numerical significance whatever, and which are merely intended to illustrate my argument, I have assumed the corona to be only from 10 to 15 times less bright than the air glare. In fig. 2, let the lines round the Sun represent lines of equal intensit of sky light, the corona being supposed to be non-existent. In fig. 3, let the Sun l eclipsed, and the lines represent lines of equal intensity of corona light. Place or image on the other, and the lines of equal intensity of the combined light of coron and air glare will be as shown in fig. 4, in which it will be seen how materially th form of the corona appears to be altered. It is to be noted also that an alteration i the state of the atmosphere by altering the ratio of air glare to corona light woul appear to alter the form of the corona as shown on the Sun light photographs.

For the purposes of the following discussion of the effect of combining the sky an corona lights on one picture, it is assumed that this particular difficulty does nc exist; that is to say, that the air glare is uniform, or, what comes to nearly the sam thing, that we are discussing a radial inequality of the corona. In dealing numericall with this problem, very little trust must be placed in figures used; but, howeve faulty they may be, they may, at all events, help to make the consideration of th subject more easy.

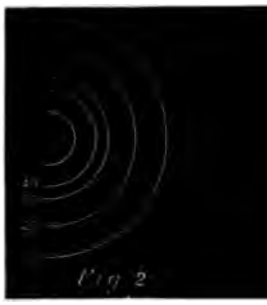

Fig. 2

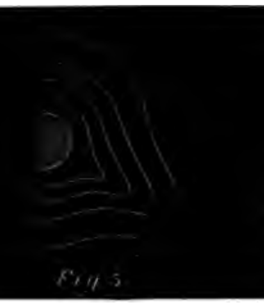

Fig. 3

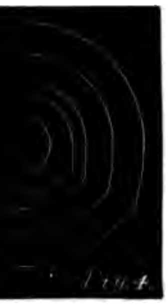

Fig. 4

Captain ABNEY informs me his experiments with regard to sky light when compar with the results obtained by Professor THORPE at this eclipse show that at 30′ fr the Sun's limb the corona, as seen at Grenada, was about 60 times less brilliant tl the air glare at the same distance from the Sun's limb, as seen at South Kensing under the most favourable circumstances, that is, with the darkest sky measured. the sky on this occasion at South Kensington was probably darker than the sk Grenada, the ratio at this latter place was probably greater than 1 to 60; bu the other hand, with a clearer sky the corona would have been brighter and the therefore less. Hence the ratio of 1 to 60 is about half way between the ratio favourable circumstances at South Kensington and the ratio at Grenada ne time of the eclipse. For the purposes of argument, let it be assumed that at G the ratio of the light of the corona to the light of the sky at 30′ from the Sur was as 1 to 50, a ratio, if the above figures are correct, no doubt considerably l the true ratio.

Captain ABNEY also informs me that experiments he has made show that a change of ½ per cent. in the density of a photograph is about the minimum shade that can be seen by the eye; or, in other words, that a photograp regarded as a drawing in which only 200 different shades can be used.

It has already been shown that to obtain the best results a photograph a a third full density should be obtained. Let us assume the most favourable circumstances, and that the development has brought the part of the picture under consideration up to half full density. Then, if there are only 200 possible shades in any photograph, there are, therefore, only 100 possible shades below this shade, which for convenience may be represented by 100. Let the light which caused this shade be also represented by 100. Now, if the shade on the photograph varied proportionately with the light, each of the 100 parts of light would be represented by one shade on the photograph, and a change of light less than 1 per cent. would not produce a change of shade on the photograph which would be visible to the eye. But there is no doubt that under certain conditions the shade on these photographs varies proportionately more rapidly than the light which it represents, and, judging from Captain **ABNEY**'s curves, before mentioned, it is not quite a fanciful supposition to assume that under the most favourable circumstances it varies twice as rapidly in parts of the photograph; that is to say, that a change of ½ per cent. in the luminosity of the object is the minimum change which would be visible on the photograph. Now, in a photograph taken in Sun light of the sky at 30′ from the Sun's limb, the 100 parts of light will consist of 98 of sky light and 2 of corona light. The sky light being assumed to be constant, the minimum visible change of ½ per cent. in the total light must be due to a change of 25 per cent. in the total light of the corona; whereas, with a photograph taken during a total eclipse, on similar suppositions, a change of ½ per cent. in the light of the corona will produce a visible change of shade on the photographs, or, in other words, the totality photographs will show 50 times as much detail as the Sun light photographs. It will be observed that, for the purposes of comparison of the photographs taken in Sun light and during totality, neither an error in the assumption of the shade to which the photographs were developed, nor in the assumption of the ratio of change of light to change of shade, nor in the number of possible shades in a photograph, would vitiate the result, as they would apply equally to both cases.

It therefore appears to me most probable that, under all circumstances, by whatever ratio the air glare is brighter than the corona, in very nearly the same ratio will the detail of the corona be obliterated in a photograph taken during Sun light, as compared with one taken during a total eclipse; that is to say, that, unless a change of shade in the corona were considerably more than 50 times as abrupt as the least change visible in the totality photographs at Grenada, it would be invisible in the photographs taken there in Sun light.

This naturally leads to the question—Are there abrupt changes of shade in the corona, or does the light diminish gradually as the distance from the Sun increases? As far as I have seen, an increase of exposure of the photographs taken during totality regularly brings with it an increased extension of the corona, as photographed; that is to say, that the light of the corona gradually diminishes in intensity from the Sun outwards. The detail of the inner parts of the corona soon gets obliterated as the exposure increases, which shows that this detail is due to delicate rather than to abrupt changes in intensity.

I conclude, therefore, that the effect of the light of the sky on photographs taken during Sun light, when the atmospheric conditions are at all comparable with those at Grenada, would be to entirely alter the form of the corona, in so far as that form is made visible by gradual changes of luminosity; and that, with regard to abrupt changes of luminosity, they would have to be exceedingly well marked before they would be visible in the Sun light photographs, whereas, as a rule, the form of the corona is indicated by somewhat delicate shading.

Thus, the possibility of photographing the corona in Sun light appears to depend on the extent to which at high altitudes, or at other localities, more favourable atmospheric conditions may be found. With clearer air, both the Sun and the corona will be brighter and the air glare less, but to what extent this change may be hoped for I do not know. It is to be observed, however, that besides air glare we have to deal with internal reflection, which increases with the increased brilliancy of the Sun. If my exposure, short as it was, was in reality sufficient for the internal reflection to produce an effect on the plate, with longer exposures and a brighter Sun, but under otherwise more favourable conditions, this might become a serious source of trouble.

The limb of the Moon has undoubtedly been seen for some time after totality (not at Grenada, where no doubt the atmospheric conditions were too unfavourable), and, if visible, it should, I think, be possible to photograph it. But we must be careful in using this as an argument in favour of the possibility of obtaining photographs of the corona in Sun light. In the first place, this phenomenon has generally been observed within a few minutes of totality. For example, Mr. S. P. LANGLEY, in 1878, at an elevation of 14,000 feet, was able to observe the limb of the Moon for 4 minutes 12 seconds after the reappearance of the Sun without taking any precautions to shield his eyes.* Mr. H. H. TURNER has kindly calculated for me that at this time only 0·068 of the Sun's surface was exposed; and, as the outer rim of the Sun has only about half the brilliancy of the central parts, it may be safely assumed that the total light emitted was not more than one-twentieth of full Sun light. If the air glare varies as the total light emitted, and if, as has been assumed for the purposes of argument, the ratio of air glare to corona were 50 to 1 in full Sun light, it would only be 2½ to 1 under these circumstances. This would, of course, make the limb of the Moon much more readily visible against the corona. This argument does not apply to the cases where the limb of the Moon has been seen long before or after totality, but as long as the cusps are fairly sharp another circumstance must be considered. In all the photographs which I took during the partial eclipse in which there is a false corona, it will be seen that it is distributed round the mass of the Sun as a centre, and does not uniformly fringe its outline. The cusps often protrude out of it, and are quite free from it. Hence, the corona will be less overpowered by the air glare near the cusps than at other parts of the limb of the Sun. In the second place, in the case of the Moon being seen against the corona, we are dealing with a sudden change of shade, and not a gradual one, as in the case of the corona. This, as I have, I think, proved, would make it more easily photographed. It would have been encouraging to have obtained photographs of the limb of the Moon during the partial eclipse; but, unless they were obtained when a large part of the Sun was exposed, it would have done little towards proving the possibility of getting photographs of the corona in full Sun light.

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* 'Nature,' November 18, 1886.

It is to be observed that many of these remarks indicating the improbability of obtaining results by this method do not hold good in the case of exceptionally well marked features of the corona, where the changes of shade are really abrupt, and especially if they are radial in their direction. Thus, the first successes that we should expect to obtain would be in such cases as that of the "remarkably formed rift" in the corona of 1883, to which such a distinct likeness is reported to have been seen in photographs taken in Sun light by Dr. HUGGINS in England.

As for hoping to obtain any photographic records in Sun light of the outer limits of the corona as visible to the eye during totality, it is, I think, out of the question, as the following considerations will show, although the actual numbers quoted may be considerably in error. Judging from my own results, 10 seconds with a bromide plate would appear to be too small an exposure for the extreme limits. Assuming that the exposure I gave during Sun light was one-fifth of a second, and that the chloride plates are 30 times less sensitive than the bromide plates, then it would take 1500 times the exposure I gave during Sun light to photograph the outer parts of the corona during totality. On this point it is instructive to compare Mr. WESLEY's drawing from the photographs with Captain ARCHER's direct drawing, probably one of the most accurate ever taken.*

6. *Should the Experiment be Tried Again?*

In conclusion, the questions naturally arise: Is the experiment worth trying again, and if so, under what conditions? As to the latter point, experience seems to show that:—

(1.) It should be tried at a station at great elevation above the sea.

(2.) It should be tried when the Sun is nearly vertical, during an eclipse of the corona by the Moon, Mercury, or Venus.

(3.) The exposure must be more varied and longer. At present I see no reason why increase of exposure should not simply increase the density of the air glare in the same proportion as that of the corona, thus gaining no advantage by the change.

Unless the air glare and internal reflection can be so diminished as to make an exposure of one second with an instrument like that used by me a possibility, I fear no good results are to be expected. As to whether the experiment should be repeated, it is to be observed that the coronagraph is admirably adapted for taking photographs of the corona during totality, for ordinary records; for reflectors have several advantages over refractors for this purpose, and the special contrivances of this instrument would all be more or less useful under the circumstances. Hence, another trial can easily be combined with what may be described as the ordinary routine work of an eclipse. Atmospheric or other conditions are now certainly unfavourable, and time only can settle the question whether this is a permanent or a temporary difficulty.* I should therefore recommend the experiment being made again in connection with other work, but not before the next occasion when the path of a total eclipse passes over high ground (say, over 7000 feet) which can be conveniently reached, and where there appears to be a reasonable chance of having a clear atmosphere. But on all occasions it would, I think, be worth while arranging that some one, not necessarily a trained astronomer, should be detailed to observe for how long before and after totality the outer limb of the Moon can be observed. This was not done at Grenada. During partial eclipses a look-out should also be kept for this phenomenon.

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* See Appendix to Captain ABNEY and Professor THORPE's paper on the Photometric Intensity of the Coronal Light during the eclipse (infra, p. 382).

Most of what I have said has been unfavourable to Dr. HUGGINS' method. But there is very material evidence in its favour, and, for the benefit of any one reading up this subject, I have given the necessary references in a footnote.†

VII. THE PRISMATIC CAMERA. BY CAPTAIN DARWIN.

1. *Description and Adjustments.*

The instrument was the same as that used by Dr. SCHUSTER, in Egypt, in 1882 (see 'Phil. Trans.' 1884, Part I.), and by Mr. LAWRANCE, at Caroline Island ('Phil. Trans.,' A, 1889).

The camera has a lens of 3 inches aperture, and of 20 inches focal length. There is no slit, and the prism, which is placed directly in front of the lens, has a refracting angle of 60°. A photograph taken with this instrument of any small object which has a bright line spectrum will show several images, each one corresponding to some

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* It is a curious fact, that the limb of the Moon beyond the Sun has seldom been reported to have been observed long before totality during the last 15 years, although before that date it is said to have been seen on several occasions. See 'Astron. Soc. Mem.,' vol. 41, 1879, pp. 25–39.

† (1.) "On a Method of Photographing the Solar Corona without an Eclipse." W. HUGGINS, 'Roy. Soc. Proc.,' No. 223, 1882.

(2.) "On some Results of Photographing the Solar Corona without an Eclipse." W. HUGGINS, 'Brit. Assoc. Rep.,' 1883.

(3.) "On the Corona of the Sun."—Bakerian Lecture for 1885. W. HUGGINS, 'Roy. Soc. Proc.,' No. 239, 1885.

(4.) "Photographing the Corona without an Eclipse." E. L. TRONVELDT, 'Observatory,' No. 117 1886.

line in the spectrum. Thus, in a photograph taken during totality, there will be several distinct images of each prominence, and, in measuring the distances between them, we are, in fact, measuring the distances between the corresponding lines in the spectrum of the prominence. A number of small prominences close together gives the appearance of a segment of a circle, while the continuous spectrum of the corona shows as a confused band across the plate. This will explain why it is difficult to obtain a clear interpretation of the various appearances shown in these photographs.

The accurate adjustment of the prismatic camera in its present form is difficult. In order that the photographic plate may be in focus for a large range of the spectrum, the camera is provided with a back which can be inclined to the optic axis of the instrument. In order to find the proper angle and position of the plate, Sun light which had passed through a collimator accurately adjusted for parallel rays was thrown into the instrument. The spectrum thus formed was focussed on the plate, and the instrument, therefore, adjusted for parallel rays entering in the direction of the optic axis of the collimator. The position of the [illegible] and end of the spectrum was then marked on the ground glass of the camera, and the collimator removed. If the instrument were then fixed on the equatorial, it would have been in focus, provided the camera were pointed relative to the Sun as it was before relative to the collimator. This was the awkward part of the adjustment; the camera was turned until the spectrum formed by the Sun occupied the position previously marked. The spectrum, of course, was blurred, owing to the finite size of the Sun, and the adjustment could be made approximately, and [illegible] was obtained. The [illegible] of the [illegible] for [illegible] of different refrangibility must be [illegible] to the [illegible] of different parts of the Sun's circumference. The instrument [illegible] is a [illegible] one, but it affords [illegible] information in every eclipse in which a number of sharply defined prominences appear.

2. Observations and Results

Only one photograph was taken during totality. The exposure was [illegible] nearly two minutes [illegible] about 40 [illegible] clouds. It has been compared with other photographs [illegible] in the same morning, [illegible]

The images of [illegible] prominences can be distinguished, [illegible] measured [illegible] prominences [illegible] the [illegible] of the [illegible] plate.

The [illegible]

designed for measuring the Transit of Venus photographs, and consists of two parallel micrometers, rigidly fixed together and capable of sliding in one direction. With one micrometer the image is observed, and with the other the distance is read off on a glass scale.

Two of the images of all the prominences are especially dense, and from their position, and by reference to previous results, these can be recognised without much doubt as being due to the Solar H and K lines. Taking these as a basis, three other lines, *h*, H (γ), and F, can be recognised in the images of prominences I. and II. with some degree of certainty. The accuracy of this assumption can best be tested by seeing how nearly the wave-lengths of three of these five lines agree with the wave-lengths of three known lines, on the assumption that the two others are known. For this purpose F and *h* were taken as known with wave-lengths 4860·5 and 4101·2 respectively. Then the wave-lengths of the other lines were calculated on the assumption that the distances measured on the plate varied inversely as the square of the wave-length. This last assumption is, however, known to be inaccurate, and a correction had to be found and applied. This was done in the following manner :— A photograph of the Solar spectrum was obtained with the prismatic camera, but placing a slit and collimator in front of the prism. The distances apart of known lines of wave-length as near as possible the same as those believed to have been found in the spectrum of the prominences were measured ; the wave-lengths of these lines were then calculated as before on the assumption that the same two lines were known. We, therefore, get in this Solar spectrum the known and the calculated wave-lengths of certain lines ; the differences between the two give corrections which, if applied to the calculated wave-lengths of the lines of the prominences for the same part of the spectrum, should give their true wave-lengths. The results thus obtained are recorded in the following Table :—

Wave-length obtained as described above.		Order of intensity, both prominences.	Names and wave-lengths, known lines for comparison.
Prominence I.	Prominence I		
4860·5	**4860·5**	3	**4860·5** F
4471·0	..	5	4471·0 *f*
4339·3	4340·3	3	4340·3 H (γ)
4101·2	**4101·2**	4	**4101·2** *h*
3966·3	3965·0	2	3968·5 H
3932·8	3930·5	1	3932·8 K
3890·0	3890·0	3	? 3887·0 H (ζ)

The three other prominences showed only in the H and K lines, and were not, therefore, measured.

It has already been explained how it came about that the instrument was shaken during exposure. The result of the shake is that there are two distinct sets of images, corresponding to two positions of the instrument. The second set of images is far more dense than the other, and was alone used in the measurements. The images in this case are lengthened out in one direction, either by vibration or through imperfect movement of the clock, and prominence No. I. presents a curious appearance in consequence. The image is wide, and is also separated into two parts by a band of white, giving the appearance of another shake, but a comparison with other photographs shows that it is in reality due to the peculiar formation of this prominence. In this instance this white band was used in measuring the distances of the images. The first set of images is much more faint than the other, and, as far as the prominences are concerned, is only visible as the H and K lines of No. I. prominence. The existence of two segments of circles has been mentioned; these were extremely puzzling at first, until it was noticed that they coincided in position with this at first unobserved faint set of images; that is to say, that if the circles had been completed they would have run through the bases of these prominences. No other rings corresponding to other lines can be seen. The H and K lines are in all cases by far the strongest, and it is, therefore, not surprising that these should be visible when the exposure is not sufficient to produce any others. But it is remarkable that, in the second or stronger set of images, the H and K lines of the same prominence should not have rings corresponding to them, for they are far more dense; that is to say, it would appear at first sight that the same object produced an image during a short exposure, and yet failed to produce one during a considerably longer exposure. It is probable from its position that this bright inner circle was covered by the Moon during the period when the one set of images was being formed, but was visible during the other exposure, which is thus presumed to be the first. It is also to be observed that the part of the limb where these rings occur was that which was least obliterated by clouds at the commencement of totality.

VIII. The Spectroscopic Cameras. By Arthur Schuster.

1. *Resolving Power of the Instruments.*

In order to form a clear idea of the amount of detail we may expect in a photograph of a spectrum, it is necessary to enter further than is generally done into the details of the construction and adjustment of the instruments used.

Lord Rayleigh has given the theory of the spectroscope for the case of a sufficiently narrow slit. It is easy to extend his results so as to apply also to the case of slits of which the width cannot be neglected. If R is the horizontal width of the beam entering the prisms (the refracting edge being supposed vertical), r that of the beam leaving the prisms, the equation $\delta i' = \mathrm{R}\,\delta i/r$ defines the angle through which the emergent beam is turned, owing to a small rotation δi of the incidental beam. If α is the width of the slit, f the focal length of the collimator, the geometrical image of the slit will subtend an angle $\mathrm{R}\alpha/fr$ at the centre of the lens of the telescope or camera, and to this we must add the width (λ/r) of the diffraction band of an indefinitely small slit in order to obtain the actual angular width of the image. The angle between two rays having a refractive index differing by $\delta\mu$, is on leaving the prisms, as shown by Lord RAYLEIGH, $t\delta\mu/r$, where t is the aggregate effective thickness of the prisms. This angle must be equal to the angular width of the slit in order that a double line may just be resolved. We then obtain $t\delta\mu = \alpha\psi + \lambda$, when ψ stands for R/f and means the angle subtended by the useful horizontal aperture of the collimator at the slit. If $\delta\lambda$ is the difference in wave-length of the two lines, we find

$$\delta\lambda/\lambda = \frac{\partial\lambda}{\partial\mu}(\lambda + \alpha\psi)/\lambda t.$$

In order to define the actual and possible power of a spectroscope, it is convenient to distinguish between the resolving power of a spectroscope and the purity of a spectrum, and to introduce the following definitions (see 'Encycl. Britann.,' "Spectroscopy") :—

The unit of resolving power of a spectroscope in any part of the spectrum is that resolving power which allows the separation of two lines differing by the thousandth part of their own wave-length.

The unit of purity of a spectrum is that purity which allows the separation of two lines differing by the thousandth part of their own wave-length.

We speak, therefore, of the resolving power of a spectroscope, which is a constant for each instrument, and of the purity of a spectrum, which differs according to the width of the slit. Both resolving power and purity vary inversely as $\delta\lambda/\lambda$, and are equal to unity if that fraction is equal to 10^{-3}. Hence, as from Lord RAYLEIGH'S investigation for narrow slits,

$$\delta\mu = \lambda/t,$$

or

$$\frac{\lambda}{\delta\lambda} = t\frac{\partial\mu}{\partial\lambda},$$

$$1000\,\mathrm{R} = t\frac{\partial\mu}{\partial\lambda}.$$

Also, from the above equations,

$$1000\,\mathrm{P} = t\lambda\frac{\partial\mu}{\partial\lambda}\Big/(\lambda + \alpha\psi),$$

or

$$\mathrm{P} = \mathrm{R}\lambda/(\lambda + \alpha\psi).$$

Here P stands for the purity, and R for the resolving power. They are both numerically equal if the slit α is indefinitely reduced. If the light is weak, narrowing the slit will not diminish the intensity until the width of the slit, as calculated from geometrical optics, is equal to the width of a diffraction band. In that case $\alpha\psi = \lambda$ and the purity is half the resolving power. Hence we shall, when light is of consideration, only be able to make use of half the resolving power of a spectroscope. Whether slits which are wider than the limit here given show the fainter lines on a photograph more distinctly owing to their increased width is a matter for investigation; but it would, perhaps, be well in a future eclipse to adopt that width of slit which makes $\alpha\psi = \lambda$.

Two spectroscopes were to be used on the present occasion, and the following numerical data will allow us to calculate their resolving power and the purity of the spectrum which they gave:—

Spectroscope I.

	cm.
Focal lens of condenser to form image of the corona on the slit	34·0
Focal lens of collimator	106·0
Aperture of „	6·0
Width of slit (α)	·003
Greatest thickness of prism	13·2
Focal length of camera lens	83·1

The beam was limited by the aperture of the collimator lens. As the full aperture of the collimator was used, we must put $\psi = 6/106$ and $\alpha\psi = 17 \times 10^{-5}$; hence, for a wave-length of 4×10^{-5}

$$\lambda/(\lambda + \alpha\psi) = 4/21.$$

With the width of slit used, we have, therefore, made use of the fifth part of the resolving power. To calculate the latter, we have, in the first place, to consider that the effective thickness of the prism used was only 11·4 cm., as the aperture of the collimator lens was not sufficient to use the full width of the prism. The dispersive power of the glass was sufficiently near to that for which Lord Rayleigh has calculated the value 1·02 cm. as the necessary thickness to resolve the sodium lines. As, according to our definition, a resolving power of ·98 is required to separate the sodium lines, we may take the prism to give a resolving power of 1 for each centimeter thickness in the neighbourhood of the sodium lines; it will vary inversely as the third power of the wave-length, and will, therefore, be 3·2 for each centimeter in the violet. We have also to reduce the resolving power in the ratio of 10 : 12, as the beam is limited by a circular aperture. We have, therefore, finally in the violet

$$R = 11{\cdot}4 \times 3{\cdot}24/1{\cdot}2 = 31,$$

$$P = 4 \times 31/21 = 6.$$

The purity was such, therefore, that near the wave-length, 4×10^{-5} two lines differing by the six-thousandth part of their own wave-length should be resolved.

Spectroscope II.

	cm.
Focal lens of condenser to form image of the corona on the slit	26·5
Focal lens of collimator for G	32·5
,, ,, telescope ,,	20·0
Aggregate thickness of two prisms	10·0
Width of slit (α)	·01

The beam was limited by the prisms, the aperture of the collimator and camera being more than sufficient to fill the prisms and transmit the beam.

The calculation is carried on as before, except that we must now take the useful instead of the full aperture of the collimator in order to find ψ. As the prisms were placed in minimum deviation for G, this can be calculated, and we find it to be 2·4 cm.; this determines

$$\alpha\psi = \frac{\cdot 01 \times 2\cdot 4}{32\cdot 5} = 74 \times 10^{-5},$$

$$P = R\lambda/(\lambda + \alpha\psi) = 2R/39\,;$$

or, roughly speaking, the purity in this case was only the twentieth part of what it might have been if the slit had been narrower. The spectroscope was intended originally for the outer parts of the corona, and the slit was, therefore, intentionally kept rather wide. There was at the time no means of measuring the width of the slit, and I did not realise how much of the resolving power was lost by widening the slit. The full resolving power in the violet was 32, and therefore the purity realised 1·6.

2. *Adjustments of Instruments.*

In order to be able to differentiate between the spectra of different parts of the corona, its image must be thrown on the slit of the spectroscope; and for that purpose the plane of the slit must be placed in the focal plane of a condensing lens. This was done as follows: an auxiliary telescope was focussed for parallel rays, and the slit of the spectroscope looked at from behind through the condensing lens, which was moved backwards and forwards until a sharp image of the slit was seen in the telescope. The method is the same as that usually employed for adjusting the collimator of a spectroscope; only that, instead of observing the slit through the collimating lens, we observe it through the condensing lens.

The backs of the cameras employed could be tilted so as to have a larger portion of the spectrum in focus at once. It would be worth while to determine by calculation the best shape of the camera lenses in order that the foci of the different rays shall be as much as possible in a straight line. An achromatic lens is not as good for the purpose as an ordinary lens, but a combination might be found which would give better results than the ordinary lenses.

3. *Results.*

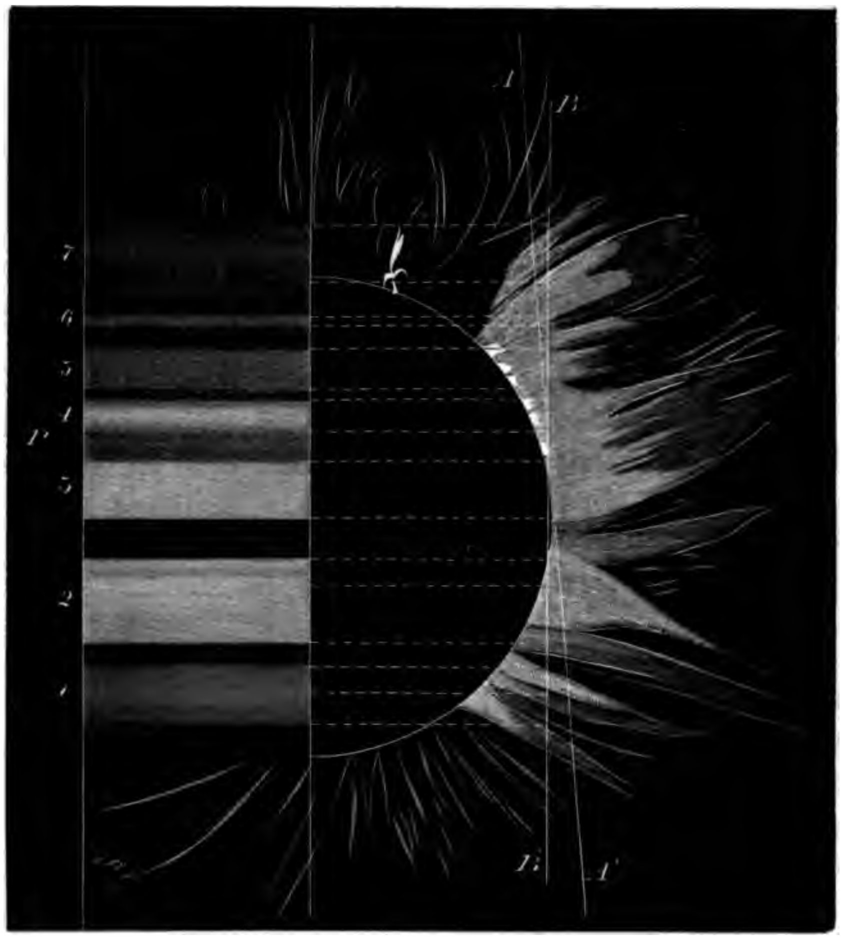
The general appearance of the spectrum of the corona, as shown on the plate exposed during the whole of totality in the spectroscope described above as No. II., is given in Plate 9, fig. 1. It has a streaky appearance, consisting of a series of bands stretching from the ultra-violet into the green, and is crossed by the bright lines of corona and prominences. The horizontal bands are parallel to the dust lines, but they must be chiefly due to the different brightness of the portions of the corona cut by the slit. This is shown by the fact that they are wider and more diffuse than the dust lines shown on the reference spectrum, and that we are able to recognise in the photograph of the corona itself the brighter portions cut by the slit. A few dust lines do actually appear, but can be easily distinguished. Fig. 5, p. 326, shows the section of the corona cut by the slit. The left-hand part of the drawing is a tracing of the streamers in the corona. The line AA′ is the line along which I had intended to place the slit. It is parallel to the declination circle, and tangential to the western limb of the Sun. It was, however, impossible to place the slit accurately in the position aimed at, and I think that, for reasons presently to be mentioned, the slit really ran along the line BB′. It must have crossed one of the prominences, for the prominence lines are very strongly marked, and this prominence will allow us to fix exactly the position of the image on the slit.

The right-hand part of the drawing gives the distribution of light and shade in the spectrum of the corona near the Solar line G reduced to the same scale as the outline of the corona. The images of the prominence appear along the line marked P. If we go from this line towards the southern side we find, in the first place, a strongly marked band, due no doubt to those parts of the corona which are closely adjacent to the Sun's limb. This band and the next are separated by an interval in which the photographic intensity is very small along the spectrum, and I believe that here the slit must to some extent have overlapped the image of the Moon.

Next comes a band marked 2 in the drawings, in which all corona lines appear very strongly. Here the slit must have crossed the most intense streamer of the Solar corona, as will be seen by comparison with Mr. Wesley's drawing. The most southerly band 3 coincides with the section of another streamer. It was the light interval between 2 and 3, and the position of this band (3), which induced me to fix on the line BB′ as the most probable line of the slit.

The part immediately above the prominence towards the north, which must be close to the body of the Sun, is comparatively light; this is partly due to a speck of dust on the slit, but the continuous spectrum of the corona here must have been weaker. Generally speaking, the spectrum all over the northern side is fainter than that towards the south. It will be noticed that the bands are nearer together in the violet than in the blue. This is due to the fact that the back of the camera was tilted in order to have the whole range of the spectrum as nearly in focus as possible. The magnifying power was, therefore, smaller in the violet than in the blue, and hence the tapering of the spectrum towards the more refrangible side.

Fig. 5.



The spectrum can be traced from a wave-length 4950 in the bluish-green to about 3700 in the ultra-violet. The maximum of actinic intensity of the coronal light was decidedly more towards the red end of the spectrum than that of Sun light. This shows that the continuous spectrum is principally due to incandescent matter of a lower temperature than that of the Sun. Although the polarisation of the corona and the appearance of Fraunhofer lines show that part of the light is due to scattering of minute particles, only the smaller part can be owing to this cause, as otherwise the maximum of actinic intensity in the spectrum would be displaced towards the violet and not towards the red end of the spectrum. The faintness of the Fraunhofer lines is further evidence in the same direction, and also makes it probable that there is not much matter round the Sun sufficiently large to scatter Sun light without polarising it.

If we now turn to the lines shown on the photographs, our attention is at once arrested by the H and K lines. They form in this, as in previous eclipses, the most intense feature of the corona. In the Eclipse of 1882, these lines were so bright that the atmosphere scattered enough of their light to make them appear over the disc of the Moon and at a considerable distance outside the corona. In the Eclipse of 1883, the same lines appeared reversed over the lunar disc; on that occasion, therefore, the light scattered by the atmosphere in front of the Moon must have been derived from some source showing absorption lines. In our photographs the same lines end sharply with the corona, and we must conclude, therefore, that, in spite of the unfavourable atmospheric conditions, there was but little light scattered by our own atmosphere in the neighbourhood of the Sun. This is confirmed by the photograph of the corona itself; for, while on previous occasions the sky light formed a bright background, on which the shadow of the wire stretched across the camera showed distinctly throughout the whole plate, the two needles do not show at all on our plates, even on those which had the longest exposure. The analysis of the light seen in front of the Moon, or in the neighbourhood of the corona, seems to give us important information on the general light emitted by it, for in three successive eclipses bright lines, dark lines, and no lines at all appeared on the photograph.

The prominence lines on the plate present a curious winged appearance towards the violet. At first sight it might seem that this is due to some defect in the spectroscope, such as a reflection on the inside of the slit; but, after careful consideration, I do not think that this is possible. One edge of the wing is perfectly sharp, and almost as black as the prominence line itself. The H and K lines in the corona seem perfectly sharp along their whole length, and do not show anywhere a trace of fuzziness such as would be produced by a defective slit. The reference spectrum shows the finest lines with perfect definition. The wing appears in all prominence lines, and, if it is not due to an instrumental defect, must be owing to a rapid motion of the matter forming the prominence. I find that an approaching motion of about 247 miles per second in part of the prominence would account for the displacement. According to YOUNG, such motions have been observed, though seldom. The limb of the Sun on which the prominence appears was receding.

Another fact of some importance is this, that, while the continuous spectrum on the northern hemisphere of the Sun is weaker than on the southern, the H and K lines are stronger. The hydrogen lines Hγ and *h* do not appear at all on the southern side, while I can, under favourable conditions, trace Hγ to some distance on the northern. It will be noticed from Mr. WESLEY'S drawing that the side on which these lines appear is the side which was rich in prominences. A similar phenomenon was shown in the photograph taken in 1882, when the hydrogen and calcium lines appeared strong in the corona on that side on which they were strong on the edge of the Sun, presumably in a prominence.

The explanation which first suggests itself cannot be the true one. We might think that, if many prominences exist over one part of the Solar surface, the corona above, which undoubtedly does to some extent scatter the light falling on it, will show the prominence lines more strongly than other parts which have no protuberances near them. No bright line could, however, be produced in this way, for the light scattered by the solid and liquid particles of the corona must have the same composition as the direct Sun light which reaches the Earth. The particles are illuminated not only by the prominence light, but by the light from the whole Solar surface beneath them. If we imagine ourselves placed near the Sun, the spectrum there seen cannot differ on the average from that which we observe here, and therefore the scattered light must also be of the same composition, and must show dark lines on a bright background, and not bright lines.

Only one other explanation occurs to me. We must imagine that sufficient matter is actually thrown into the corona from the prominences to show the lines. This does not seem surprising if we look, for instance, at the large prominence on Mr. WESLEY'S drawing. The life of such a prominence is short; had the eclipse taken place a few hours later, it might not have appeared as a prominence; but would it not have left behind it a sufficient quantity of calcium and hydrogen to show their characteristic lines? We need only consider the remarkable effects produced in our atmosphere by the Krakatoa eruption, and reflect that matter is projected into the Solar atmosphere on a vastly larger scale, to see that the coronal spectrum, as we observe it, may contain injected calcium, which only very gradually sinks back again into the Sun.

According to this view, we must separate the true coronal spectrum, which will be seen evenly all round the Sun, from the spectrum due to prominence matter, which will differ much in intensity during different eclipses.

The strongest of the true corona lines has, according to my measurements, a wavelength 4232·8. It is slightly less refrangible than the calcium line 4226·4 which forms a conspicuous feature in the Solar spectrum. That it is not identical with it is proved not only by the measurements, but by the fact that on the corona photograph the Fraunhofer line appears faintly by the side of the corona line. As the well known green corona line is always present in the spectrum of the chromosphere, it is very likely that this new line, which appears so strongly on our photographs, should frequently make its appearance there. YOUNG, in his excellent book on "The Sun," gives, indeed, a line 4233·0 as one of about 30 lines which appear there on "very slight provocation." It therefore seems highly probable that the two lines are identical, and that the comparatively weak Fraunhofer line 4233·0 is a corona line.

A faint Fraunhofer line has just been mentioned as appearing in the corona spectrum. The appearance of dark lines has been a source of considerable trouble in the reduction of the photographs. After all the measurements had been made, it struck me that some lines which I had put down as corona lines were really only the intervals between Fraunhofer lines, and I had to subject the photograph to a further careful examination. The effects produced by the overlapping of a spectrum of dark lines over one of bright lines is very complicated; especially, apparently, some of the weaker Fraunhofer lines can be traced, while some of the stronger ones do not make their appearance. I believe that this is due partly to the overlapping of bright and dark lines, but principally to an optical effect of contrast.

A bright line shows black on the negative and is bounded on both sides by an apparently lighter background. This is a well-known contrast effect. The H and K lines, for instance, seem to be surrounded by a lighter band, which follows the contour not only of the lines, but also of the wing by the side of the prominence. If, now, a Fraunhofer line happens to be by the side of a bright line, the contrast is strengthened, and both the bright and the dark lines appear more distinctly than they otherwise would. This is the only simple way in which I can explain some of the appearances of the photographs.

The triplet with a wave-length 4026·0, 4029·7, and 4036·8, which is represented in fig. 1, Plate 9, is a case in point; the group of Fraunhofer lines at 4031, about, is weaker than the strong Fraunhofer line 4045, and yet is much more easily visible on the eclipse photographs. This I believe to be due simply to the fact that the two lines 4029·7 and 4036·8 set off by contrast the intervening group of Fraunhofer lines.

Such contrast effects may, in some cases, have materially affected my measurements. I have made these as carefully as I could, and I believe that the great majority of the lines I have put down as corona lines are really such, but some of the weaker lines may be due to the optical effects I have just described.

Possibly the true spectrum of the corona may be still further complicated owing to the following cause:—The base of the corona gives us evidently a strong continuous spectrum, and it is possible that the lines of the outer corona may therefore appear as dark lines in a bright background. Captain ABNEY found some Fraunhofer lines reversed over the face of the Moon, while the G band was absent. The cause I have suggested may account for this. All these considerations show how very careful we must be in the interpretation of photographs of the coronal spectrum.

I had intended to have made a careful drawing of all I can see on the photographs. Figs. 2 and 3, Plate 9, are specimens of certain portions of the spectrum on a scale 40 times the original.* I had, however, to give up the work, as I found it too trying to the eyes. The length of spectrum represented in fig. 3 is in the original about 2 mm.

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* The distances between the horizontal bands are magnified on a slightly different scale in the two drawings, so as to make them equal, while as before explained, the real distances decrease towards the violet.

By holding the photograph against a bright sky, and examining it with a lens of about half an inch focal length, I can see what I have tried to represent in the drawing. But I could only carry on the work about an hour at the time, and it was always followed by strong pain and neuralgia in the eye, lasting sometimes for several days. I am sorry, therefore, not to be able to accompany this report with a complete set of drawings. The two specimens which I can give will, however, give a fair idea of the appearance of the lines on the photograph. The band reaching from a wave-length 4318·4 to a wave-length 4323·7 has a curious shape. It is broad near the centre of the field, where it is widest, and there nearly covers the bright space on the least refrangible side of the G band, which comes out so strongly in photographs taken on a small scale. If it was of equal width throughout, I should have taken it for that band, coming out by contrast between the Fraunhofer line G and the group of lines at 4324. But the manner in which the line becomes thinner towards the outer parts of the corona shows that it must be a real band.

As we go from this band towards the less refrangible side, we come to very complicated markings until we reach the Hγ prominence. These markings I believe to be due to an overlapping of Fraunhofer lines and corona lines; whether the difference in the appearance on the northern or southern side is real or not I cannot be certain. The line at 4378·1 is the most conspicuous one in this part of the spectrum. By its side we can trace the Fraunhofer line 4383. The series of lines here seem to widen at the base of each bright band of the corona, and they are weak on the northern side.

I have not been able to trace with certainty a difference in the lines of the spectrum of different parts of the corona, except that already mentioned of the calcium and hydrogen lines. The group of lines at 4076 (fig. 3, Plate 9) at first sight looks as if there was such a difference; but we have here possibly only one broad band, and the lighter appearance in the centre may be a defect in the photograph, or it may be due to a Fraunhofer absorption line, which ought to be here, or, finally, to a reversal of a corona line against a hotter background.

The results obtained with Spectroscope I. are much less satisfactory. The plate had a very narrow escape during development. Owing to the hot damp weather the films were often found to detach themselves from the glass in the developing bath; but we had not found any difficulty when the bath was kept cool by ice, and when the film was soaked with alum. Captain DARWIN had kindly offered to do the whole of the photographic part of the work for me, and for this, as well as for continuous assistance in other ways, I have to offer my sincere thanks. The plate in question was developed during the afternoon of the eclipse. For some time nothing appeared, and when at last the image showed itself the plate at once began to frill at the edges. It was only by repeated treatment with ice and alum that Captain DARWIN saved the plate, but the image is very faint. What this is due to, I cannot say with certainty. The plate was one of Captain ABNEY's red end plates, and probably was less sensitive in the blue and violet than those used in the other spectroscope, which also were supplied by Captain Abney. The spectroscope was in good adjustment, and the reference spectrum taken with it on the day of the eclipse is perfectly sharp. If the development could have been carried further without danger, a better image might have been obtained. The plate shows in a very striking way the displacement of actinic intensity towards the red in the coronal spectrum. We can recognise on the plate the absorption of the G band and the strongest corona line, and thus fix the extension of the spectrum. The reference spectrum reaches from F into the ultra-violet, and is very strong between *h* and H. It shows, in addition, a band in the yellow or red, which, however, is very faint. The corona spectrum begins about F, is strongest between F and G, and falls off very rapidly in intensity about *h*. It can be barely traced between *h* and H. The lines shown on this photograph will be given in the annexed Table.

I must now describe the method adopted to determine the position of the lines which appear on the photographs. In 1882 a reference spectrum had been taken on the same plate previous to totality, but that reference spectrum was found to be of very little use in measuring the lines. It had to be near the edge of the plate in order not to interfere with the spectrum of the corona; and, owing to the curvature of the lines, we cannot directly compare the position of a line coming from the centre of the slit with one from near one of the edges. In order to compare accurately two spectra on the same plate, they must be in contact with each other, which would be impossible unless part of the coronal spectrum were sacrificed. A reference spectrum taken on a different plate is as useful as one taken on the same plate, if precautions are taken to fix well the position of the plate in the holder.

All good photographs of the corona which have been taken hitherto show the calcium lines coincident with H and K, and we can take these always as the starting point for our measurements. The distance of these lines from any unknown line will be the same in the centre of the reference spectrum as in the centre of the coronal spectrum, provided no shrinkage of the film has taken place during development. But, if a shrinkage has taken place, there is no reason to suppose that it is uniform all over the plate, and it may, therefore, be just as different in the reference spectrum and the coronal spectrum, if these are on the same plate, as if they are on different plates. Besides the H and K lines we can generally recognise some other feature in the coronal spectrum, consisting either of another prominence line or some well-marked Fraunhofer line; and in that case interpolation becomes easy with the help of the lines on the reference spectrum.

If we add to these considerations the danger of exposing to Sun light, or even day light, a plate which is to be used during the eclipse, merely for the sake of the reference spectrum, I think it will be conceded that I was justified in taking the reference spectrum on a separate plate. That this danger, which I had pointed out before the eclipse, is not imaginary, is shown by the failure of the Carriacou photographs.

In the present instance the hydrogen lines Hγ, *h*, and those in the ultra-violet, appear in the prominence spectrum as well as H and K, and they give us sufficient data to determine the wave-lengths of the lines which appear in the spectrum of the corona. The measurements were taken by means of a dividing machine reading to ·002 mm. The following Table gives in centimeters the distance of three of the hydrogen lines from H, as measured on the reference spectrum and on the spectrum taken during the eclipse.

	Reference spectrum.	Eclipse spectrum.
h	·3187	·3184
Hγ	·7943	·7930
F	1·5362	1·5382

h and Hγ appear as prominence lines, and their distances from H are practically the same on the two plates. About F there is more doubt. There is a faint mark on the eclipse spectrum which I had originally taken to be the image of the prominence which I expected to find there, but which may easily be an accidental spot on the photograph. The distance of this spot from H is 1·5450 or the tenth part of a millimetre further than F. The discrepancy seemed to me to be too great, and on examining the photograph again I found on the more refrangible side of the spot a distinct absorption line, which, when measured, gave 1·5382 as distance from H. Considering the faintness of the image near F, the measurement seemed now to agree, as well as could be expected, with the reference spectrum, on the supposition that F shows as an absorption line in the corona. I conclude, therefore, that the distance of a given line from H on the eclipse plate is the same as that of a corresponding line on the reference plate. In order to find the wave-lengths of the lines in the corona I proceeded as follows.

The distances of the following lines were carefully measured on the reference spectrum from H:—

	λ
Fe	4004·9
Fe	4045·1
h	4101·2
Ca	4226·4
Hγ	4340·1
Fe	4666·5
F	4860·7

Assuming the wave-lengths of F and of H to be given, we can, from the observed positions, calculate by some interpolation formula the wave-lengths of the intervening reference lines. The formula I used is the usual one

$$\mathrm{D}\left(\frac{1}{\lambda_1{}^2} - \frac{1}{\lambda_x{}^2}\right) = \alpha\left(\frac{1}{\lambda_1{}^2} - \frac{1}{\lambda_2{}^2}\right),$$

where λ_1 and λ_2 are the two known wave-lengths, λ_x that to be determined. D is the distance measured on the plate between λ_1 and λ_2, α that between λ_1 and λ_x. The wave-lengths thus found are approximate only; but, by comparing the calculated values with those found in Ångström's map and given above, we can construct a table of corrections for the reference lines. A curve was drawn having the calulated wave-lengths as abscissæ and the corrections as ordinates. This curve was found to be quite regular in shape. The same interpolation formula was now used to calculate approximate wave-lengths for the corona lines, and the corrections were read-off from the curve. I have found this combination of interpolation by calculation and by a graphical method to be very convenient. In order to show the accuracy which may be obtained in this way I give the following example:—

I left out originally the reference line 4666·5 altogether. There was, therefore, a very large gap between the lines F and Hγ. The curve had to be drawn, more or less, as a straight line between the points corresponding to the two lines. Treating the reference line, then, first like an ordinary corona line, that is, finding its approximate wave-length by calculation and the correction from the curve, I found 4667·0 as its wave-length, being very near the correct value. The curve was improved by making it still further agree with this line, and, as there is no gap of equal magnitude in other parts of the spectrum, I concluded that now the errors due to interpolation are negligible. It is more difficult to give an idea of the possible error due to faulty measurement. Most of the important lines were measured several times, and, from the way in which the wave-lengths agree, I should say that an error of 1·5 near F and 1 near H in Xth metres will not often occur in the stronger lines. The position of the weaker lines is, of course, more uncertain.

As the strong corona line 4232·8 had to be used as the starting point for the measurements of the second plate, considerable trouble was taken to find its position. Perhaps the best idea of the accuracy which can be reached in such measurements can be got from a comparison of the individual measurements. The following then are in centimetres, the distances from H of the corona line and of the calcium line 4226, as measured on the corona plate and on the reference spectrum respectively. The measurements were taken at different times during the last two years. Those marked * were taken by my assistant, Mr. Arthur Stanton.

Distance from H.

Corona line.	Calcium line.
·5920	·5810
·5920	·5810
·5950	·5752
·5930	·5780
·5930	·5790
·5920	*·5804
·5930	*·5816
·5930	*·5804
·5902	·5810
·5910	·5824
·5930	·5822
*·5904	
Mean . ·5923	Mean . ·5802

Reduced to wave-lengths, the difference is 6·4, and, taking the wave-length of the calcium line as 4226·4, I arrive at the adopted wave-length 4232·8.

The plate taken with Spectroscope I. was more difficult to reduce. When the G absorption and the strong corona line had been recognised, the distances of the other lines were measured, and the corresponding wave-lengths found by help of the reference spectrum. All measurements of this plate were taken by Mr. STANTON, as my eyes could no longer stand the strain. The measurements are very difficult, as the band which has left a mark on the plate is exceedingly narrow, and it is sometimes impossible to distinguish accidental spots from real lines. Mr. STANTON has found a number of apparent absorption lines in the spectrum. Most of these agree with strong Fraunhofer lines, and this justifies the belief that his measurements are substantially correct as giving the positions of certain markings on the plates; but it is quite possible that specks of dust or other accidental marks have occasionally been taken for corona lines.

The following Table gives a list of the lines which have been measured on the two plates. For reference, the corona lines observed in 1882 and 1883 have been added. A query is attached to those wave-lengths taken in 1883 which have not been included in Captain ABNEY'S final list. The lines which are printed in thicker type are those I have measured repeatedly, and about which I can speak therefore with greater confidence; but all lines given have been measured at least twice. In the column headed intensity—(6) represents the highest intensity, and (1) the lowest. The intensities refer to the appearance of the lines in the centre of the spectrum, that is, on the prominence line. The numbers given are, of course, very approximate only, as the judgment about intensities much depends on the fatigue of the eye and other circumstances. Down to intensity (3) the lines are easily visible; (2) means more difficult to see, but not doubtful; the lines marked (1) are doubtful.

	Spectroscope II. Tangential slit.	Intensity.	Spectroscope I. Radial slit.	Eclipse 1882.	Eclipse 1883.	Remarks.
1	3923·9	2				
2	**3933·0**	6	..	3933	3934	K.
3	3942·3	2				
4	3950·1	2	..	3948		
5	3958·8	2				
6	**3968·1**	6	..	3968	3969	H.
7	3975·4	3				
8	3979·7	1				
9	3985·0	1	3936	..	3986 ?	
10	3990·0	2	..	3992		
11	3998·4	2	4003	..	..	An absorption line shown here in Spectroscope I., 3997, reversed across Moon's disc in 1883.
12	4004·0	1	..	..	..	
13	4009·7	3	4008			
				4015	4016	Strong in 1882.
14	**4026·0**	3	..	..	..	There is a group of Fraunhofer lines in the Solar spectrum about 4031, which is apparently visible between these lines; hence the actual positions of the lines are a little uncertain.
15	**4029·7**	4	..	..	4031 ?	
16	**4036·8**	4	..	..	4037 ?	
17	..	..	4047	4044	4045 ?	Apparently double in Spectroscope I.
18	4054·8	2	4050			
19	**4056·7**	3	4054	4057	4056	
20	**4063·5**	3	..	..	4064	The centre line shows only in the outer parts; possibly covered by Fraunhofer line 4071 in the centre. This line reversed across Moon's disc in 1883; possibly a broad line showing reversal in centre.
21	**4067·7**	1	4071	4067	..	
22	**4075·7**	3	..	..	4075	
23	**4084·2**	4	4087	4085	4085	4084 extends outwards in Spectroscope I. The two lines 4084 and 4089 are the most striking objects in this part, and look like band fading towards red (see fig. 3, Plate 9).
24	**4089·3**	4	4090			
25	**4099·7**	2				
h	4101·2	..	..	4101	..	*h* prominence line, presence in corona towards North uncertain; not visible in corona towards the South.
26	4105·2	2				
27	4112·5	2				
28	4122·7	2				
29	4128·0	1				
30	4138·0	2	..	..	..	There was a group of lines here in 1883, but the measurements do not seem to agree very well. Spectroscope I. appears to show an absorption line at 4143.
31	4144·2	2	..	..	..	
32	4148·9	4	..	..	..	
33	4156·2	3	..	..	..	
34	4165·0	3	4163			
35	4169·7	3	4166	4168	4169	
36	4173·6	2	..	4173 (short)	..	Absorption line shown in Spectroscope I. at 4180.
37	4177·3	3	..	4179		
38	**4183·5**	3	..	..	4185	

	Spectroscope II. Tangential slit.	Intensity.	Spectroscope I. Radial slit.	Eclipse 1882.	Eclipse 1883.	Remarks.
39	**4189·2**	3				
40	**4195·0**	3	4198	4195	4192	Extends outwards in Spectroscope I.
41	**4203·5**	3	..	..	..	Shows as absorption line in Spectroscope I.
42	**4211·8**	3	4212	4212	4213	
43	4216·5	3				
44	**4222·6**	3				
45	**4232·8**	5	4233	4224	4227	
46	4237·9 ?	1	..	..	..	Doubtful.
47	**4241·0**	4	..	4241	..	Shows as absorption line in Spectroscope I.
48	**4247·2**	4	..	..	4248 ?	There was evidently a group of lines here in 1883, but the measurements given in the Report of that eclipse do not agree very well at this point.
49	**4253·6**	3	4251	4252	4255	
50	4258·5	3	..	..	..	
51	4268·5	3	..	4267	..	
52	4275·5	2	..	..	..	
53	4280·6	4	..	..	4279	
54	4286·3	3	..	4289	4291	
55	4293·9	2	4293	..	..	These lines are probably only the intervals between the groups of lines which make up the G band.
56	4301·0	2	..	..	..	
57	4305·5	2	..	..	..	
58	4310·2	1	..	..	..	
59	**4318·4** **4323·7**	4 4	.. ..	.. ..	.. ..	A band broad in the centre and narrowing in the outer parts of the corona.
60	**4332·1** **4339·7**	.. ..	.. ..	.. ..	.. ..	The photograph here shows complicated markings (see fig. 2, Plate 9). The radial spectroscope shows an apparent absorption at 4336.
61	**4346·9**	2	..	..	..	This line is strongly marked in the most southerly band, but this is possibly accidental.
62	**4354·7**	1	..	..	4353	
63	**4365·4**	3	..	..	..	Triplet very clearly marked in bands (2) and (3). The line 4378·1 is very conspicuous. 4370 is marked short and winged in 1882.
64	**4372·2**	3	..	4370	4370	
65	**4378·1**	4	..	..	4377 ?	
			4389	..	..	Extending outwards as an absorption. There is a strong Fraunhofer line at 4384.
66	**4387·6**	2				
67	**4395·8**	2	..	4395	..	This line shows strong in band (1).
68	**4402·2**	3	..	4401	4400 ?	These bands present a curious appearance in Spectroscope II., being very broad near the middle and weak. 4422 and 4427 appear strong in Spectroscope I. 4401 was short in 1882.
69	**4411·3**	2	..	4414	..	
70	**4418·3**	1	4422	..	..	
71	**4427·5**	3	4427	..	4427 ?	4427·5 seems rather stronger in the northern part of the corona.
72	**4434 8**	2				
73	4445·8	1	..	4442	4449 ?	
74	4452·9	3				
75	4460·7	2				
76	4468·5	3	..	..	4465 ?	

	Spectroscope II. Tangential slit.	Intensity.	Spectroscope I. Radial slit.	Eclipse 1882.	Eclipse 1883.	Remarks.
77	4474·4	2	..	4473	4473	
78	4482·7	2	4481	..	..	Extending outwards in Spectroscope I.
79	**4485·6**	3				
80	4493·4	1	4491	..	..	Extends outwards in Spectroscope I.
81	4498·5 }	3	..	4501	4501	Marked double in 1882.
82	4505·4 }	2				
83	4508·9	2				
84	4515·6	2	..	..	4518 ?	
85	4520·7	1	4520	4526		
86	4530·0	1	4527	..	..	Shows as an absorption line in Spectroscope I.
87	4536·1	2				
88	4541·3	1				
89	4547·8	2	4550	..	4546 ?	This and the succeeding two lines are very broad at the base of each horizontal band.
90	4557·2	2	4563	..	4555 ± 3 ?	Marked in Spectroscope I. as the beginning of a band.
91	4570·2	2	..	..	4571 ?	The continuous spectrum here is very strong, and the line difficult to measure.
92	4579·7	1	..	..	4577 ?	
93	4588·8	1	4586			
94	4596·2	2	4593	..	..	Extends outwards in Spectroscope I.
95	4605·9	2	..	..	4604	
96	4616·9	2				
97	4621·7	1	4620	..	4620	
98	4627·9	4	4631	..	4636	
99	4644·0	3	..	..	4642	
100	4652·1	3	4652			
101	4657·9	3				
102	4667·6	1				
103	4678·9	2	..	..	4673	
104	4686·0	1	..	..	..	Broad.
105	4696·0	..	..	..	4695	Very uncertain.
106	4708·0	..	..	..	4706	
107	4725·0	..	..	4721	4717	
108	4733·0	..	..	..	4730	
109	4746·0	..	..	..	4738	
110	4796·0	..	..	..	..	
111	4881·0	..	..	..	..	
112	4834·0	..	..	..	..	
113	4877·0	..	..	..	..	
114	4891·0	..	..	..	..	
						End of visible spectrum at 4948.

If we compare together the measurements taken during different eclipses, we find that the agreements are good, especially those taken with Spectroscope II. in 1886 compared with those taken in 1883. The resolving powers in the two cases were about equal, and double that used in 1882. Nearly all lines found in 1883 have a representation of intensity (3), or above in 1886. Our data as yet are insufficient to judge whether there is any great difference in the spectra during different eclipses. The slit in 1883 was radial; that in 1886 was placed tangential to the Sun's limb, and this would account for some difference in the relative intensities of lines, for some of the lines in the radial spectroscope fade undoubtedly more rapidly as we go outwards in the corona than others. A number of lines are wide in those parts of the corona which show the continuous spectrum, strongly thinning out where that spectrum is weak. Nevertheless, I cannot help thinking that, if the line 4232·8 had been as strongly marked in previous eclipses as it has been in this, it would have attracted special notice. On the other hand, a line 4015, which was marked strong in 1882, and which was present in 1883, does not appear in our photographs. There is, therefore, some ground to believe that the spectrum of the corona differs much on different occasions, although we want further evidence to settle the point definitely.

I have collected in the following Table the most persistent lines in the spectrum of the corona; that is to say, those which appear in all three eclipses.

1886.	1883.	1882.
4056·7	4057	4056
4084·2	4085	4085
4169·7	4169	4168
4195·0	4195	4192
4211·8	4213	4212
4232·8	4227	4224
4253·6	4255	4252
4372·2	4370	4370
4402·2	4401	4400
4474·4	4473	4473
4498·5 4505·4 }	4501	4501 (double)
4725	4718	4721

It seems curious that the discrepancy is greatest for our strongest corona line. The line 4227 in 1883 may have been the calcium line. That our corona line is less refrangible is proved not only by the measurements which are given above in detail, but can be shown by simply placing the corona spectrum film to film against the reference spectrum. If the H lines are made to coincide, the corona line is shown distinctly to be by the side of and not above the Fraunhofer line belonging to calcium.

For the greater part of these lines the agreement is as good as can be expected.

Amongst the stronger lines the following have been observed in 1886 and 1883, but not in 1882:—

1886.	1883.
4029·7	4031
4036·8	4037
4063·5	4064
4075·7	4075
4183·5	4185
4247·2	4248
4280·6	4279
4378·1	4377
4427·5	4427
4468·5	4465
4627·9	4636
4547·8	4546
4557·2	4555±3
4570·2	4571

Here again the agreement is good for the greater number of the lines.

If we endeavour to trace coincidences between the corona lines and the lines of known elements, we meet with serious difficulties. Owing to the multitude of lines, accidental agreements will be frequent, and no certain conclusions can be drawn unless we can trace a number of coincidences, or, at any rate, discover some group repeated with all its characteristic features. Nor must we forget that the green corona line, which, before the Egyptian Eclipse, was the only known corona line, has no representative in the spectrum of terrestrial substances, and the same holds for a large number of the lines occurring most frequently in the Solar chromosphere.

It should, therefore, not surprise us if we cannot recognise any of our elements in the corona.

I have been much puzzled, however, by a series of coincidences, which, although I have finally come to the conclusion that they are accidental, yet seem to deserve being mentioned in this report. If we look at photographs of the line spectrum of nitrogen, the most striking features in the violet are as follows:—

Nitrogen.		Corona.	Manganese.
1. A very strong line at	3994·5		
	4025·3	4026·0	4029·5
2. A triplet, the least refrangible being the strongest	4034·4	4029·7	4031·8
	4041·7	4036·8	4040·6
	4228·9	4232·8	4227·0
3. A triplet, in which the two least refrangible are the strongest	4236·4	4237·9 ?	4234·6
	4240·6	4241·0	
4. A group of lines, having its two strongest lines at	4628·9	4627·9	
	4641·2	4644·0	

Now almost the three strongest corona groups occur exactly at the same places as the groups 2, 3, and 4. When we compare the individual lines within each group, however, the agreement is not so satisfactory, and the defective coincidence in group 3 argues strongly against its reality. The strongest corona line is at 4232·8, falling in the middle between the two nitrogen lines; but the line 4228 is really a weak nitrogen line, and while the other two are about equally strong and so broad, at any rate at atmospheric pressure, that they would not be separated on our photographs, but show as a broad band with 4239 as centre, I conclude that the strongest corona line can have nothing to do with nitrogen; and this makes the other coincidences very doubtful, especially as the strongest of all violet nitrogen lines at 3994·5 has no representative in the corona spectrum. Nevertheless, I think it would be worth while to look for the strong green nitrogen line in the spectrum of the corona on the next occasion. I have also examined the spectrum of oxygen, finding a number of curious coincidences which are also very likely accidental only.

Finally, there are some of the corona lines which seem to lie very near lines of manganese. As Mr. LOCKYER has shown that the spectrum of manganese plays an important part in cosmical spectroscopy, the coincidences deserve careful consideration. I have, therefore, added a column for the manganese lines in the above Table. The strongest manganese lines in the violet are 4235·0 and 4230·7; the mean of these two numbers is 4232·85, or exactly at the place at which the strongest corona line is placed. I think that our opinion as to the presence of manganese in the spectrum of the corona must depend on the question whether the two manganese lines could possibly, with the instruments used, look as a single line. My own opinion is against such a view. Our photograph resolves lines which are quite as close or even closer together than the two lines in question, which are quite sharp. The corona line on our plate has a certain width filling a space between 4231 and 4235·0, that is to say, it fills exactly the two spaces between the two manganese lines, but does not overlap them. It is strongest near its centre. From the width of the slit used, I calculate that an indefinitely thin line would, in this part of the spectrum, cover a space of about 2·6 units, so that the manganese lines should be separate, and reach respectively from 4228·4 to 4232·0, and from 4233·7 to 4236·3. The corona line should, therefore, show very decided signs of duplicity, and I cannot reconcile its actual appearance with the supposition that it is the representative of the two manganese lines. For the present, then, our attempt to identify corona lines has only led to negative results.

4. *Summary of Results and Suggestions.*

In conclusion, I give a summary of the principal results of the spectroscopic camera, together with a few suggestions which may prove useful to future eclipse observers.

Summary of Results.

1. The continuous spectrum of the corona has the maximum of actinic intensity displaced considerably towards the red, when compared with the spectrum of sky light. This proves that it can only in small part be due to light scattered by small particles.

2. While on the two previous occasions on which photographs of the spectrum were obtained lines showed themselves outside the limits of the corona, this was not the case on this occasion. Hence there must have been less light, due to the scattering in our atmosphere.

3. Calcium and hydrogen do not form part of the normal spectrum of the corona. The hydrogen lines are visible only in the parts overlying strong prominences: the H and K lines of calcium, though visible everywhere, are stronger on that side of the corona which has many prominences at its base.

4. The strongest corona line on the present occasion was at λ = 4232·6; this is probably the same line as 4233·0 often observed by YOUNG in the chromosphere.

5. Of the other strong lines the positions of the following seem pretty well established:—

4056·7. **4084·2**. [illegible], **4169·7**, **4196·0**. **4211·8**.
[illegible]. [illegible]. **4372·2**. [illegible]. [illegible]. [illegible].

The lines printed in thicker type have been observed also at the Caroline Island and Egyptian Eclipses.

6. A comparison between the lines of the corona and the lines of terrestrial elements has led to negative results.

Suggestions concerning the Spectroscopic Arrangements at future Eclipses.

1. In order to distinguish better any differences in the spectra between different parts of the corona, a larger image should be thrown on the slit. A lens of 4 or even 5 feet focal length might be employed with advantage. The aperture of the lens need not be larger than that required to fill the collimator with light.

2. The width of slit should be equal to $f\lambda/R$, where R is the useful aperture of the collimator lens, λ the wave-length and f the focal length of the collimator. In order to prevent difficulties due to dark lines &c., f should be about 4 or 5 feet.

3. A resolving power of about 2 in the yellow, if full use is made of it, seems sufficient. This can be obtained either by one large prism or two small ones.

IX. PHOTOGRAPHIC RESULTS OBTAINED AT CARRIACOU ISLAND.

BY E. W. MAUNDER.

The work allotted me in the observation of the eclipse was purely photographic, and was intended to be, in its general character, a duplication of that undertaken by Dr. SCHUSTER. The photographs which I was to take were to be both of the corona itself and of its spectrum. For the former I was provided with a lens of about 4½ inches aperture, corrected for the photographic rays, and having a focal length of about 5 feet. The diameter of the image of the Moon on the photographs, which were taken in the primary focus, was therefore about six-tenths of an inch. For the photographs of the spectrum I had two spectroscopes, the second of which was only provided immediately before the instruments were packed up for shipment. The first spectroscope had two prisms, each 1·75 inch in height and 2·5 inches in base, and with refracting angle of 62°; and it was used in conjunction with a condensing lens of 3·5 inches aperture, and focal length of 17·5 inches. The second spectroscope had one prism, 2·6 inches both in height and base, and with refracting angle of 60°; the condensing lens used to throw an image of the Sun on the slit of this spectroscope was 3 inches in aperture, and had a focal length of 14·5 inches. These three instruments were all attached to the same frame, which was mounted equatorially and supplied with clock-work. The polar and declination axes and the R. A. and declination circles were those of the Corbett Equatorial of the Royal Observatory, Greenwich; the driving-clock also belonged to the same instrument, but the stand to which these were attached was made specially for the expedition. It was a tripod stand, composed of pieces of angle iron bolted together, and was found to be light and portable, and at the same time strong and steady. In addition to the camera and spectroscopes, a telescope of 3·6 inches aperture and 5 feet focal length, together with its finder, was mounted on the same stand; and a lens of 1 inch aperture and 4 feet focal length, with a little screen in its primary focus, was attached to the side of the coronal camera as a finder.

Owing to a delay in the selection of the equatorial mounting to be assigned to my use, and to the fact that the second spectroscope was added to the equipment as an afterthought, the preparation of the entire instrument was thrown so late that it was completed only just in time to be packed for shipment, and I had no opportunity, even for a moment, to test its performance as a whole, or of the different parts separately, except the coronal camera, until my arrival at Carriacou. On setting up the instrument there, it was at once seen that the driving-clock drove in the wrong direction, the Corbett equatorial having been last used in the Southern hemisphere. By the help of one of the artificers of the "Bullfrog" this was altered, and the driving of the clock rendered fairly good. Its actual rate was not determined, as the necessary alterations were not completed until the day before the eclipse. But the gearing of the hour circle into the driving-screw of the clock remained loose and unsatisfactory, and it was not found possible to remedy it. The spectroscopes, too, when put together, proved to require many alterations in small details, which cost the most unceasing labour, but which were all effected very satisfactorily before the day of the eclipse.

The programme which I was to carry out with the above instruments comprised the taking of one photograph of the spectrum of the corona during totality with each of the two spectroscopes, and of seven photographs of the corona itself with the four-foot coronal camera. The slit of the two-prism spectroscope was to be [illegible] so as to point east and west, and to lie across the centre of the Sun. [illegible] prism spectroscope was to lie north and south, and to [illegible] of the Sun.

The expedition reached Grenada on the afternoon of [illegible] the afternoon of the following day [illegible] PERRY and myself were taken [illegible] H.M.S. [illegible] daybreak on August [illegible] Carriacou [illegible] miles north of Grenada [illegible] Grenadines. [illegible] of the island and made the necessary examination [illegible] regarding other portions of the island [illegible] site for our observing station. [illegible]

[illegible]

[illegible]

The entire programme of the eclipse had also been rehearsed on two occasions, viz., the mornings of Thursday and Saturday, August 26 and 28. Friday, August 27, was wet.

On the day of the eclipse the Sun rose behind cloud, and the sky was generally overcast, though with breaks here and there. A smart shower fell shortly before the eclipse, which necessitated a hasty closing of the observing huts, but it passed off before totality began, and the total phase was observed in what appeared to be an entirely clear space of sky. An alarm clock, ringing at every tenth second, was set up in the observing hut, and the timekeeper started the clock at the moment totality commenced, and called out at every tenth second the number of seconds yet remaining before the first re-appearance of Sun light. The clock face had been numbered for 205 seconds, and the total phase was over about one second and a half after the last had been counted, so that, estimating half a second for the delay in starting the clock, the duration of totality must have been 3 minutes and 27 seconds.

As totality approached I watched the Sun in the finder of my telescope, and gave the word "Start the clock" to the timekeeper as the last ray of Sun light disappeared. He started the striking-clock very promptly, and called the seconds very sharply and clearly throughout the eclipse.

I drew back the slide of the camera of the two-prism spectroscope first, then that of the single-prism spectroscope, and then proceeded to expose seven plates upon the corona itself with the five-foot camera, the several plates being exposed for the following times :—

First plate	0·2 second.
Second plate	2·0 "
Third plate	10·0 "
Fourth plate	40·0 "
Fifth plate	7·0 "
Sixth plate	4·0 "
Seventh plate	0·2 "

The fourth plate was exposed at the word "120 seconds," *i.e.*, 85 seconds after the commencement of totality, and closed at the word "80 seconds." The seventh plate was exposed before the word "20 seconds." It would have been quite possible to have obtained at least two more short exposure plates had I had them ready, but I had not judged it wise to attempt more than the seven of the original programme, as I had not been able to manage more than that number during the rehearsals, but I found that I was able to work more rapidly and collectedly during the eclipse itself than during the preliminary drills.

At "10 seconds" the timekeeper gave the word "close cameras," and I closed the single-prism spectroscope first, and the two-prism spectroscope afterwards. Both were closed some seconds before the end of totality. I was able to look up at the corona during the exposure of my plates, and I watched it through the finder during the 40 seconds exposure of the fourth plate. I saw no trace of a red or rosy tint in either chromosphere or prominences, but I remarked two exceedingly bright and beautiful prominences of the intensest silver whiteness. The taller of these is very well shown on some of my photographs.

The light during totality was feeble, but was just barely bright enough to enable me to read the programme which I had written out in a bold round hand, and had pasted on the top of the coronal camera.

After totality the Sun was covered by light cloud almost immediately, but a photograph was secured to give the direction of the two needles which were fixed in the east and west sides of the coronal camera, close to the sensitive plate; and later on the Sun was brought on the bottom of the slit of each of the spectroscopes and the plates were re-exposed for a second in order to secure a reference spectrum.

The photographs were not developed until after the return of the expedition to England, when Captain ABNEY kindly consented to undertake the operation. No ice could be obtained at Carriacou, and many of the best trial plates, taken for the purpose of ascertaining the focus of the different cameras, had been spoiled or completely lost by the heat. It was, therefore, thought unwise to run the risk of developing the eclipse photographs at Carriacou, and the plates were accordingly securely sealed up, and brought home undeveloped.

Of the seven photographs of the corona taken with the five-foot coronal camera, five proved to be good, one showed some deformation of the image, and the seventh was spoiled. The spoiled plate was the fourth in order of exposure, and was exposed for 40 seconds; the accident which rendered it useless was brought about in the following manner:—Mr. DRUMMOND, the owner of the estate on which we had fixed our observing station, and who had been our most self-sacrificing host, had looked through the little telescope attached to the coronal camera during the first 80 seconds of totality, but immediately on the exposure of the plate in question he stepped down and I took his place. Unfortunately, in making the transfer in the semi-darkness, the instrument received a severe jar, a jar rendered the more serious by the unsatisfactory character of the gearing of the R. A. circle alluded to above. The clock, however, drove the telescope very satisfactorily, both before and after this occurrence. The other plates were placed, with the photographs of the other observers, in the hands of Mr. W. H. WESLEY, Assistant-Secretary of the Royal Astronomical Society, who has prepared a drawing from the collation of the entire series. The plates exposed upon the corona were supplied by Captain ABNEY, and were 3¼ inches by 4¼ inches.

The two spectrum plates were also supplied by Captain ABNEY, and were 1⅝ inch by 4¼ inches in size. Both these unfortunately proved to be useless, for, on development, the coronal spectrum was found to be masked by an ordinary Solar spectrum.

It appears most probable that, whilst taking the reference spectrum, I inadvertently exposed the plates to full Sun shine for a moment or so; for it does not appear possible that the exposure at the actual time of the eclipse can have been prolonged beyond the duration of the total phase. It is to be much regretted that the attempt was made to secure anything beyond the coronal spectrum upon the same plate with it, and that any instrument not absolutely necessary should have been mounted on the same stand as the cameras and their accessories. But for this mistake I should probably have had to report the success of all the nine photographs instead of that of only six of them.

X. DESCRIPTION OF THE ECLIPSE AND DRAWING OF THE CORONA. BY CAPTAIN IRWIN C. MALING, COLONIAL SECRETARY.

The Total Eclipse of the Sun on the 29th August, 1886, was observed by me from Prickly Point, Grenada, West Indies, the station selected by Captain L. DARWIN, R.E., and Mr. A. SCHUSTER, F.R.S., of the Eclipse Expedition; they kindly requested me to take charge of the disc, and the following are the results of my observations:—

Previous to the commencement of totality my eyes were covered for 10 minutes to enable the sight to be as strong as possible; I had, however, scarcely begun my drawing when a small drift of cloud passed over the eclipse, hiding it for about 40 seconds, after which time it was perfectly clear, and I was enabled to continue my observations. The Moon was surrounded by a bright halo resembling that painted round the heads of saints in old pictures, from which long streams of light extended, varying much in length, form, and apparent density. The longest ray was on the upper right side. It was of a bright pale yellow, fading into white at the extreme point; it appeared to be about two and a half times the diameter of my disc.

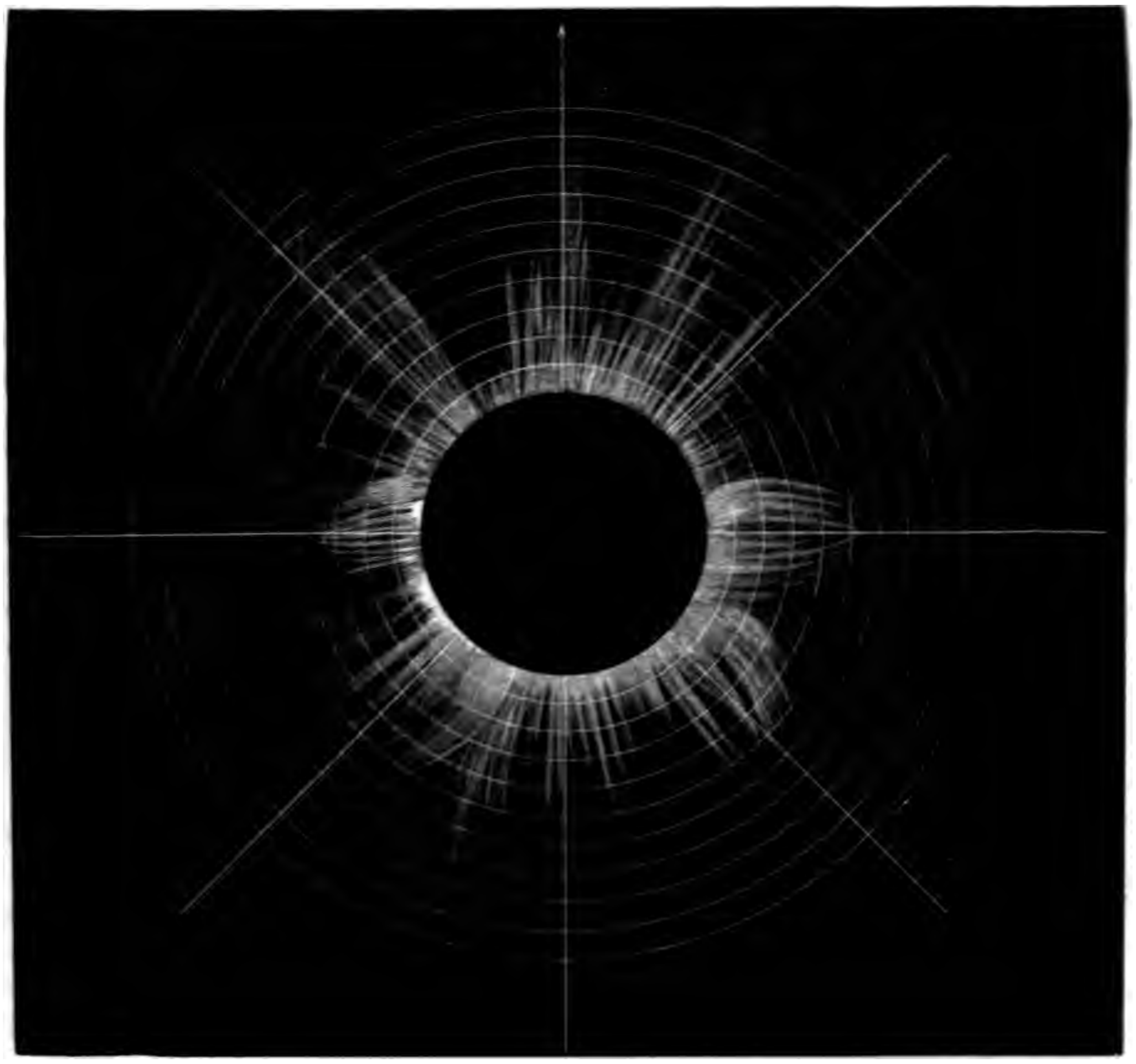
The next longest, and I admit most beautiful, streamer was on the left side, in about the 315°, counting from zenith to the right, and immediately above a small red prominence. This ray differed from its companion on the right, inasmuch that it was of a conical shape, dense along the edges and upper curve, and gradually thinning towards the corona. Its colour appeared to be of a whitish-yellow, and the centre seemed to be hollow, as if one could see through it. This description also applies to the smaller conical shaped rays on the right and the lower part of the corona.

I further observed two small prominences of a red salmon colour on the left of the Moon. The upper one was round, the lower irregular and angular, apparently in contact with the Moon, or immediately contiguous to it.

It will be observed that a small space between the long conical streamer and zenith is left bare. I was unable to complete my drawing, owing to totality being over.

I attach the original drawing in chalk done on the spot. It has not been touched in any way since.

Zenith.



XI. ON THE PHOTOGRAPHS OF THE CORONA OBTAINED AT PRICKLY POINT AND CARRIACOU ISLAND.

BY W. H. WESLEY.

The drawing from which the plate (Plate 10), has been engraved was made from a series of 7 negatives, taken by Mr. MAUNDER, and 5 by Professor SCHUSTER. In the original negatives the diameter of the Moon's disc is ⅝ inch, and the drawing has been made to a scale of 2¼ inches for the Moon's diameter. The following is a brief description of the individual plates.

Mr. MAUNDER'S *Negatives.*

Plate 1. Exposure 0·2 sec. Corona well defined, but not extending further than 7′ from limb. Three prominences on N.E. limb, and lower part of great prominence on N.W.

Plate 2. Exposure 2 secs. Corona well defined, but not extending further than 11′ from limb. Same prominences visible as on Plate 1.

Plate 3. Exposure 10 secs. Corona can be traced on N.W. to nearly a Lunar diameter from limb, but is extremely ill defined, showing scarcely any detail. Negative so dense that prominences can hardly be made out.

Plate 4. Exposure 40 secs. Corona of great extent, but ill defined. There has been much shake, and two separate images, about 13′ apart, are superposed upon the plate.

Plate 5. Exposure 7 secs. Corona can be traced on N.W. to about 27′, but detail very imperfect and indefinite. Not quite so dense a negative as Plate 3.

Plate 6. Exposure 4 secs. Greatest height of corona 22′. Dense negative, but detail very ill defined.

Plate 7. Exposure 0·2 sec. Scarcely a trace of corona. Plate fogged, but prominences on N.W. and W. limb perfectly defined and better seen than on any of the plates.

Professor SCHUSTER'S *Negatives.*

Plate 1. * Exposure 15 secs. (?) Lower portions of corona only are just visible. Two prominences on N.E. limb, and lower part of great prominence on N.W.

Plate 2. * Exposure 15 secs. (?) Corona slightly more shown than in Plate 1. One prominence visible on N.E. limb, and lower part of great prominence on N.W.

Plate 4. Exposure 20 secs. (?) Corona of greater extent than in any of the plates of this series, reaching on the N.W. to a height of nearly 26′. Details of coronal structure very well shown. Negative dense near limb; prominences not very distinct.

Plate 5. Exposure 15 secs. (?) Coronal detail well shown, but extension somewhat abruptly cut off at a height of about $\frac{1}{3}$ of a Lunar diameter. Dense near limb; prominences not well seen.

Plate 6. Exposure 5 secs. (?) Somewhat thin negative; details of corona in N. and S. polar regions very clearly defined, but a superposed image (a few minutes of arc from the principal image), with a trace of the re-appearing crescent, has blotted out the corona on the W. side.

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* The corona was behind a film of clouds while these photographs were taken.

to the west. In the centre of this mass is the tall prominence before referred to, and over this prominence the coronal rays bend towards each other on either side. The base of this group is encroached upon by the broad equatorial mass, which appears to overlap it. The group bounding the northern rift on the east shows but little structure. The synclinal group to the west of the great southern rift is nearly radial, narrow, and conical, and extends to a height of quite 24′ from the limb. The corresponding group to the east extends not much more than half this height, and is broad and depressed towards the equator.

On comparing the corona of 1886 with those of other years, it appears decidedly different from any previously photographed. The great northern and southern rifts extending for some distance along the limb, and the character of the rays filling the rifts, recall to some extent the more extreme form shown in 1878. The depression of the main coronal mass towards the equator, however, even on the eastern side, is not nearly so great as in 1878, and the western side is totally different in character. In fact, the western half of the corona of 1886 shows a striking resemblance to the corresponding side in 1871, but the compressed eastern half in 1886 and the wide polar rifts have nothing in common with the corona of 1871.

In 1875 there was the same decided want of symmetry between the two sides of the corona, but in this case it was the western half which was more depressed towards the equator. In 1875, also, there was a greater tendency towards the extreme polar depression of 1878.

The corona of 1886 has no resemblance to that of 1882, which had no conspicuous polar rifts. Almost equally dissimilar was the corona of 1883, with a single great rift at the north pole only, opened at an angle of about 90°, but hardly extending to the limb, and very unsymmetrically placed with regard to the Sun's axis. Of the Eclipse of 1885, I believe the only successful negatives were those taken by Mr. RADFORD, at Wellington, N.Z., now in the possession of the Royal Society, but which have not yet been published. I am not certain about the orientation of these, but, as far as I can judge, they present no resemblance to the corona of 1886.

In conclusion, it may be said that this corona occupies a middle place between the extreme forms of 1871 and 1878.

X. *Report of the Observations of the Total Solar Eclipse of August* 29, 1886, *made at the Island of Carriacou.*

By the Rev. S. J. PERRY, *S.J., F.R.S.*

Received April 5,—Read May 5, 1887.

[PLATE 11.]

THE astronomers appointed by the Committee of the Royal Society to proceed to the West Indies to observe the total eclipse of the Sun on the morning of August 29, sailed together from Southampton in the R.M.S. "Nile," Captain GILLIES, on July 29, and, after a fair passage, anchored at Barbados at daybreak on August 11. A committee meeting on board had partly fixed our plans with regard to the stations of observation, so that, when we found two of H.M.'s gunboats awaiting our arrival in the roadstead, the instruments of Mr. MAUNDER and of the Rev. S. J. PERRY were, after consultation with the commanders of H.M.'s vessels, at once transferred to the "Bullfrog," whilst the remainder of the instruments found a ready berth on the deck of H.M.S. "Fantôme," which, being the larger of the two gunboats, was reserved for the observers destined for Grenada and its immediate vicinity. Both the war-vessels started the same morning for Grenada, Mr. LOCKYER and Dr. THORPE sailing on board the "Fantôme," in order to secure the earliest possible interview with the Governor of the Windward Islands. The rest of the astronomers left the same evening in the R.M.S. "Eden," Captain MACKENZIE, and, after touching at St. Vincent, arrived at Grenada early on the afternoon of the 12th. The private luggage of Mr. MAUNDER and of the Rev. S. J. PERRY was immediately placed on board H.M.S. "Bullfrog," where they received the heartiest welcome from Captain MASTERMAN, R.N., who devoted the best part of his own cabin to Father PERRY, and found a comfortable private cabin for Mr. MAUNDER. Captain ARCHER, R.N., had also arrived at Grenada in command of H.M.S. "Fantôme"; and the "Sparrowhawk," a surveying vessel, commanded by Captain OLDHAM, R.N., was anchored in the harbour of St. George, her officers having been placed by the Hydrographer of the Admiralty at the disposal of the expedition. Previous to our arrival Governor SENDALL, most ably assisted by Captain MELLING, had personally inspected most of the best sites for the astronomical observations, collected all existing records of the weather, and designed huts for the protection of the instruments. Carriacou and Green Island were told off for the northern station, to be occupied by Father PERRY and Mr. MAUNDER, assisted by the officers and men of H.M.S. "Bullfrog" and by Sub-Lieutenant Helby, of H.M.S. "Sparrowhawk." It was thought, however, more advisable not to separate the members of this party by a distance of some twenty miles, and, therefore, the more northerly island of Carriacou was fixed upon as the site best suited for both observers.

The 13th of August was spent in packing the huts and getting them on board, and on the 14th H.M.S. "Bullfrog" left early for Carriacou, and cast anchor in Hillsborough Bay the same afternoon. Immediately on our arrival we received a visit from the resident magistrate, Mr. Roche, and from the harbour master, Mr. Isaacs, who both offered us every assistance in their power. On landing we paid our respect to the venerable Canon Petretto, whom Governor Sendall had specially named, along with Mr. Roche, as most anxious to render us every possible aid. From information received from these gentlemen and from the resident physician, Dr. Archer, we concluded that the southern shore of the island might provide an excellent site for the observations, as well as good anchorage, and a fair prospect of landing safely our heavy instruments. A note to Mr. Drummond, the owner of an estate in the south of the island, met at once with a cordial response, and we were invited to make "The Hermitage" our home during our stay at Carriacou.

On August 15, it being Sunday, we lay at anchor off Hillsborough, but the next morning, in spite of a heavy sea, which formed part of the cyclone that destroyed five churches and many houses at St. Vincent, only forty miles to the north of Carriacou, we steamed round to Tyrrel Bay, and took up our final position close by the estate of Mr. P. Drummond. With some difficulty we found our way through the coral reefs in the captain's cutter, and were met on the shore by the land agent of Mr. Drummond, who pointed out the best spot for landing and the most accessible road to "The Hermitage." This building stands on the summit of a ridge 200 feet above the level of the sea, the land stretching in a long promontory towards the South-West between Tyrrel and Manchioneal Bays. On examining carefully the bearing of the neighbouring hills we found that none would at all interfere with our view of the Sun on the morning of the eclipse, Chapeau Carré being sufficiently remote from the East point, and the others still more so; and, as everything else was as favourable as we could expect, we fixed upon this ridge as our station of observation. The following bearings of some of the chief objects in view serve to fix our position very accurately. The observations were made by Mr. Maunder, the angles being reckoned from the true North, through East:—West end of Sandy Island 2°·5, Chapeau Carré 43°·2, Eclipse Peak 86°·0, centre of Frigate Island 168°·3, peak at entrance to Tyrrel Bay, N. side, 329°·5.

The site chosen for our huts was at the summit of the ridge, about 300 yards from "The Hermitage," and towards the ENE. The ground in the immediate vicinity was fairly level, and the foundations for the equatorials excellent. A number of labourers and a bullock-cart were hired for the following day, and soon after daybreak the men of H.M.S. "Bullfrog" began to land our heavy packages on the sandy beach.

The native workmen carried some of the lighter pieces to the top of the ridge as soon as they were landed, but most of the instruments had to be placed on the bullock-cart and dragged up the steep, rough road at great cost of time and labour, the men aiding the bullocks when necessary. The ship's carpenter, with a shore party of seamen and marines, was soon at work clearing the ground and erecting the observing huts, and these had to be fixed rather firmly in the ground, in order to resist the storms of wind so violent in these islands. The dwelling of Mr. DRUMMOND, thanks to his self-sacrifice, afforded excellent accommodation for the two astronomers and for Lieutenant HELBY; but the incessant attacks of the gallinippers, which brought on an incipient fever, rendered it necessary for Mr. MAUNDER to sleep on board H.M.S. "Bullfrog" during our stay at the island. A small cottage, consisting of two rooms, and situated close to the main dwelling, was also placed at our disposal. One of the rooms served to keep our packages dry, and the other, at the expense of the sailroom of the gunboat, was excellently fitted up as a developing room for photography. A bed of concrete was laid in one of the observing huts for the photoheliograph of Mr. MAUNDER, and concrete was also used to fix firmly in position the legs of the strongly-built tripods on which stood the equatorials of JONES and ALVAN CLARK. The landing of the instruments commenced on the 17th, thus leaving twelve days before the eclipse for the erection of huts and instruments and for all necessary preparations. The exact bearing of the polar axes of the equatorials was determined by observations of Polaris and of δ Ursæ Minoris, and we were ready on Monday, the 23rd, to commence the testing of our instruments. The Simms transit-theodolite from Stonyhurst Observatory was of great use for observing altitudes of the Sun by which to rate our chronometers, and also for determining the positions of the disks erected to obscure the inner corona for those who had undertaken to make sketches of the outlying streamers during totality. These disks were fixed firmly on the top of each hut, with sight-holes on uprights placed at a convenient distance on the side opposite the rising Sun. Captain MASTERMAN and Paymaster OSBURN kindly volunteered to observe and sketch these faint, delicate objects.

In the course of the morning of the 23rd Father PERRY adjusted the grating of his spectroscope, and obtained a very perfect spectrum. H.M.S. "Fantôme" arrived the same day from Grenada, and we learnt from Captain ARCHER that Mr. LOCKYER had just established himself at Green Island, the station appointed at first for Mr. MAUNDER.

On the 24th the weather in the early morning was all that could be desired, and the Sun could have been observed under the most favourable circumstances had the eclipse occurred on that day, although later in the morning there was a succession of heavy showers.

The detailed plan for the morning of the eclipse was definitively settled on the 25th, the assistants were chosen, and everything made ready for a complete rehearsal on the morrow. The plan finally adopted was the following:—

1. The Jones 4-inch equatorial, provided with a Hilger solar prism and power of 110, was destined for observations of first and last contacts; but, as it was raining heavily both at the beginning and at the end of the eclipse, this notice of the preparation for contact observations will amply suffice.

2. After contact, the solar prism was to be dismounted, and a large direct-vision spectroscope by BROWNING substituted in its place, to be used during totality as an analysing instrument by Lieutenant PASCOE, R.N., assisted by Dr. ARCHER.

3. Dr. WRIGHT, R.N., had charge of the Hilger direct-vision spectroscope, mounted on its own stand, and to be used as an integrating instrument. The observations with this spectroscope, and also those undertaken by Lieutenant PASCOE, were intended to supplement any results obtained with the grating attached to the 5½-inch Alvan Clark equatorial.

4. Mr. MAUNDER's work with the photoheliograph and spectroscopic cameras, which will form the subject of a separate report.

5. Drawings of the streamers of the outer corona, to be made with the aid of disks obscuring the inner corona. The sketches of Captain MASTERMAN, R.N., and of Paymaster OSBURN, R.N., will be appended to this report, with their own remarks explanatory of the nature of the results obtained.

6. Spectroscopic observations with a Rowland grating attached to the 5½-inch Alvan Clark equatorial. The telescope to be pointed by Lieutenant HELBY, and the readings taken by Father PERRY.

This equatorial of ALVAN CLARK was the instrument used by the Rev. T. WEBB in the preparation of his well-known work on "Celestial Objects." The glass was one of those guaranteed by Mr. DAWES, and it would be difficult to surpass it in excellence of defining power. The mounting was not comparable with the quality of the glass, and no driving clock was attached. Mr. WEBB used only a slow motion in R.A.; but, as it was necessary to vary the position of the slit of the spectroscope during totality, in order to place it successively on different parts of the corona, it became imperative to provide a slow motion for N.P.D. before taking the instrument to the West Indies. This addition was made by COOKE, of York, and he also arranged the clamps so that the telescope might be fixed firmly in every direction. The absence of clockwork to drive the instrument made it necessary to have an assistant to point the telescope, and therefore Lieutenant HELBY, of H.M.S. "Sparrowhawk," was chosen for this work. The grating used with this equatorial was kindly lent by the authorities of South Kensington; but a direct-vision spectroscope, constructed by HILGER, was also provided, by which an equal dispersion could be obtained, and which might replace the grating in case of accident. The plate of the grating was polished and figured by J. A. BRASHEAR, and the parallel lines, 14,438 to the inch, were ruled on Professor ROWLAND's engine at Baltimore in 1884. The plate was mounted by HILGER on a student's spectrometer belonging to Stonyhurst. The combination answered very well, and gave little trouble in the adjustments.

The work expected from this instrument by the *Committee of the Royal Society* was an examination of the spectrum of the inner corona immediately before and after totality, and a search for the bands of carbon during totality. The observations before and after totality were to serve as a test of the accuracy of Mr. LOCKYER's theory concerning the concentric layers of the solar atmosphere, in which selective absorption is supposed to take place. It is important to know whether nearly the whole of the Fraunhofer lines are produced in the layer observed by Professor YOUNG close to the photosphere in 1870, or whether they are due to the combined absorptive action of successive layers, each producing its own characteristic lines. If the latter hypothesis be the true one, then the layers nearer the Sun's centre, being hotter than those outside them, should produce brighter lines. These, therefore, would be the first to come into view as the eclipse approaches totality, and they would also be the most enduring after totality. Belonging, as they are supposed to do, to the inner layer, they should be short and bright, and not increase in length, but only in relative intensity, as the darkness became greater. The other lines, belonging to layers farther removed from the Sun's centre, would be invisible at first, owing to their want of intensity, but they would gradually come into view as the darkness increased, and always appear less brilliant and longer than those which preceded them. Immediately after totality the previous order would be, of course, reversed, the longest lines, which are also the faintest, disappearing first, and then the others, according to their length, leaving the shortest and brightest in view, until even these are at last overpowered by the returning light of the photosphere. In 1882 the Egyptian observations had favoured this theory, and Mr. TURNER was asked to repeat in 1886 the observations previously made in the F. region, whilst Father PERRY watched the same phenomena in the region on the less refrangible side of *b*. The plan adopted at Carriacou was that Lieutenant HELBY should keep the slit of the spectroscope radial on the centre of the solar crescent for eight minutes both before and after totality, whilst Father PERRY watched the changes in the bright line spectrum of the inner corona. To enable the assistant to point the telescope with very great accuracy, a cap with a white enamelled surface had been closely fitted to the slit, and on this cap two sets of parallel lines at right angles to each other had been most carefully ruled, the distances between the lines being one-tenth of the projected diameter of the solar disk. The cap could be fixed only in one position, so there was no possible danger of one set of lines not being in exactly the same direction as the slit, and still less of the slit being partially covered. A clear image of any visible corona was thus secured, and the assistant could see perfectly whether the required portion of the image fell upon the slit.

The remaining work expected from this instrument was a search during totality for the two principal bands of the carbon spectrum. In 1883 *Professor* TACCHINI had thought he glimpsed the carbon bands, and some few previous observations rendered their existence in the coronal spectrum not improbable. It was evidently of very great importance to test thoroughly so interesting a fact, and Professor TACCHINI joined the British Expedition at Southampton with the intention of placing, if possible, this question beyond the region of doubt. The instructions to Lieutenant HELBY were to place the slit of the spectroscope exactly on the inner edge of the corona at the commencement of totality, and then to move it successively to distance 0·1, 0·2, 0·3, 0·4, and 0·5 of a diameter from the dark surface of the Moon, repeating afterwards at the Sun's pole what first was done near his equator. To avoid rotating the spectroscope, the slit was placed radial at the solar equator, and tangential at the poles. The same portion of the spectrum remained always in the field of view during the whole of the observations, and embraced rather more than the distance from W.L. 5600 to *b*, comprising, therefore, the positions of the two principal bands of the carbon. To fix accurately the place of any lines visible, photographs of the solar lines in the portion of the spectrum required had been taken at Stonyhurst on plates stained with eocine, and on others kindly sent by Captain ABNEY, R.E. As these did not come out distinct enough to use safely with a feeble illumination, a number of the principal lines in the field of view were measured with a micrometer, and then mapped on a large scale and reduced photographically to the scale required. Transparent scales graduated to tenths of millimetres were also prepared, so as to be ready for any change that might be required.

On August the 26th, between 6 and 7.15 A.M., we had the first all-round practice with every instrument in position, with all hands on shore who were to take part in the observations on the 29th, and each thing done just as if the eclipse had been taking place. The time was called every ten seconds by ROBERT SMITH, A.B., coxswain of the captain's cutter, in a loud and distinct tone, that could be heard easily by all present. JOHN COLLUM, signalman of H.M.S. "Bullfrog," and other reliable seamen, noted down the observations as these were called out, and affixed the corresponding times. All was found to work well, and not a few useful lessons were learnt for the morning of the 29th. Later on in the day, whilst observing with the grating, I found the heat so intense that I was forced to leave the instrument for a time and retire to the house. On my return I perceived at once that some inquisitive looker-on had been gently feeling the grating with his greasy finger, probably to ascertain its degree of smoothness. My dismay at first was great, as I was afraid I might have been obliged to abandon the grating in favour of the Hilger direct-vision prisms, with which I had supplied myself in case of accident. I removed part of the roof of my observing hut, in order to test the grating thoroughly, and I was satisfied at last that the spectrum showed no signs of being in the least affected by the stain left by the finger on the surface of the ruled metallic plate. I resolved, therefore, to retain the grating; but, finding that the second order of spectrum with a power of 4 gave a more brilliant picture than the first order with the power of 6½, I made up my mind to adopt the second order for the day of the eclipse, although this necessitated the sacrifice of the photographs of the solar lines, which had to be replaced by a scale divided in tenths of millimetres.

The early morning of the 27th was cloudy and showery, and all practice impossible at the eclipse hour, but the Sun was observed later on in the day, and the position ot the various points of the limb accurately determined for all circumstances that might arise in the use of direct or inverted images, of solar prisms, or of projections.

On August the 28th the sky was quite clear an hour before sunrise, and, observing the Sun at the time when first contact was to take place on the morrow, I found the definition very good, but the low altitude made the limb somewhat unsteady. I was then using the power of 110 on the Jones 4-inch equatorial. Changing my instrument, I then took a number of readings of the solar lines, using the first and second orders of the spectrum with the powers of 6½ and 4 respectively. As I again found the second order with power 4 to be much more distinct than the other combination, I resolved to adhere to my intention of adopting the second order for any observations on the morrow.

At 2 A.M. on the 29th, the morning of the eclipse, not a cloud was to be seen, and at 4 A.M. the stars were still shining brightly in every direction, although a slight breeze had sprung up from the South. The wind then shifted gradually towards the East, and at 5 A.M. clouds were fast beginning to appear. Soon it became but too evident that rain was falling at no great distance to the North-East of our station, and heavy clouds began to show themselves in the direction of the rising Sun. External contact took place in the midst of rain, and the first glimpse of the Moon was obtained through the Jones equatorial when one-third of the Sun's surface was already obscured. The clouds then cleared off rapidly, and we could safely uncover our larger instruments. The sky, however, remained only fairly good until the near approach of totality. This was particularly unfortunate, as the interval between first contact and totality would have been most valuable for testing the more delicate adjustments of our instruments, and for preliminary observations.

As soon as the Sun's image could be seen upon the cap of the slit, Lieutenant HELBY placed the centre of the radial slit on the middle of the outer arc of the solar crescent, and kept it there as steadily as possible; but no bright lines came into view between W.L. 5600 and *b* until one minute before totality, when the first line seen was 1474 K., which stood out very brightly, and then followed almost immediately a number of bright lines close by *b*, on the less refrangible side. I estimated their number at about fifteen. They seemed to be of different lengths, but I did not see them long enough to judge of their relative intensities. The height to which 1474 K. extended from the photosphere might be about 8′ of arc. The exact position of this line and the general position of the group were fixed by the lines of the solar spectrum, which had been under my eye for some time previous. I never moved the grating, or the viewing telescope, during the observations, so that everything was in excellent adjustment the whole time, and the field of view was well known. The captain's coxswain was counting the time aloud during the whole of totality, the seconds being taken from Mr. Maunder's clock.

When totality commenced, the slit of the spectroscope was radial on the inner edge of the corona, near the centre of the line where the thin crescent had just been visible; no carbon bands could be perceived. The slit was then moved successively to distances 0·1, 0·2, 0·3, 0·4 and 0·5 of a solar diameter from the Moon's dark limb as 190^s, 180^s, 170^s, 160,^{s} and 150^s were called by the coxswain. All this time I kept my eye steadily at the viewing telescope, but could see nothing of the carbon bands. The slit was then moved to the vicinity of one of the Sun's poles, and placed tangentially on the inner edge of the corona at 130^s before the end of totality. Afterwards it was gradually shifted away from the Moon's limb, the distances being 0·1, 0·2, 0·4, and 0·5 of a diameter, at 120^s, 110^s, 100^s, and 90^s respectively, and in none of these positions could I catch the slightest trace of the bands of carbon.

Thinking it hopeless to continue any longer the search for carbon, and wishing to be prepared in good time for the observation of the bright lines at the end of totality, I asked Lieutenant Helby to place the slit at once radial at the point of re-appearance of the photosphere. Whilst this was being done, I took up a powerful binocular, which I had placed for this purpose close at hand, and viewed for a moment the eclipsed Sun. The upper rays on the western limb, to the right and left of the vertical line, were by far the longest streamers, and were situated almost at right angles to each other, a third, but shorter, ray appearing between them to the left of the vertical diameter. These rays were all well defined, and the one most to the North of West was curved on both sides like a leaf. On the Eastern, or lower, limb the rays were irregular and less extended than in the West. I did not notice any rays near the poles, but my view was scarcely more than an instantaneous glance.

At this moment Lieutenant Helby lost the solar image from an irregular movement of the telescope, but I was able to recover it almost immediately. Whilst thus replacing the Sun upon the slit, I obtained a hasty view of the corona upon the white enamelled cap, and this picture far surpassed in beauty anything I had seen before, although my binocular is an excellent instrument. The details of the streamers, and the short red prominences, were exceedingly well defined, showing the splendid quality of the Alvan Clark objective and the purity of the sky at the moment.

When the coxswain had called out 20^s, the slit being radial near the point of re-appearance, I saw a large number of lines flash out in my limited field of view: there might have been fifty altogether between W.L. 5600 and *b*. This lasted only a very short time, and after totality no lines were seen, as the rising wind interfered considerably with the steadiness of the telescope, and in a few minutes we were again deluged with rain. The darkness was never much less than that of a fair moonlight night, but during totality the light was not equal to that of a full Moon in a clear sky. Heavy rain prevented the observation of last contact.

Since the observations were taken I have frequently wished that my equatorial had been supplied with clockwork, which would have enabled me to dispense with the aid of an assistant for fixing, in each case, the position of the slit. I have not any reason to doubt but that the gentleman who so kindly aided me on this occasion did his duty as perfectly as could be expected; but it is not such an easy matter for one who has had only a few days' acquaintance with the slow-motion rods of an equatorial to keep the slit of a spectroscope on any precise point of a celestial object. The strength of the wind and the imperfections of the slow-motion rods added in the present instance to the difficulty of following the Sun exactly with the R.A. rod, and of changing the N.P.D. as required. I am forced to the conclusion that the time lost by the unaided observer in placing the slit of his spectroscope would be more than compensated by the security he would feel, that he was viewing exactly the desired point of the object. Without clockwork this is not practicable; but I should never think of again attempting eye observations with a spectroscope during a total eclipse without a clock to drive my equatorial, and then, if a grating was used, I should certainly dispense with any assistant at the telescope.

The two main questions to be answered by the spectroscopic observations at Carriacou were: (1.) Does the absorption, which produces the Fraunhofer lines, take place mainly in a single layer of the solar atmosphere, or in concentric layers? (2.) Does carbon exist in the corona? As far as the above results may afford any satisfactory evidence on these two points, I should be inclined to say that the difference in the length of the lines observed before totality on the less refrangible side of *b* seems somewhat to strengthen the view that the absorption takes place in concentric layers. And the search for carbon tells us that, if present, its spectrum was not strong enough in 1886 to make any appreciable effect upon the retina, when the eclipsed Sun was viewed through so powerful a diffraction spectroscope as that used at the island of Carriacou. It may, perhaps, be established, by later observations, that the intensity of the carbon spectrum varies in each eclipse, and may have some direct connection with the amount of solar activity.

I should mention in conclusion that the diameter of the solar image on the slit plate of my spectroscope was 20 millimetres, the width of slit used $\frac{9}{100}$, and its effective length $6\frac{82}{100}$. The dispersion was sufficient to enable me to see b_3 and b_4 very distinctly separated.

Subjoined are the Sketches of the Coronal Streamers with explanatory notes by Lieutenant Commander J. MASTERMAN, R.N. (Plate 11), and Mr. F. W. OSBURN, R.N. (p. 362).

Notes on the Solar Eclipse of 29th August, 1886, observed at Carriacou.

H.M.S. "Bullfrog," under my command, was ordered to convey the northern division of the expedition for observing the eclipse to Carriacou, an island about twenty miles N.W. of Grenada, and render what assistance she could; I and all the other officers of the ship offered our services for the observations, which were accepted.

On arriving at Carriacou, on the 14th August, we were cordially welcomed by Mr. ROCHE (the magistrate), and hospitably entertained by Mr. DRUMMOND, the owner of the property on which we selected a spot to erect the huts. This was a small plateau, on the summit of a steep ridge 175 feet above the level of the sea. The sea was on each side of the ridge, and Mr. DRUMMOND's house was 200 or 300 yards off. The beach in the bay where the ship anchored was very suitable for landing the instruments.

I undertook the observation of one of the disks; it was mounted on the hut used by Mr. MAUNDER. The height of disk above eyepiece was 11 feet 10½ inches, the angle subtended by the disk being 72′; the diameter of the disk was 9 inches; the cross-bar was 1¼ inch thick; the uprights were 1 inch thick; the length of uprights was 2 feet 10½ inches; the horizontal distance of the eyepiece from the disk was 33 feet 7 inches; the apparent diameter of the disk from the point of observation was equal to 2¼ solar diameters.

On the 26th and on the 28th I practised the observations under as nearly as possible the same conditions that we should be under on the day of the eclipse. I was prevented from doing so on the 27th by the weather.

The hole in the eyepiece I had increased to ¼ inch diameter.

At 10 minutes before totality my eyes were bandaged with a thick black handkerchief, without any pressure on the eyeballs, but totally excluding the light. The bandage being taken off at the commencement of totality, I looked through the eyepiece, but found the adjustment not correct, and lost some time in correcting it before I could commence my observations.

The sketch I took of the phenomenon, as I saw it, together with a copy (which differs only in being a little more finished and shaded a little darker), accompanies this.

The first things that caught my attention were the two rays of light marked A and B; they seemed to be of exactly the same length, and each to make an angle of about 45° with the vertical, in length 1½ diameter of the disk from the disk, or 4 solar diameters from the Sun's periphery. The next thing that I observed was the ray that I have marked C, very bright, but small and partially hidden by the cross piece; the observed extension was under 2 solar diameters from the Sun's limb. The two remaining rays, D and E, on the Sun's eastern limb, were very faint, and they seemed to fade into the general surrounding light instead of tapering away to a point, as the others did.

At the end of the observation I am certain that B was considerably longer than A, though, as I said before, they appeared to be exactly similar at first. I should be inclined to attribute this to the fact that the Sun was altering a little in azimuth, and towards the end of the observation I had given all the correction that the apparatus admitted of, and therefore the Sun got nearer to the side of the disk.

I found that the Sun altered in altitude so rapidly that it was difficult to keep pace with it with the rough arrangement for altering the elevation of the eyepiece; a small rack and pinion would have been a great boon.

A better arrangement for making correction in azimuth and a longer range would be an advantage.

The cross-piece, 1¼ inch wide, interfered with the observation; a thin metal rod or stout wire would have been better.

I observed with the disk until the last moments of totality, and only just looked off in time to see the burst of bright light at the second internal contact. I at once made the (original) sketch which I send, and saw nothing of any other phenomena, as, two minutes after, everything was obscured in clouds and we were in a drenching rain.

I had the tides both before and after the eclipse measured, and found the rise and fall was normal, 18 inches.

J. Masterman,
Lieutenant Commander, H.M.S. "Bullfrog."

Grenada, 1*st September*, 1886.

NOTES DESCRIPTIVE OF DRAWING (PLATE 11) BY LIEUTENANT J. MASTERMAN.

A and B.—Very distinct, especially B. They each formed an angle of 45° with the vertical. B appeared to increase in length, not in brilliance; A did not.

C.—Distinct, but small. I could not detect any sign of C above the batten, though, if the upper edge had been inclined to the horizontal at the same angle as the lower edge was, I think I must have done so.

D and E.—Very faint indeed, merging into the luminosity which was faintly apparent between all the shoots of light; but, faint as this luminosity was, it had a distinct definition, which had the appearance of going about the points of the shoots.

Copy of sketch of the phenomenon as observed by me with one of the disks mounted on the roof of a hut.

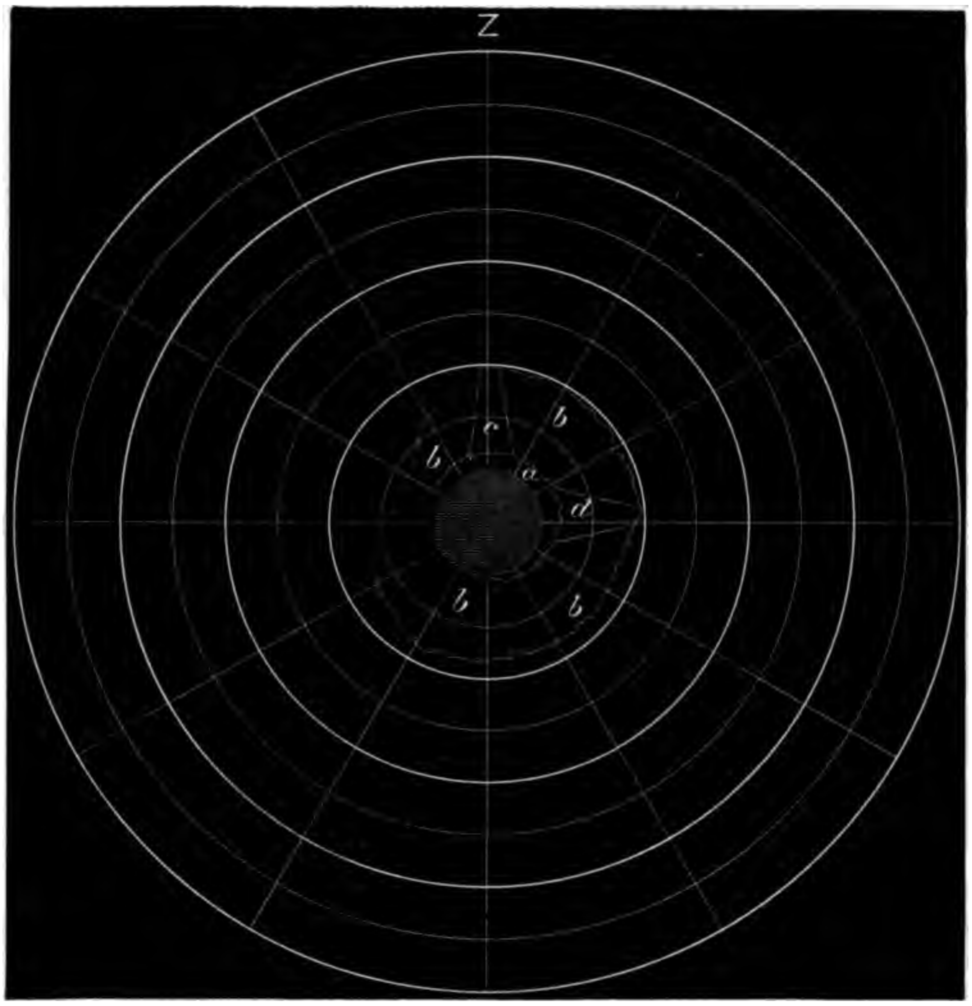
Dimensions:—Disk, 9 inches diameter; cross batten, 1¼ inch; upright battens 1 inch, length 2 feet 10½ inches.

J. Masterman.

Drawing by Mr. F. W. Osburn, R.N., of H.M.S. "Bullfrog."

Horizontal distance from eyepiece to disk, 26 feet 10 inches; height above level of eyehole, 9 feet 4 inches; diameter of disk, 9 inches; angular diameter, 1° 31′; breadth of upright batten, 1 inch; breadth of cross batten, 1¼ inch; height of disk above ridgeway of roof, 2 feet 10½ inches.

Note.—At the commencement of totality, when my eyes were first unbandaged, I found, on looking through the eyepiece, that the eclipsed Sun was about 4 diameters above the disk, and some time was wasted in shifting the eyepiece down as far as it would go, when the Sun and corona were just covered by the disk; but before totality was over, the corona was again visible over the top of the disk.



Marked *a*.—Around three parts of the disk there appeared a band of bright light, sharply defined and irregular in shape, its broadest part on the right, and gradually diminishing until, at the left lower corner, it ceased altogether.

Marked *b*.—Entirely surrounding the disk I observed an irregularly shaped field of very faint light, standing out at its widest part nearly 1 diameter beyond the disk.

Into the field above mentioned I observed two very faint rays extending, one towards the zenith (but so faint as to be hardly discernible), and the other to the right of the disk (which I saw more clearly), on each side of the horizontal support; these rays are marked respectively *c* and *d*. They appeared about equal in length, *i.e.*, 1 diameter of the disk.

Francis W. Osburn.

XI. *On the Determination of the Photometric Intensity of the Coronal Light during the Solar Eclipse of August 28–29, 1886.*

By Captain W. DE W. ABNEY, *C.B., R.E., F.R.S., and* T. E. THORPE, *Ph.D., F.R.S.*

Received February 7,—Read February 14, 1889.

Introduction.

ALTHOUGH it has long been suspected that the amount of light emitted by the corona, as seen at various Solar Eclipses, may vary with its configuration, [illegible] no attempts to measure its intensity appear to have been made prior to the Eclipse of December 22, 1870. On that occasion Professor PICKERING employed an arrangement constructed on the principle of BUNSEN's photometer. It consisted of a box 9 inches wide, 12 inches high, and 6 feet long, within which a standard candle could be moved backwards and forwards by means of a rod. One end of the box was covered with a piece of thin white paper, on which was a greased spot about half an inch in diameter. The box was adjusted so that the rays from the corona were normal to the plane of the paper and the lighted candle was moved backwards and forwards within the box until the greased spot was no longer visible. From a [illegible] of observations made during the period of totality of this eclipse Mr. W. [illegible] acting under Mr. PICKERING's direction found that the standard candle [illegible] placed at distances varying from [illegible] inches from the paper before the visibility of the greased spot was reduced to a minimum [illegible] Coast Survey Report [illegible] p. 172.) The observations were much interrupted by clouds and are also probably affected by irregularities in the rate of the burning of the candle. The mean of all the readings was [illegible] inches, hence the light of the corona [illegible] was approximately equal to [illegible] of a standard candle at a distance of [illegible]

A precisely similar arrangement was used by [illegible] during the Total Eclipse of July 29, 1878. [illegible] observing at Virginia City, [illegible] found from eight observations that the candle had to be placed at a distance of [illegible] inches from the screen before the minimum of visibility of the greased spot was obtained.

During the same eclipse [illegible] intensity of the corona light [illegible] means of a photometric arrangement [illegible] intended to measure the [illegible] first to draw an outline of the corona [illegible] several points along a solar radius extended to the outer limits of visibility; and third, to draw one or more iso-photal lines which should give the contour of the corona for varying degrees of illuminating power. The method was to project the image of the corona upon the screen of a BUNSEN'S photometer in which glass ground to slight opacity replaced the greased paper; and instead of one translucent spot there was a large number, in order that several hundred samples of coronal light taken from different portions of its surface might be observed simultaneously, and compared with the standard light by drawing iso-photal contour lines.

In the apparatus as finally arranged the screen consisted of a piece of perforated cardboard covered by a sheet of oiled paper. A number of translucent spots separated by opaque interspaces was thus obtained, the spots being sufficiently close together to allow the projected image to be seen with but little loss of distinctness. The light from the corona was reflected from a heliostat and transmitted through a photographic lens of long focus, the aperture of which could be diminished, if required, by means of a cat's-eye diaphragm, and passed down a dark chamber about 41 feet in length, at the end of which was a box carrying the screen and standard candles. Within the box was a railway, on which ran a wagon mounted on brass wheels and bearing the lighted candle, the distance of which from the screen was recorded on a wooden rod placed immediately in front of the observer. The whole apparatus stood on four piers, and it was so arranged that any vibration caused by wind should not be communicated to the lens producing the coronal image. The focal length of the lens was 11·27 metres; and the diameter of the solar focal image was ·104 metre.

The corona as actually observed was excessively faint, and could only be seen to an extent of less than 1′ from the Moon's edge. The coronal light was so feeble that it was found impossible to measure its intensity at several points along an extended radius.

From his observations Professor LANGLEY concludes that the light from the corona at 1′ from the limb of the Moon was equal to that of the standard candle at a distance of 1 metre. From the photometric value of the candle as compared with diffused sunlight, LANGLEY further found that the intensity of the coronal light about 1′ from the limb of the Moon was ·0000132 of that of mean sunlight; at 3′ from the limb it was ·0000000244. Assuming the intensity of mean sunlight to be 500,000 times greater than that of moonlight, the corona at 1′ from the Moon's limb was six times the intrinsic brightness of the Moon; at 3′ it was but one-tenth the intrinsic brightness of the Moon. (Professor S. P. LANGLEY'S Report, p. 211, 'Washington Observations' for 1876, Appendix III.)

The photometric observations made during the 1878 Eclipse have also been discussed by Professor W. HARKNESS, of the United States Naval Observatory (*loc. cit.*, p. 386). Combining the observations, he concludes that the total light of the corona was ·072 of that of a standard candle at 1 foot distance, or 3·8 times that of the full Moon, or ·0000069 of that of the Sun. It further appears from the photographs that the coronal light varied inversely as the square of the distance from the Sun's limb.

Probably the brightest part of the corona was about 15 times brighter than the surface of the full Moon, or 37,000 times fainter than the surface of the Sun.

It would further seem that the corona of December 22, 1870, was 7¼ times brighter than that of July 29, 1878.

Description of Methods Adopted during the Eclipse of August 28–29, 1886.

The instruments used by us for the measurement of the coronal light on this occasion were three in number. The first was constructed to measure the comparative brightness of the corona at different distances from the Moon's limb. The second was designed to measure the total brightness of the corona, excluding as far as possible the sky effect. The third was intended to measure the brightness of the corona, together with the brightness of the sky in the direction of the eclipsed Sun.

In a paper by one of us, in conjunction with General FESTING,[*] it was shown that light of any colour can be measured for luminosity in terms of light of any other colour, provided always that the last-named light can be rapidly altered in intensity, so that at one time it is evidently below the intensity of the light to be compared, and immediately afterwards that it is evidently above it. The oscillations of the intensity, if then gradually diminished, finally give the value of the coloured light in terms of the luminosity of the light of which the intensity is rapidly changed.

In a more recent paper[†] it has been shown that the light of a glow lamp may be used for measuring the intensity of any other light by making a rapid change in the resistance of the circuit. In the photometric measures which are now to be described this plan was adopted for ascertaining the value of the coronal light in terms of a Siemens unit. Before continuing the description it may be well to note that the Siemens unit is very nearly 0·8 of a standard candle. This unit has the advantages that the area of the burner is fixed; that the flame used in the photometric measures can always be made of exactly the same height; that the thickness and shape of the flame are practically invariable; that the material producing the flame can be obtained in commerce; and that any slight impurity in it has no practical effect on the value of the light emitted. Neither the effect of the temperature at the time of trial nor the variation due to difference in barometric pressure has been thoroughly tested, but there are presumably but slight differences due to these causes. At all events, there is nothing to prevent its employment for the object we had in view. One experiment may, however, be quoted as regards the luminosity of the flame when the temperature was varied some 20°. The lamp was carefully adjusted so that the tip of the flame just touched the gauge supplied with the instrument, and its value taken against the glow lamp, which was kept at a bright yellow heat by a current passing through it. A large photograph of the flame was also taken. The lamp was then warmed from 55° to 75°; the flame became longer, but when turned down to the height of the gauge the same value was obtained against the glow lamp as before within 2 per cent. Another photograph was taken of the flame from the same position, and the two compared. The flames in both cases were equal in dimensions.

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[*] 'Phil. Trans.,' 1886, "Colour Photometry," ABNEY and FESTING.

[†] 'Roy. Soc. Proc.,' vol. 43, 1887. ABNEY and FESTING.

In the paper last referred to it was also shown that either the BUNSEN or the RUMFORD method of photometry could be adopted. The method of RUMFORD is undoubtedly better than that of BUNSEN when the lights are very different in colour, as in the latter method there is a certain thickness of translucent material through which both lights have to pass, and only after such passage can equality of illumination be estimated; and if the paper employed for the screen is coloured in any degree, this must of necessity affect the results. The light of the corona and that of the glow lamp are very different in colour, the former being stronger in the blue end of the spectrum than the latter. It must be recollected that the greatest luminosity is in the yellow of the spectrum in both cases, and, though the blue end of the spectrum alters the hue, it has very small effect on the luminosity. This being the case, it was thought that no error of any magnitude would be introduced by adopting the BUNSEN plan, since the brightest part of the two spectra would be compared with one another.

It was evidently impracticable to adopt the RUMFORD method in the apparatus in which the intensity of different points in the corona had to be measured. For this purpose a telescope by SIMMS, lent by the Astronomer Royal, was employed. The object glass had a focal length of 78 inches and an aperture of 6 inches, thus forming an image of the Moon ·76 inch in diameter. The image was received on a circular white screen contained in a photometric box and placed exactly in the focus of the object glass. In the centre of the screen was traced a circle of the diameter of the image of the Moon, and during the observation the Moon's disc was made to fall exactly within the circle. As the telescope was equatorially mounted with clockwork, the image was kept stationary within the circle. The screen was of RIVES' paper of medium thickness, and round the pencil-circle a series of small grease spots about $\frac{1}{8}$ of an inch in diameter had been made. There was some difficulty in preparing these small grease spots, but a method was eventually devised which answered admirably. Faint pencil lines were drawn radially from the centre of the circle, and the places where each spot was to be produced were marked with a dot. White blotting paper was soaked in spermaceti, any excess being avoided. Small discs, $\frac{1}{8}$ inch, were punched out, and these discs were put centrally on the dots. Blotting paper was next placed over them when in position, and a hot flat-iron was passed over them. The blotting paper and the small discs were then removed, and clean blotting paper and the flat-iron again applied to remove any slight excess of spermaceti. The screen now presented the appearance shown in fig. 1.

Several screens were made and tested. The test consisted in causing a glow lamp on one side of the screen to balance a glow lamp placed at the same distance on the other side by means of a variable resistance in the circuit. The spots, if correctly made, become invisible at the same time. The majority of screens fulfilled this condition, and the best as regards uniformity of size of spot and freedom from grain in the paper were selected for use. The screen, as will be seen from the figure, was mounted in a circular frame, which could be rotated so as to bring the spots into any desired angular position. It could be removed at pleasure by releasing it from the buttons which held it in position.

Fig. 1.

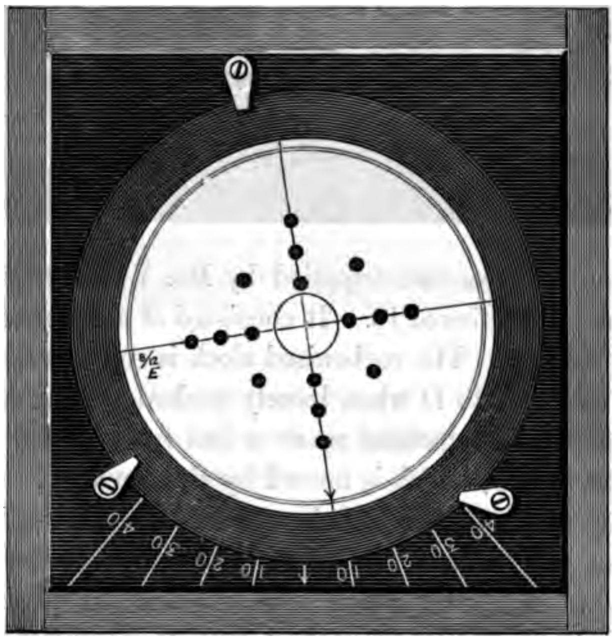
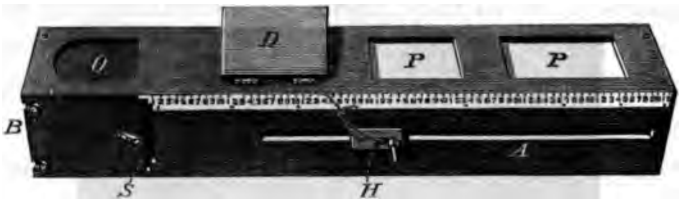


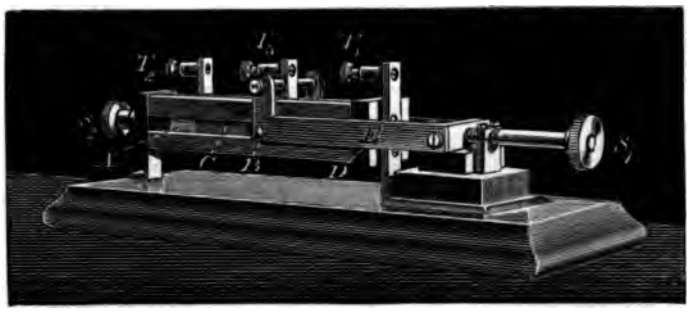
Fig. 2.



To hold the screen a box was constructed, as shown in fig. 2. It was made as light as possible, panels of card (as at P, P) being used instead of wood when practicable. The glow lamp to be employed was fixed in a holder inside the box; this could move along the slot A, and be fixed by a thumbscrew, H, in any desired position. (It may here be remarked that the plane of the filament was at right angles to the axis of the tube.)

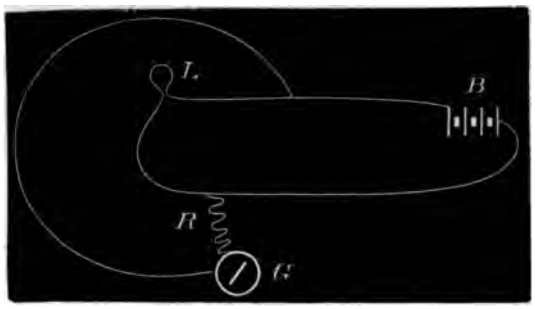
At the end B was an aperture into which the sliding tube of the telescope fitted; at D was a door, which could be opened to adjust the lamp. The screen shown in fig. 1 was inserted at S, and held in position by means of buttons. At O was an opening, which was covered by a black velvet bag into which the head of the observer was inserted during the time of observation. As before said, the image of the Moon was accurately focussed on the screen inserted at S, and was viewed through the opening at O. The wires to the lamp passed through the slot; the carbon-resistance (fig. 3) and also the galvanometer should have been introduced into the circuit.

Fig. 3.



The carbon-resistance used was one supplied by Mr. **Varley**, and the description is taken from a paper already referred to. It consisted of a series of pieces of carbonised cloth, more or less in contact. The carbonised cloth is represented by C (fig. 3), which fills the whole length from A to D when loosely packed. At B is a plate to which T_3 is attached, and which can be separated more or less from a fixed metal plate to which T_1 is connected by the arm E, which is moved by the screw S_1. At A is an insulated block, carrying another plate to which T_2 is attached, and A can be carried backwards or forwards by means of the screw S_2. For some purposes the main current can be brought in at T_3, and leads be taken from T_2 and T_1, thus forming part of a Wheatstone bridge. During the eclipse the terminals T_1 and T_3 were used.

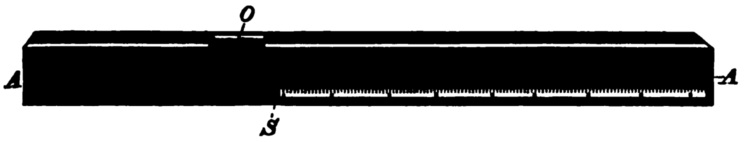
Fig. 4.



The connections were made as shown in fig. 4. The current from 10 cells of a carefully made up Grove battery, B, was passed through the lamp L. A shunt, including the galvanometer, G, and the resistance, R, was made. The brightness of the lamp was thus increased by adding more resistance to the shunt. Consequently, in the measures made, the highest readings of the galvanometer showed the lowest intensity of light. It would have been better had the ordinary plan of putting the galvanometer and resistance in the main circuit been adopted, but when once the value of the lamp illumination by the plan actually employed had been ascertained for varying resistances it became a matter of no moment. The galvanometer used in this case was one of THOMSON'S ammeters, made more sensitive by fixing a permanent magnet alongside the usual magnet, so as partially to neutralise its magnetism. By this plan a very small change in current gave a large deflection, or at all events a deflection which was readable. By reproducing these deflections under exactly similar conditions the illuminating value of the lamp could be measured in the ordinary way.

The second instrument, which we shall call the integrating box, for measuring the total coronal light with as little light from the sky as possible, was constructed on the same principles. It consisted merely of a long deal box coated internally with lampblack, in which a screen with a large grease spot was inserted at S. There was a similar slot for the lamp as in the other instrument. The end A was, however, open, and during the eclipse it was placed at such an angle that the axis of the tube pointed to the centre of the Moon. The aperture, O, for making the observations was in this case also covered with a black velvet cloth, under which the head of the observer was placed.

Fig. 5.



The third piece of apparatus consisted of an ordinary Bunsen bar photometer, 60 inches in length, with movable disc, made by Messrs. ALEX. WRIGHT and Co., of Westminster. As originally arranged, it was fitted for two standard candles; for the purpose of the eclipse observations, these were replaced by a small glow lamp.

As the plan of the photometric work contemplated by us depended for its execution upon such assistance as we were able to get out at Grenada, it was arranged that we should take advantage of the kind offer of service made by Captain ARCHER and the officers of H.M.S. "Fantôme," which had been told off to assist the expedition, and make the observations at some spot in convenient proximity to the anchorage of that vessel. As the latter end of August falls during the hurricane season in the West Indies, it was desirable to moor the "Fantôme" in the most secure anchorage in the island, viz., in Clerk's Court Bay, which is at the south end of Grenada. It appeared from the charts that a suitable station might be found on the southern end of Caliveny Island, distant about 1½ mile from the spot which would be made use of as the anchorage. Caliveny Island was accordingly included in the list of stations provisionally selected by the Eclipse Committee of the Royal Society, and submitted to the Governor of Grenada. Mr. SENDALL and Captain HUGHES were kind enough to visit the spot, and they reported that a fairly good station might be obtained on the extreme end of the island, but that difficulties might be experienced in landing the apparatus. Captain ARCHER deemed it prudent, therefore, to make a preliminary survey of the place before the "Fantôme" left St. George. No landing was practicable on the leeward side of the island, and, although two or three places were met with on the other side, they could only be counted upon during fine weather. Moreover, as the greater part of the island is covered with dense "bush," the transport of the instruments to and from the station would be very laborious and tedious. There was the further difficulty that it was well nigh impossible to make one's way through the tangle of bush at night, when much of the work of adjustment of the equatorial would have to be done. And, lastly, there was the possibility, even if the instruments were successfully set up, that the noise of the surf and the driving spray in bad weather might seriously interfere with the work of observation.

For these reasons we decided to abandon the Caliveny site, and, after a careful examination of the neighbourhood, we selected a station near a little creek on Hog Island, to the westward of the bay. The position was fairly good; during dry weather it was indeed all that could be desired. The ground was about 10 to 15 feet above the sea level, and was close to a shelving, sandy beach, readily accessible and generally free from swell. In bad weather, another landing could be obtained round a point to the north, with only a few hundred yards of bush to be got through. The position had a good eastern horizon, the sun rising behind the lowland running out to Point Egmont, which here subtended an angle of less than 1°. Its position, as taken off the Admiralty Chart, was lat. 12° 0′ 4″ N., long. 61° 43′ 45″ W.

The "Fantôme" left St. George on the 17th August, and came into Clerk's Court Bay in the afternoon of the same day. All the apparatus was safely got to shore before nightfall, and the positions for the base of the equatorial and for the tents of the party undertaking the integrating work were decided upon. Early next morning the erection of the wooden hut to shelter the equatorial was begun, and a concrete base for the stand made upon the rock, which was found at a depth of a few inches below the surface of the soil. Before the surface of the cement was finally set it was carefully levelled, the base put into position, and the mounting of the telescope proceeded with. The integrating apparatus was placed in a small marquee tent, a few yards to the north of the hut. Small slabs of concrete were also made in convenient situations, to carry the galvanometers, &c. As the photometer box attached to the equatorial added considerably to the length of the apparatus, it was necessary that the hut should be of somewhat larger dimensions than that generally adopted by the rest of the expedition; otherwise it was very much of the pattern of that which the Governor had caused to be constructed prior to our arrival at St. George, and which answered admirably in all respects. Round the hut and tents a deep trench was cut, with an outfall leading down the slope towards the sea to carry off the rain-water collected by the roofs. Our chief difficulties, indeed, were due to the frequent rains and constant humidity. At times the ground became worked into a sort of quagmire. By the 19th everything was in fair adjustment, and during the subsequent ten days the various members of the party were assiduously practised on all available occasions in their duties. The driving clock of the equatorial gave trouble in the outset, but it was eventually got into order, and on the day of the eclipse, and for some days previously, ran sufficiently well. The constant dampness of the ground, and consequent absence of dust, probably contributed to its good behaviour.

The duration of the total phase of the eclipse at Hog Island was about 230 seconds. We found that a simple and sufficiently accurate method of informing the party as to how this time was speeding, and of the amount of time still left at intervals of 15 seconds before the end of totality, could be obtained by observations made with a 14-second sand-glass, such as is employed on shipboard in heaving the log.

The arrangement of the party was as follows :—

Integrating box	Lieutenant ANGUS DOUGLAS.
Galvanometer	Mr. WEBB.
Bar photometer	Lieutenant BAIRNSFATHER.
Recorder.	Mr. ROBERT JACKSON.

These instruments were placed close together in a small marquee. Each observer had in addition a man to charge and connect up a battery of Grove cells for the glow lamps.

Equatorial photometer . .	Professor T. E. THORPE.
Galvanometer	Mr. H. A. LAWRANCE.
Disc observations	Captain ARCHER.

On the day before the eclipse the following instructions were issued to the observers in charge of the integrating apparatus :—

Instructions to Lieutenant DOUGLAS. (*Integrating Box.*)

1. The eclipse begins at 6hr. 12m. L.M.T. Totality commences at 7hr. 10m., and lasts about 230 seconds.

2. It is desirable that on the morning of the eclipse you should be ashore not later than 6 A.M. (It is assumed that your integrating box, leads to galvanometer, glow lamp, and resistance apparatus are left in position over night.) Ascertain that the box is in proper azimuth, and test by the Abney level that the inclination is about 19°.

3. See that the connections to the glow lamp are properly made, and that the galvanometer is levelled on the cement foundation and is in adjustment.

4. Give instructions to have a battery of 7 cells in readiness for you, not later than 6.30: 6 cells to be connected up, the seventh to be reserved in case of accidents. See that you have actually 6 cells connected up before you begin.

5. At 6.30 connect up all leads, see that your lamp works properly, and that the intensity of the light responds to the screw of the resistance apparatus.

6. The back edge of the wooden piece carrying the lamp may be conveniently placed at Division 50 of the graduated scale. This, with the resistance apparatus open, will probably give you more light than you require. If at the moment of totality, and with the lamp full on, the coronal light is greater than that of the glow lamp, push up the lamp to Division 30. If you have occasion to move the lamp from 50 (which is very improbable), be very careful to note the particular distance from the screen at which you place it.

7. At 7·0 request Mr. WEBB and the man who records his readings to take up their stations at the cement slab. Take up your own position at the integrating box. Mr. WEBB's position to be such that he readily hears your command to read.

8. As the light decreases just before totality, that is between 7.0 and 7.10, turn the light up or down with the screw, so as to follow the decrease, so that after the moment that totality begins you may be able to begin your comparisons with the least possible delay.

9. Intimation that totality has begun will be given to the party by Quartermaster FOLLETT, who will call out 230. At intervals of 15 seconds he will call out the number of seconds still to elapse before totality ends.

10. When you have made your adjustment by the screw as carefully as you can, call out "read" to Mr. WEBB; do not turn the screw again until you hear him give his reading to the man who records. Again work the screw, make a second adjustment, and again call out "read," and again wait until Mr. WEBB has given his reading before you begin again. If this point is not attended to, the needle will be in such rapid oscillation that it will be impossible to get an accurate galvanometer reading.

11. Experience shows that 12 readings may be taken in 100 seconds, but it is not advisable that you should attempt to make more than 12 comparisons during "totality." Recollect that a few readings carefully and deliberately done are worth far more than a large number made hurriedly. Do not touch the position of your lamp at the end of your observations.*

12. Be careful not to fatigue your eye by looking too much at the Sun during the first stage of the eclipse. Insist that all talking ceases after 7.

Instructions to Lieutenant BAIRNSFATHER. (*Bar Photometer.*)

1. The eclipse begins at 6hr. 12m.; totality commences at about 7hr. 10m., and lasts about 230 seconds.

2. It is desirable that you should be ashore not later than 6 A.M. (It is assumed that your photometer bar, leads to galvanometer and to glow lamp, stand for galvano-

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* This instruction was given in order that the position of the lamp might be verified after the observations were concluded.

meter, &c., are left in position on Saturday evening.) Ascertain that your photometer is in the proper azimuth ; test by the Abney level that its inclination is about 19°. Fasten the front rod securely down, so that on running the lantern backwards and forwards it works smoothly and without shaking the bar unduly.

3. See that the connections to the glow lamp are properly made ; next see that the galvanometer is levelled and in adjustment.

4. Give instructions that a battery of 7 cells should be in readiness for you not later than 6.30. Only 6 cells are to be connected up, the seventh to be in reserve in case of accident.

5. At 6.30 connect up all leads, see that your lamp works properly, and take a reading of your galvanometer. Go round to the back of your tent and see that you have actually 6 cells connected up.

6. At 7.0 request Mr. JACKSON to take up his station at the end of your table, and get ready to record. Go round to the galvanometer again and carefully note the reading of the needle, which Mr. JACKSON is to record. At 7.5 take up your position at the photometer bar.

7. As the light wanes follow up with the lantern, so as to get it into position with the least possible delay after the moment totality begins.

8. The preliminary drill has shown that 12 sets of double readings may be taken during the duration of totality. Do not, however, aim at doing more than 7 or 8 sets (in all 14 or 16 readings). A few readings carefully and deliberately done are worth far more than a large number taken very hurriedly.

9. When you have taken your last reading let the lantern remain as you placed it for the reading, in order that after the eclipse its position may be verified. Be careful to note on which side the middle point of the photometer bar the readings are taken.

10. There will probably be sufficient light during totality to see readily the numbers on the photometer bar; but, in case you have difficulty, it will be advisable to have a lighted lantern in readiness under your table.

11. At the end of totality, and therefore at the conclusion of your readings, again note the position of the galvanometer needle and record the deflection.

12. Be careful not to fatigue your eye by looking too much at the Sun during the first stage of the eclipse. All talking to cease at 7.0.

Observations on the Day of the Eclipse.

The general character of the weather, as noted during the ten days prior to the date of the eclipse, rendered it very doubtful whether any photometric observations would be at all possible at the time of totality.

The following Table will serve to indicate the condition of the sky at about the hour of the eclipse on successive days from August 17th to the 28th.

The amount of cloud is given on the scale from 0 to 10.

	Cloud.	
Aug. 17	5	Sun seen through haze. Clouded at times.
„ 18	3	Sun unclouded at time of totality.
„ 19	3	„ „ „
„ 20	2	„ „ „
„ 21	4	Sun seen through thin cloud.
„ 22	4–5	Sun seen through haze. Much rain in night.
„ 23	5	„ „ „ „
„ 24	5–6	Sun frequently clouded.
„ 25	8–9	Sun clouded over; much rain at times.
„ 26	6–7	Sun seen through faint clouds.
„ 27	8–9	Dense clouds. No Sun; much rain at times.
„ 28	7–8	Hazy at time of totality. Strong east wind with showers at times.

The observing party left the "Fantôme" shortly before daybreak on the morning of the eclipse, and in a short time everything was in readiness for the observations. The sky was almost completely clouded over. A light breeze from the E.S.E. drove up sluggishly moving cumuli in great detached masses, some as high as 40° above the horizon, *i.e.*, double the height at which the Sun would be at time of totality. The high land of Grenada was completely enveloped in cloud, and heavy rain was falling in the middle and over the western slopes of the island. Over the low land of Point Fort Jeudy, across the bay, behind which the Sun would come up, was a mass of cumuli, with flattened bases, seemingly motionless. At 6.15 there was a slight shower of rain, and at 6.18 the partially eclipsed Sun was seen for a few seconds. In spite of the fact that the Solar disc was being rapidly obscured, the clouds were gradually breaking up into detached masses. By about 6.40 the greater portion of them had drifted away to the North. The equatorial was then put on the Sun, and the gradually diminishing crescent observed by reflection from the photometric screen. The clock went fairly well, and no adjustment was necessary to keep the limb in contact with the pencilled line of the image on the screen; indeed at no time during the totality was the edge separated from the circle by more than the thickness of the pencil line itself. The shadow of the Moon was plainly visible on the white disc. The moment of totality was 7hr. 10m. 14.6s. (L.M.T.), the calculated time as determined by interpolation from the data furnished by Mr. HIND for Caliveny and Point Salene, was 7hr. 10m. 18s. With respect to the observation, it may be stated that it was determined by means of a chronometer, the error and rate of which had been ascertained from observations of double altitudes made during the preceding 10 days by Captain ARCHER.

After a few moments' observation of the corona, the structure of which was admirably pictured on the white screen in the equatorial, the photometric comparisons were begun in the order previously fixed upon.

In all, 15 comparisons were made out of the 16 originally intended. At about 60 seconds from the calculated end of totality, a dark cloud swept over the corona, rendering all further observation impossible.

Reduction of the Observations.

The lamp used in the equatorial photometer had the values given in Table I. in Siemens units, reduced to a distance of 1 foot from the screen.

TABLE I.

Measures of potential.	Intensity of light reduced to 1 foot from screen.
24	·005
23	·007
22	·010
21	·0145
20	·020
19	·027
18	·036
17	·050
16	·066
15	·089

The lamp was 21·25 inches from the screen during the eclipse; hence the real values of the lamp were higher, but, as it is an inconvenient distance at which to compare the values obtained by the different instruments, an uniform distance of 12 inches from the screen has been adopted.

The same remark applies to the lamp used by Lieutenant DOUGLAS. The measures of potential in intensity of light are given in Table II.

TABLE II.

Measures of potential.	Intensity of light reduced to 1 foot from screen.
10·0	·0024
9·0	·0036
8·0	·0040
7·5	·0055
7·0	·0074
6·5	·0104
6·0	·0135
5·5	·0183
5·0	·0258
4·5	·0340

During the eclipse this lamp was really 21·6 inches from the screen, and, as in the former case, the actual values of the light were higher.

The numbers in the above tables are graphically represented in the following curves, from which the results of the observations were measured.

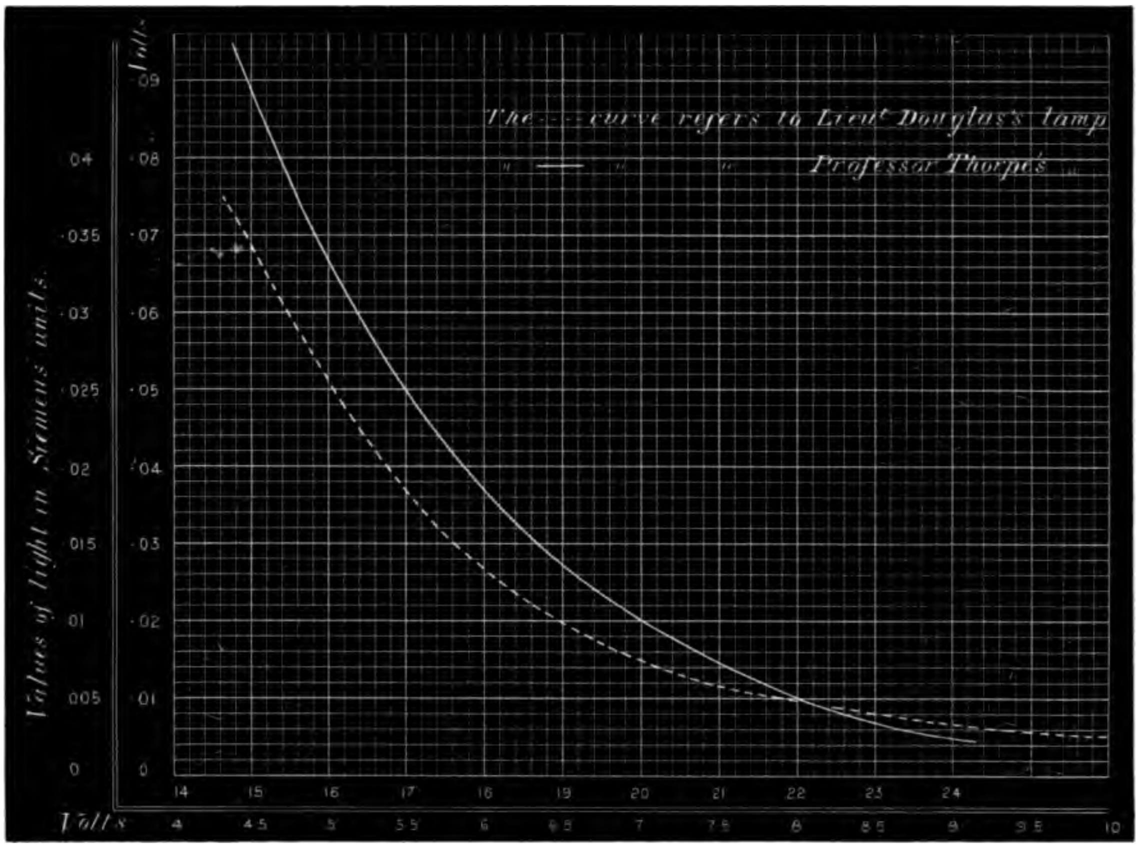


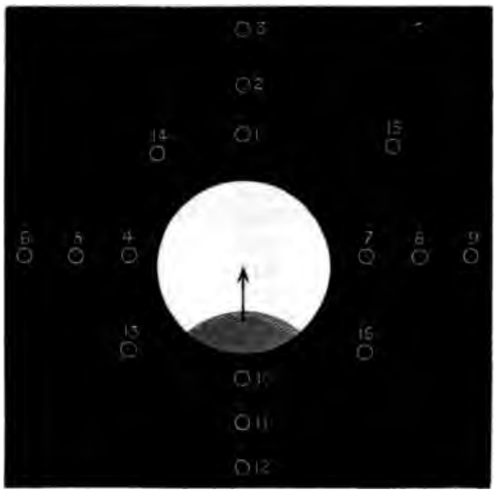
Table III. gives the value in light of the readings on the equatorial photometer. The numbers of the grease spots in Column I. correspond with the order in which they were read. Column II. gives the calculated distance of each spot from the Sun's centre in terms of the Solar diameter. Column III. gives the readings on the voltameter; and Column IV. the corresponding value of the light in Siemens units. Column V. gives the approximate time in seconds after the beginning of totality when the several readings were made.

Table III.—Readings on the Equatorial Photometer reduced to the Values of Light Intensity.

I.	II.	III.	IV.	V.
				seconds.
1	1·55	15·8	·070	10
2	2·66	18·3	·033	20
3	3·66	20·3	·019	30
4	1·61	16·5	·058	40
5	2·55	17·5	·043	50
6	3·44	19·4	·024	60
7	1·25	19·8	·021	70
8	2·16	21·4	·012	80
9	2·60	24·0	·005	90
10	1·11	18·0	·036	100
11	2·16	21·5	·013	110
12	3·16	23·5	·005	120
13	2·16	23·0	·007	130
14	2·33	23·5	·005	140
15	2·33	23·5	·005	150

Fig. 6 shows the position of the several spots as seen on the photometer screen. The arrow and shaded segments show the apparent direction of the Moon's path across the Solar disc.

Fig. 6.



The numbers correspond with the order in which the readings were made.

Table IV. gives the value of Lieutenant Douglas's readings, and Table V. those of Lieutenant Bairnsfather. It must be remembered that Lieutenant Douglas's instrument measured the total light from the corona with only a small portion of the light from the sky, whereas the bar photometer as used by Lieutenant Bairnsfather measured the light from the corona together with that from a large portion of the sky. Hence the readings on the bar photometer are necessarily higher than those obtained by the integrating box.

TABLE IV.—Readings on the Integrating Box reduced to Values of Light Intensity.

Voltameter readings.	Value of light at 1 foot from screen in Siemens units.	Approximate time when readings were made from beginning of totality.
5·4	·0197	15
6·2	·0122	30
5·9	·0142	45
7·0	·0075	60
6·8	·0085	75
7·1	·0070	90
7·3	·0065	105
7·3	·0065	120
7·7	·0054	135
7·8	·0051	150
8·3	·0045	165
8·8	·0040	180
8·9	·0035	195
9·3	·0030	210
9·4	·0027	215
9·4	·0027	220

TABLE V.—Readings on the Bar Photometer reduced to Values of Light Intensity.
Value of the lamp = 0·133 unit.

Distance of lamp from screen in inches.	Equivalent value in Siemens unit at 1 foot.	Approximate time of reading.
33·5	·0160	50
34·4	·0152	70
36·2	·0137	90
39·1	·0118	110
39·6	·0115	130
44·1	·0093	150
46·4	·0084	170
47·8	·0079	190
48·0	·0078	210
47·8	·0079	220
48·5	·0077	230

It will be noticed that Lieutenant BAIRNSFATHER'S first reading was made when 50 seconds of the totality had passed. The delay was due to the circumstance that it was found necessary to diminish the number of cells connected up after totality had begun, and hence a new reading of the galvanometer was needed before the observations could be commenced. The actual time of beginning could only be very approximately known, but from trials made subsequently it is probably accurate to within 5 or 6 seconds.

It will also be observed that both Lieutenant DOUGLAS and Lieutenant BAIRNSFATHER continued to read after the Moon and corona were actually obscured by cloud. Of course neither of the observers was able to notice the fact of the obscuration without looking up from his instrument. The passage of the cloud was, however, readily noticed on the screen of the equatorial photometer, and the time of obscuration during the phase of totality was noted. It occurred, as already stated, at about 1 minute from the calculated end of totality.

The observations of all these observers show in the clearest manner that the results are affected by haze after the first 60 or 70 seconds of totality. Thus, the first six readings on the equatorial photometer are fairly concordant, but after 1 minute had elapsed the light intensities begin to decrease rapidly. Thus, spots 7 and 10, which were considerably nearer to the limb than spots 1 and 4, show a considerably less intensity, and the differences are far greater than could be accounted for by any possible variation of local intensity in coronal light.* Lieutenant DOUGLAS's readings also show a sudden drop at about 60 seconds from the beginning of totality, and they continue to decrease steadily and in precisely the same manner as those of Lieutenant BAIRNSFATHER.

This result is indeed what might have been anticipated. It must be remembered that the air was practically saturated with moisture; a slight shower had fallen even a few minutes before totality, and the lowering of the temperature consequent on the obscuration of the Solar disc would inevitably cause the gradual precipitation of moisture from air already charged to saturation.

If we assume, therefore, that we are justified in regarding the first six observations made with the equatorial photometer as valid, we obtain the following curve as showing the photometric intensity of coronal light at varying distances from the Sun's limb expressed in terms of Solar semi-diameters from Sun centre.

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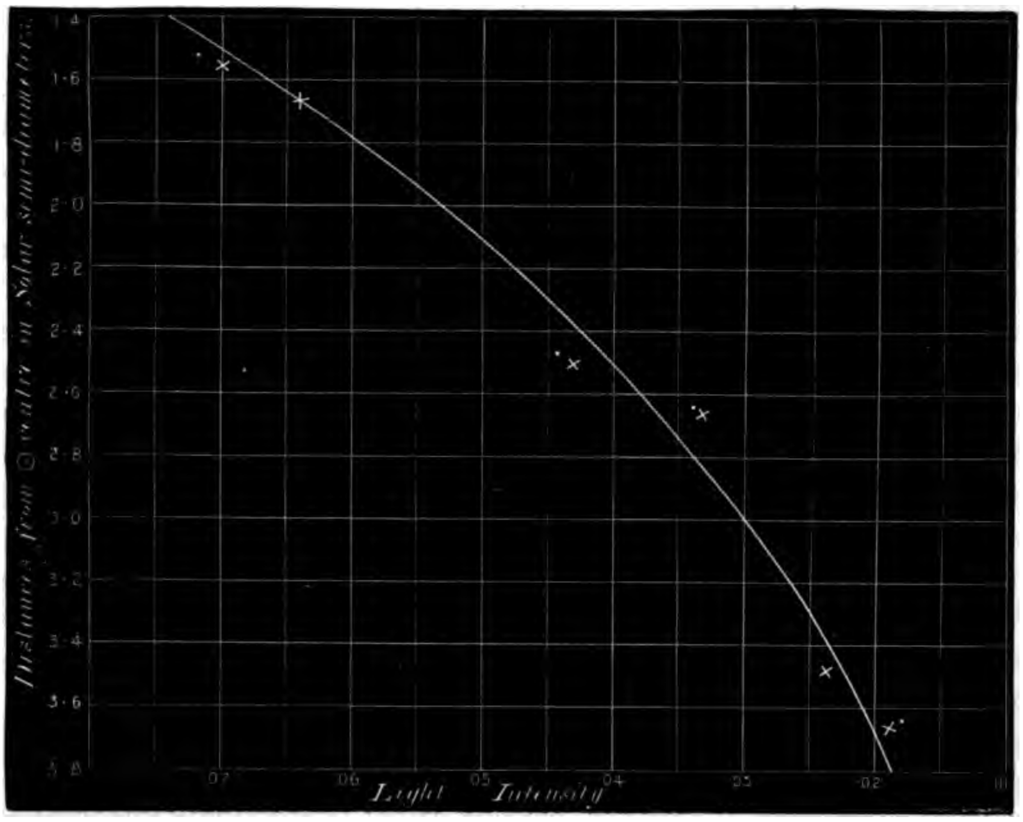
* The photographs show that no such variations in local intensity were present.

Fig. 7.



Curve showing relation between photometric intensity of coronal light and distance of corona from Sun's limb.

It will be quite obvious, from the character of this curve, that the diminution in intensity does not vary according to the law of inverse squares. To show the degree of departure we have calculated the intensity, on the basis of the law, at several points and compared them with the values taken from the curve at the same points.

Distances in Solar semi-diameters.	Photometric intensity.	
	Observed.	Law of inverse squares.
1·6	·066	·066
2·0	·053	·042
2·4	·043	·029
2·8	·034	·022
3·2	·026	·016
3·6	·021	·013

In comparing the observations of Lieutenant DOUGLAS and Lieutenant BAIRNSFATHER with those made during the 1878 Eclipse it must be remembered that the conditions of observation on the two occasions were widely different. The observations in the West Indies were made at the sea-level, in a perfectly humid atmosphere, and with the Sun at no greater altitude than 19°. Professor J. W. LANGLEY observed from the summit of Pike's Peak, which is 14,100 feet above the sea, in a relatively dry atmosphere, and with the Sun at an altitude of 39°. Dr. J. C. SMITH's position at Virginia City, Montana, was about 6,000 feet above the sea-level, and the Sun's altitude at the time of observation was about 44°. If we have regard then to the extinction of the coronal light in the Earth's atmosphere, it follows, *cæteris paribus*, that the observations during the 1886 Eclipse should show a much lower photometric intensity than those of 1878, even if the intrinsic brightness of the corona was the same on the two occasions.

It will be a matter of remark that the brightness of the corona at the various points measured is very inferior to that of the Moon's surface. The value of the light from the full Moon has been variously estimated, but it is not an unfair estimate to take it as ·02 of a candle at 1 foot distance. Supposing the whole surface of the Moon to be of equal brightness, the brightness of the Moon's image on the photometric screen used in the equatorial telescope compared with moonlight itself would have been very closely 60 × ·02 candle, or 1·2 candles. No matter what part of the image fell on the grease spots, it would have required this illumination to have made the grease spots disappear. If we take the highest reading of the corona measured, we find that it is ·07 of a Siemens unit, or about ·06 candle. It thus appears that the brightness of the brightest measured part of the corona (1·55 Solar semi-diameters) was 200 times less bright than that of the surface of the Moon, whilst the furthest spot at 3·66 Solar semi-diameters, having a value of ·019 Siemens unit, or ·015 candle, was only $\frac{1}{800}$ of the brightness.

The highest value of the coronal light, measured in the ordinary photometer, was about ·02, or equivalent to that of the full Moon. It is evident therefore that, even when taking into account the greater angular area which the corona occupied, the brightness of the corona was very much greater close to the limb than elsewhere. The photographs which were taken show this fact. A dense image of the Moon might be secured on plates such as were used at the eclipse in the $\frac{1}{20}$th of a second. The photographs which were taken of the corona varied in exposure from half a second to 100 seconds. Even with the first named exposure the corona close to the limb was much over exposed, showing the intense brightness of that part, whilst the image of the highest part of the corona measured was hardly visible. Owing to the variable quantities of cloud during totality, it would be unfair to try to make any comparison between the brightness of the corona at the poles and at the equator, though, had no cloud intervened, the arrangements adopted would have enabled such to be done. Nor is it worth while to endeavour to establish any law for the decrease of brightness of the corona in terms of its distance from the limb. We have, however, in the measures given, obtained results which will be of use in comparing the brightness of the corona on this occasion with that of other future eclipses; and we believe that measurements of the brightness obtained by the plan adopted, or by photography, will become a necessary observation in determining what connection, if any, the Sun-spot periods have with the coronal phenomena.

We have to express our acknowledgments to Mr. H. A. LAWRANCE for the very ready and efficient aid which he rendered in mounting the equatorial, and in assisting in making the photometric measurements.

To Captain ARCHER and his officers, Lieutenants DOUGLAS and BAIRNSFATHER, we are also greatly indebted for the lively and eminently practical interest which they manifested in the work of the expedition generally, and especially for their zealous co-operation in the particular observations which had been entrusted to us. Indeed, the entire crew of the "Fantôme" laboured in the most willing and cheerful manner, often under circumstances of considerable personal discomfort, to promote the success of the expedition in every possible way.

APPENDIX.

Observations made by NEWCOMB'S *Method on the Visibility of Extension of the Coronal Streamers at Hog Island, Grenada, by Commander* ARCHER, *R.N.*

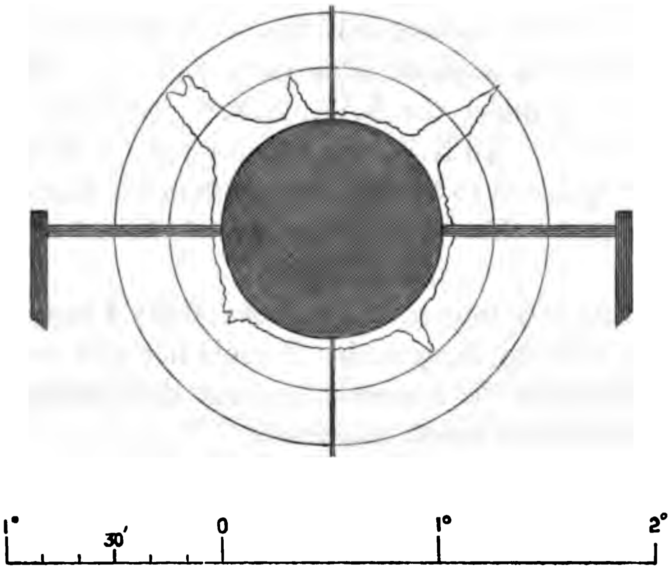
1. A 9-inch disc was erected on a pole fastened to the N.W. corner of the observatory, much in the manner suggested in the instructions; the distance of the eye-piece from a plumb-line hanging under the centre of the disc frame was exactly 40 feet.

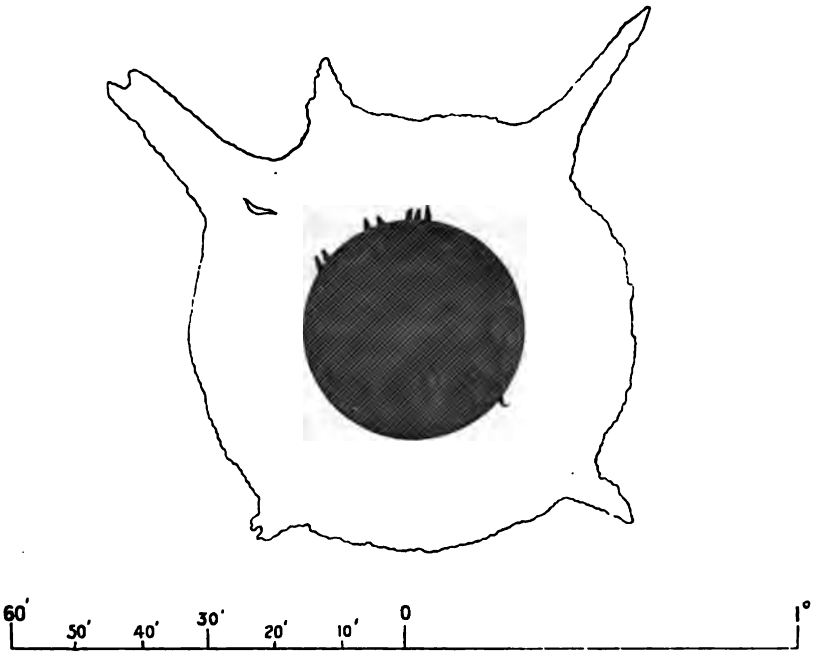
2. As a means of measuring the length of the streamers of the corona, I had fitted two copper circles concentric with the disc, supported on a light iron cross; the diameter of the inner circle was 1½ disc diameter (13½ inches), that of the outer one 2 disc diameters (18 inches).

3. In order to localise the position of the streamers, I decided to look upon the disc as a compass card, with the North point vertically upwards; the arms of the iron cross therefore indicated the cardinal points.

4. The iron crossbar which carried the eye-piece I had lengthened to 18 inches, in order to increase the limits of adjustment, and also to admit of the Sun being observed when at the proper altitude (18° 48′) on several preceding days; owing to this circumstance, I feel considerable confidence in the Sun's having been nearly concentric with the disc at the beginning of totality, as on two previous days I timed the exact moment when the Sun's centre was behind the centre of the disc, and found the calculated altitudes at the time were between 18° 48′ and 18° 49′, the calculated altitude at commencement of totality being 18° 47′ 51″. The position of the eye-piece was adjusted on the previous day, and required no further alteration.

5. I found the concentric wire circles of great advantage, as the streamers, when judged by eye alone, appeared to be much longer than they really were; and, had it not been for this device, I feel sure I should have over-estimated the distance to which they extended.





6. Half an hour before the commencement of totality I sat down in the shade, and a quarter of an hour before I had my eyes bandaged with a double thickness of black cloth; on the signal being given, I removed the bandage and observed through the

eye-hole a narrow fringe of very bright white light all round the wooden disc, with marked extensions to the N.E. and N.W., which both just reached the outer wire, though the outer portion was very faint. The N.W. one was rather forked, the N.E. one pointed. To N.N.W. was a fainter thin extension, which reached to the first wire; to S.E. was a similar one, also reaching to the first wire; and to S.W. was a very small double one, hardly reaching half way to the first wire.

7. I then looked through a ship's telescope I had on a stand near me, and I observed red prominences under the N.W., N.N.W., and S.E. extensions, and also some prominences at North. To N.W. was a long tongue of flame about the colour of a candle flame, which appeared to be disconnected from the Sun's limb, and to extend to about 10′ from the limb; this estimation was made by aid of the afore-mentioned wires.

8. On a cloud passing over towards the end of totality I looked at the stars overhead, but there were so many in sight that I could not pick out what they were; I should say that, judging by the contrasts between their brightness, some of them must have been of the third or fourth magnitude.

9. At the eye-hole a linear inch on the disc subtended an angle of 68 minutes; thus, supposing the Sun were concentric with the disc at the beginning of totality, the streamers to the N.W. and N.E. would be about 45′ from the Sun's limb, those to N.N.W. and S.E. about 30′, that to S.W. about 22′, and the general ring of light of the corona probably about 18′.

10. I observed a light film of cloud to be passing over the corona for several seconds before it was hidden by the cloud.

11. Explanation of the diagrams (p. 383):—

The upper figure represents the wooden disc with its supports and wires, and the irregular ring of the corona and streamers showing round it, as seen at the commencement of totality. The second figure is a reproduction of the first on a larger scale, with the disc removed; the dark circle represents the Moon, and the black dots show the prominences that were noticed. A scale of degrees and minutes is attached to each figure. The vertex is towards the top of the paper.

Robert H. Archer, *Commander.*

H.M.S. "*Fantôme,*" *Grenada,*
30th August, 1886.

XII. *Report of the Observations of the Total Solar Eclipse of August* 29, 1886, *made at Grenville, in the Island of Grenada.*

By H. H. TURNER, *M.A., B.Sc., Fellow of Trinity College, Cambridge.*

Communicated by the Astronomer Royal.

Received February 23,—Read March 15, 1888.

I. GENERAL ARRANGEMENTS.

THE eclipse party was landed at St. George's, Grenada, on Thursday, August 12. On Saturday, August 14, I proceeded with Professor TACCHINI to Grenville, a village on the east coast, near which it had been decided that we were to take up our stations. Our instruments and baggage, with two huts constructed by a local carpenter at St. George's, were placed on a sloop which was towed round the coast as far as Grenville Bay by H.M.S. "Fantôme"; but, as it was unadvisable for a large ship to attempt to enter the bay, the sloop went in alone, and deposited its freight on the jetty. I proceeded to Grenville by the Grand Etang Pass, which runs nearly east and west across the island from St. George's to Grenville, being anxious to determine whether there was any chance of returning with heavy baggage by the overland route. However, it soon became evident that this would be very difficult, if not impossible, the pass being little more than a bridle-path in certain portions.

The party—consisting of Professor TACCHINI and myself, Lieutenant SMITH, of H.M.S. "Sparrowhawk," a quartermaster from the same ship, and an artificer from H.M.S. "Fantôme"—found very comfortable quarters at Boulogne, the house of Colonel DUNCAN, which is situated about two miles from Grenville Bay, along a fair road.

After some delay, caused by very heavy rain on Monday, August 16, which made the roads nearly impassable for a time, the huts and instruments were carted from the bay to Boulogne, and erected very quickly on a site close to the house, longitude 4h. 6m. 30s. W., latitude 12° 8'·5 N. One bell tent was also erected for contingencies, though this was almost unnecessary, as our instruments were unpacked and the cases left in the "buchan," a kind of superior barn for storing cocoa, which was then nearly empty and was kindly placed at our disposal by Mr. ST. GEORGE, Colonel DUNCAN's representative.

The huts, which had been constructed before our arrival under the direction of his Excellency Governor SENDALL, required very little modification. They were ten feet square, with a gabled roof, the ridge running north and south. The door for entrance was in the western face. The eastern roof consisted almost entirely of two shutters, which could easily be thrown open with a pole; and, after this, a large portion of the eastern face (which, in my own case, was enlarged still further) could be opened, turning about hinges which ran horizontally across the face. This gave practically a clear view eastwards from the sea horizon up to the zenith, so that it was possible to work on the Sun from sunrise to midday. The instruments inside were further protected from drippings by mackintosh covers. My own instrument consisted of the Simms equatorial No. 1, with a grating spectroscope; and a 4-inch telescope by Simms, mounted as a part-counterpoise on the same polar axis. The whole weight was about 15 cwt.; and, as my work was not to be photographic, and no very great steadiness was necessary, it was found sufficient to place the base plate (an iron plate 3 feet in diameter) on the soil, after removing the turf.

After Tuesday, August 17, the mornings were generally fine, and everything was soon in order for observation. Lieutenant Smith rated the chronometer (Arnold and Dent, 965) by sextant observations of equal altitudes of the Sun with an artificial horizon; the local time of totality was found to be correct within a few seconds.

In taking a photograph of the portion of the spectrum under observation, some difficulty was found in preventing the film from becoming detached during development or washing. I found at last that drying the plate thoroughly by the stove before taking the photograph prevented this detaching.

The night of August 28–29 was beautifully fine, but at sunrise clouds began to gather, and the observations of the eclipse were made under considerable difficulties. The following notes will give some idea of the circumstances :—

	h.	m.	s.	
From sunrise to	6	30	0	Cloudy.
	6	30	0	Sun appeared.
	6	31	30	Cloudy.
	6	33	15	Clear.
	6	34	15	Cloudy.
	6	41	20	Clear.
	6	45	30	Slight shower; Sun visible.
	6	47	0	Shower passed.
	6	48	0	Another shower.
	6	49	5	Shower passed.
	6	52	20	Showery.
	6	55	0	Shower passed.
	6	56	30	Cloudy.
	7	0	20	Clear.
	7	10	10	Cloudy for ten seconds.
	7	12	0	Totality.

h.	m.	s.	
7	14	30	Light cloud passing.
7	15	5	Quite cloudy.
7	15	50	End of totality; cloud lifted for five seconds.
7	16	0	Quite cloudy.
7	20	0	Clear.
7	23	15	Cloudy.
7	24	10	Clear.
7	25	17	Clouded up for some time.

After the eclipse the instruments were quickly dismounted, and, with the huts, were placed on board a small steamer, the "Waltham," which, on Wednesday, September 1, conveyed us back round the south coast to St. George's, stopping at Prickly Point on the way, to pick up a lighter containing the instruments and baggage of the southern party. I returned to England with some other members of the expedition by the mail which left Grenada on the following Sunday.

II. OBSERVATIONS.

(*a*) *The Order of Appearance of Certain Bright Lines of the Chromosphere and Inner Corona.*

In the programme arranged before the expedition left England, I was directed to attempt the confirmation of Mr. LOCKYER's observations before and after totality during the Egyptian eclipse of 1882. These are described briefly, with promise of further details, in the 'Roy. Soc. Proc.' for 1882 (pp. 291 *et seq.*), and it will be sufficient here to reproduce the following paragraphs from this paper for convenience (*ibid.* p. 296, §§ 10–13). The observations were intended as a test of two rival hypotheses.

"On the old hypothesis the construction of the Solar atmosphere was imaged as follows:—

"(1) We have terrestrial elements in the Sun's atmosphere.

"(2) They thin out in the order of vapour density, all being represented in the lower strata, since the Solar atmosphere at the lower level is incompetent to dissociate them.

"(3) In the lower strata we have especially those of higher atomic weight, all together forming a so-called 'reversing layer, by which chiefly the Fraunhofer spectrum is produced.

"The new hypothesis necessitates a radical change in the above views. According to it, these three statements require to be changed, as follows:—

"(1) If the terrestrial elements exist at all in the Sun's atmosphere, they are in process of ultimate formation in the cooler parts of it.

"(2) The Sun's atmosphere is not composed of strata which thin out, all substances being represented at the bottom; but of true strata, like the skin of an onion, each different in composition from the one either above or below.

"(3) In the lower strata we have, not elementary substances of a high atomic weight, *but those constituents of all the elementary* bodies which can resist the greater heat of these regions.

* * * * * * *

"It was stated in (6): While discussing the conditions of observation, that whether we were dealing with strata of substances extending down to the Sun, or limited to certain heights, the spectral lines would always appear to rest on the Solar spectrum, and that the phenomena would *in the main* be the same. This, however, is true in the main only; there must be a difference, and this supplies us with a test between the rival hypotheses of the greatest stringency.

"For take three concentric envelopes, A, B, C, so that only A rests on the photosphere, B rests on A, and C on B. The stratum B, being further removed from the photosphere than the stratum A, will be cooler, its lines will be dimmer, and the lines of C will be dimmer than the lines of B, and so on. So, if we could really observe the strata, *the longer a line is, i.e.*, the greater the height at which the stratum which gives rise to it lies, *the dimmer the line will be.*

"Now, our best chance of making such an observation as this is during a total eclipse. We do not see the lines ordinarily, in consequence of the illumination of our air. As during an eclipse, before totality, the intensity of this illumination is rapidly diminishing, the lines first visible should be short and bright, and should remain short, while the new lines which become visible as the darkness increases should be of gradually increasing length.

"Further, the short lines which first appear should be lines seen in prominences and not in spots, and relatively brighter in the spark than in the arc, while the longer lines added should be lines affected in spots and *not* in prominences."

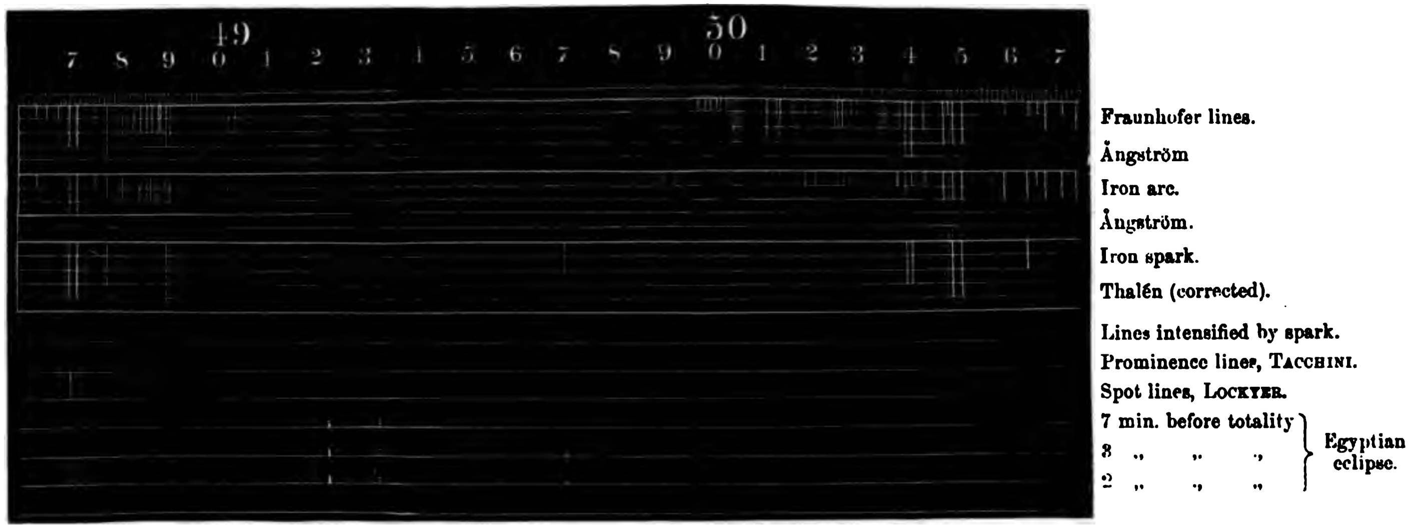
The manner in which these expectations were realised is shown at once by the subjoined diagram from the same paper (fig. 1).

For convenience of reference I have used small letters to denote the lines shown in the lowest line of the diagram, beginning with *a* on the left. Thus—

The two lines at 4870–71 are called *a* and *b*.
" " " " 4890 " " *c* and *d*.
" three " " 4918–23 " " *e*, *f*, and *g*.
" two " " 4932–33 " " *h* and *i*.
" one " " 4956 " " *k*.
" " " " 4970 " " *l*.

In this nomenclature, then, lines *g* and *l* are seen by TACCHINI in prominences,

Fig. 1.



The Ba line at λ 4933·4 is a line seen thirty times by Young at the maximum spot period, and not recorded by Tacchini at the minimum. The lower longer line not seen till five minutes afterwards is a Te line λ 4932·5.

and the other lines (excepting *h* and *i*) are seen by LOCKYER in spots. In the Egyptian eclipse—

g and *i* . . . . were seen 7 minutes before totality;
k and *l* . . . . ,, ,, 3 ,, ,, ,,
and all the other lines ,, ,, 2 ,, ,, ,,

In making my own observations, the slit of the spectroscope was placed radial to the limb at the cusp some minutes before totality. The point of the cusp was brought on to the slit, and the motion of the Sun allowed to carry it gradually away. The Solar spectrum, as seen in the spectroscope, thus became narrower as the point of the cusp approached, and finally disappeared. It was hoped to catch the bright lines just at this point, the dark lines of the vanishing Solar spectrum serving as a set of fiducial marks for the identification of the bright lines. But there was also inserted in the focus of the eyepiece a photograph of this region of the spectrum taken the day before. Some twenty minutes before totality it was suspected that, in the increasing darkness, the lines of the photograph would not be well seen, and the places of the lines were therefore carefully marked by scratching the film with a sharp penknife; the scratches were seen to accord well with the proper lines in the Solar spectrum. But in the actual observations it was not found necessary to refer to the photograph, and the identifications mentioned below were referred to the vanishing spectrum. For the period of the observations the cusp does not travel very rapidly along the limb,* so that the slit was nearly radial until a few seconds before totality. I now subjoin the observations, as recorded at the time by an assistant to my dictation.

Time by chronometer.			
h.	m.	s.	
6	55	30	No bright line visible at cusp.
6	56	30	Cloudy; no bright line visible up to the time clouds appeared.
7	0	20	Clear again.
7	7	45	F line appeared.
7	8	55	*g* appeared; very short.
7	10	10	Cloudy for ten seconds.
7	11	30	*g* and *i*.
			Immediately after many lines appeared.
7	12	0	Totality.
7	15	50	End of totality; cloudy.
7	20	50	Only F, *g*, and *i* visible at times.

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* The rate of motion is very nearly inversely proportional to the duration of totality. For a three-minute eclipse the angular distances of the cusp from the point of disappearance are respectively 75°, 60°, 51°, 33½°, at 4½ mins., 90 secs., 54 secs., 18 secs. before totality.

7 21 45	*g* still suspected, and *k*.
7 22 28	*k* certainly visible.
7 23 15	Cloudy.
7 24 10	Clear again.
7 24 42	No line visible.
7 25 17	Cloudy for some time.

In addition, I wrote the following note immediately afterwards:—

"Just before totality all the lines were glimpsed, but all were much shorter than was expected; *g* and *i* were certainly seen before this, but no others. F was very bright from time noted, and prominent after totality. After totality was over, cloud still remained for nearly a minute. Then F line only was prominent, but *g* and *k* were afterwards seen; *i* was not seen; *k* appeared to remain longest."

The telescope used was one by Simms, of 6 in. aperture and 6 ft. focal length. To this was attached a grating spectroscope by Hilger; collimator, 1½ in. aperture and 18 in. focal length; telescope, 1½ in. aperture and 9 in. focal length; magnifying power of eyepiece, 10. The grating kindly lent to me by Captain Abney was by Rutherfurd, and contained 17,200 lines to the inch. The second order of spectrum was used.

It will be seen that these observations confirm Mr. Lockyer's in only a few points. The phenomena were apparently of greater intensity in Egypt; the bright lines *g* and *i* appeared there 7 minutes before totality, the first of which I saw 3 minutes and the second only 30 seconds before totality. The second stage of the appearance of *k* and *l* in the Egyptian observations is not represented at all in my own, being indistinguishable both from the third stage and from the general flashing out of many bright lines which took place just before totality, and is represented in my own notes by "many lines appeared."

Bright lines are recorded after totality for nearly 6 minutes; and, as the appearance of *k* was unexpected, it was looked at carefully, with the result noted, "*k* certainly visible." I could not help feeling some doubt afterwards as to the observation; and it is possible that it is really spurious, and due to the straining of the eyes, to the imagination, or other causes. But it is only differentiated from the other observations by the fact of coming rather later in a somewhat exciting half-hour, and I have left the record untouched. It is perhaps only natural that the clear atmosphere of Egypt and the great altitude of the Sun, as compared with the vapour-laden air of the West Indies and an altitude of only 18°, should modify the particular phenomena under discussion. And, again, it must be remembered that in 1882 the Sun's activity was nearly at its maximum; whereas in 1886, judging at least by the paucity of spots, the minimum was appearing somewhat prematurely. This may have had a considerable effect on these phenomena.

This diminution of the period of time over which the phenomena were distributed, and especially the concentration of the later changes in the spectrum into the space of a few seconds, made a most essential difference in the observation. The record of the Egyptian eclipse gave rise to the hope that the observations could be made somewhat leisurely—which, indeed, is almost essential if the eye is to carefully compare faint lines (the few early lines were too faint on this occasion to be held steadily by the eye, but could be seen by glimpses in the manner familiar to observers of faint objects) in different parts of the field. Instead of this, I found that the phenomena were sudden, and, with the few exceptions mentioned, the change which the spectrum underwent was confined to what has been called "the flash," as I understand it. Mr. LOCKYER seems to have had the good fortune to see this "flash" in stages, extended for analysis. For a complete confirmation of his results it is possible that better conditions may be necessary than those of the 1886 eclipse.

III. OBSERVATIONS DURING TOTALITY.

During totality I had been directed by the Committee to examine the corona, with a view to the detection of currents. For this purpose, a 4-in. telescope by SIMMS was attached to the same mounting as the 6-in. used for the spectroscope. The power used was 140. With this instrument I made a careful examination of the corona all round the limb. It did not seem to me to vary essentially in appearance from point to point. The structure was radial, and on following the rays outwards from the limb I could not detect any appearance of curvature, to join another ray in the form of a loop. I believe such forms were represented in some of the naked-eye drawings. The great prominence was a striking feature ; it seemed to me of a rosy tint throughout. Concerning the particular object of the search—indications of any sort of current —I can only report a negative result.

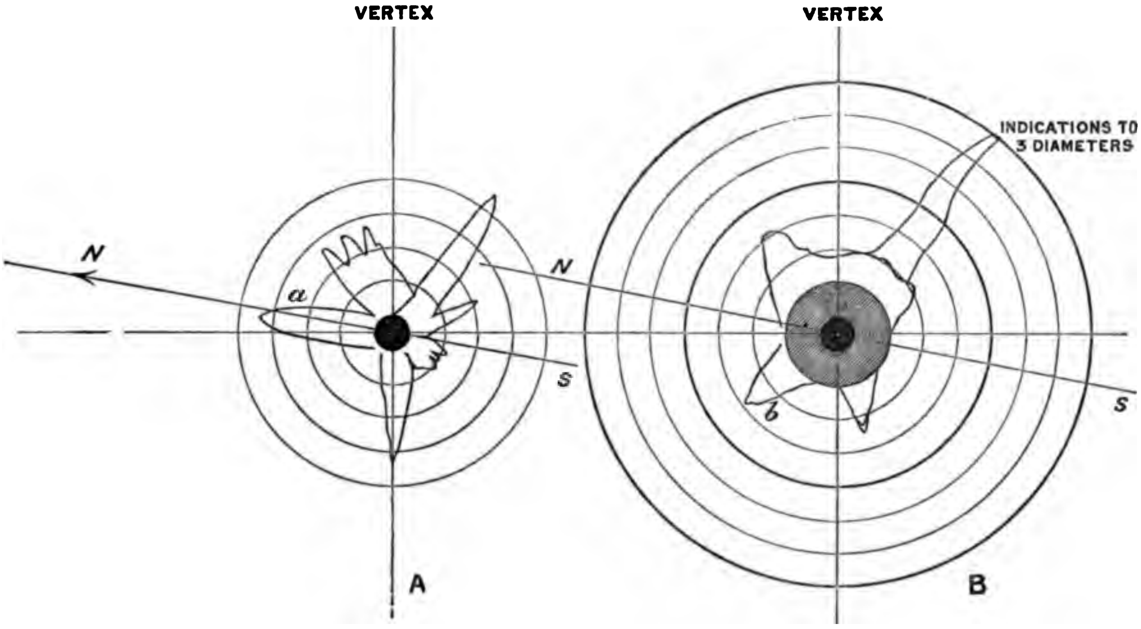
There was, to my eye, scarcely any distinguishing feature in *structure* by which the orientation could be recognised ; though some of the rays were, of course, longer than others—that is to say, I looked specially for the structure characteristic of the poles, and failed to notice any very marked difference from the structure in other parts of the circumference.

I then returned to the spectroscope, with a view to examining the brightness of the lines at different distances from the limb ; but the eye examination had taken some considerable time, and the clouds which obscured the Sun for the last minute of totality were already approaching.

IV. DRAWINGS.

The drawing marked A was made by Mr. ST. GEORGE with an opera glass. His eyes were not specially made sensitive before totality.

The drawing B was made by Lieutenant SMITH, of H.M.S. "Sparrowhawk," from naked-eye observations; but his head was wrapped in a black mackintosh 15 minutes before totality commenced.



Mr. St. George's drawing. Lieutenant Smith's drawing.

The black centre represents the Moon; the shaded circle in B represents the disk (three times size of Moon) which obscured the brighter portions of the corona.

Long. of station, 4h. 6m. 30s. W.; lat. 12° 8′·5 N.

No attempt is made in either drawing to give details, but merely the distances to which coronal extension could be traced, as estimated in diameters of the central black disk, which in the first case represents the Moon, and in the second a disk which was so placed as to screen the brighter portions of the inner corona from the observer's eyes, and subtended an angle of 3 diameters. It will be seen that the chief discrepancy in the drawings is in the orientation of the rays marked respectively *a* and *b*, in one of which there would seem to be some error; otherwise the correspondence is remarkably good, except that Lieutenant Smith obviously traced the extension much further than Mr. St. George. It may be mentioned that special rehearsals were conducted on the two days before the eclipse, in drawing on such skeleton forms of concentric circles pictures of coronæ held before the eye for 3½ minutes. The two gentlemen mentioned above were found to reproduce the direction of the rays very accurately, and, as regards distance, Mr. St. George seemed to be liable to slightly over-estimate the extensions.

XIII. *Revision of the Atomic Weight of Gold.*

By J. W. **Mallet**, *F.R.S., Professor of Chemistry in the University of Virginia.*

Received April 15,—Read May 9, 1889.

Until lately gold ranked among the elementary substances of which the general properties had been well ascertained, but in regard to the atomic weights of which our knowledge was least satisfactory. That this constant should be determined as accurately as possible for gold was desirable in view of its bearing on the precise place assigned the metal in the "periodic" classification of the elements based on the ideas of **Newlands**, **Odling**, **Mendelejeff**, and L. **Meyer**. Furthermore, an exact knowledge of the atomic weight of gold might be conveniently applied in the determination of the atomic weights of some of the other elements. A practical laboratory reason for desiring to possess a trustworthy value for this constant was also presented by the facility with which gold compounds of many organic substances may be prepared, and the ease with which their composition may be ascertained by simple ignition in the air and weighing of the residual gold, the results leading to a knowledge of molecular composition when the atomic value of the weight of the metal obtained is assumed to be known.

For the last three years and a half I have been occupied, during a large part of such time as has been available for original work, in devising and carrying out experiments aiming at the redetermination of the constant in question. The difficulties met with have been greater than were at first looked for, and have led to much time and labour being consumed in attempts to overcome them. About two years ago, when this work was already well under way but still in progress, there appeared the results of experiments aiming at the same end, by **Krüss** in Germany and by **Thorpe** and **Laurie** in England—experiments made with the care and accuracy of modern methods, and apparently deserving of much confidence. My own work, however, was continued, as we cannot have too many careful independent determinations of atomic weights by different workers, and as I had used to a considerable extent other processess than those on which the newly published determinations were based, while the chemists named had employed, in the main, one and the same method. A preliminary notice of my work was read in the Chemical Section of the British Association at the Manchester meeting of 1887. The details of my experiments and the results which I have reached are now laid before the Royal Society.

Earlier Determinations of the Atomic Weight of Gold.

In the work of L. MULDER, 'Historisch-kritisch Overzigt van de Bepalingen der Æquivalent-Gewigten van 24 Metalen,' Utrecht, 1853, and in the recent papers of KRÜSS and of THORPE and LAURIE, there are abstracts of reports upon a number of experiments by chemists of the earlier part of this century bearing on the value to be assigned to the atomic weight of gold, such as those of PROUST,* RICHTER,† DALTON,‡ THOMSON,§ OBERKAMPF,|| PELLETIER,¶ FIGUIER,** and JAVAL;†† but of these none deserve any attentive consideration at the present day, the methods used having in some cases been such that accurate results could not be expected from them, and the actual figures obtained in other cases differing so widely from each other that no importance can be attached to them.

Before the year 1887 but two chemists—BERZELIUS and LEVOL—had published the results of experiments furnishing fairly admissible data for calculating the atomic weight in question.

A. *Experiments of* BERZELIUS, 1813.‡‡—The method adopted was the precipitation of a solution of auric chloride by metallic mercury, determining the quantities of mercury dissolved and gold thrown down. In the original paper but a single experiment is reported, but later the author appears to have made a second,§§ so that for the two MEYER and SEUBERT, in their recalculation of the atomic weights of the elements,|||| give as the sums of the amounts of mercury and gold found to be equivalent to each other 24·240 grm. of the former and 15·912 grm. of the latter. Taking these quantities to represent the ratio between the weights of three atoms of mercury and two atoms of gold, we have for the weight of the single atom of the latter (H = 1)

If we assume Hg = 199·8 (L. MEYER and SEUBERT||||). . 196·73
" " " 199·712 (F. W. CLARKE¶¶). . . . 196·65

This method recommends itself as advantageous on several grounds, and the experiment deserves repetition as soon as the atomic weight of mercury becomes

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* 'Journal de Physique,' vol. 62, 1806, p. 131; 'N. GEHLEN, Journal,' vol. 1, 1806, p. 477.

† 'SCHWEIGGER, Journal,' vol. 7, 1813, p. 43.

‡ 'New System,' part 2, 1810, p. 253.

§ 'First Principles,' vol. 1, 1825, p. 440.

|| 'Annales de Chimie,' vol. 80, 1811, p. 140.

¶ 'Annales de Chimie,' vol. 15, 1820, pp. 5, 113.

** 'Annales de Chimie,' vol. 19, 1821, p. 177.

†† 'Annales de Chimie,' vol. 17, 1821, p. 337; 'SCHWEIGGER, Journal,' vol. 33, 1821, p. 238.

‡‡ 'Stockholm, Kgl. Vetensk. Akad. Handl.,' 1813, p. 185.

§§ BERZELIUS, 'Lehrbuch,' vol. 3, p. 1212, str. 70.

|||| LOTHAR MEYER u. KARL SEUBERT, 'Die Atomgewichte d. Elemente aus d. Originalzahlen berechnet,' Leipzig, 1883, p. 191.

¶¶ "The Constants of Nature," 'Smithsonian Miscellaneous Collections,' Washington, D.C.

known with greater certainty than at present. But until this condition is fulfilled the result for gold cannot be depended upon as of the first rank in exactness. In any renewed attempt to apply this method several questions would have to be examined as to the precise nature of the solution used, and of the reaction itself.

[BERZELIUS* also precipitated gold by means of a known quantity of phosphorus from a solution of the chloride used in excess, and his results, as calculated by F. W. CLARKE,† lead to the atomic weight 195·303 for gold; but this process appears ill adapted to give very exact results, even in such hands as those of the great Swedish chemist, and the value obtained is certainly too low in the light of more modern researches.]

B. *Experiments of* BERZELIUS, 1844.‡—In these experiments potassium auri-chloride, which, it was found, could not be completely dried without loss of chlorine, was ignited in hydrogen, and the residue was treated with water to dissolve potassium chloride, the quantity of which was determined, as well as that of the metallic gold left undissolved. Five experiments were made, and the aggregate amounts obtained of potassium chloride and gold were 3·7800 grm. and 9·9685 grm. respectively. These figures, if we assume K = 39·03 and Cl = 35·37, give for the atomic weight of gold 196·20, the lowest result from one of the individual experiments being 196·11, and the highest 196·27.

Among possible sources of error in this process we may note as deserving consideration the conceivable retention by the potassium auri-chloride of hydrogen auri-chloride, and the difficulty of directly determining with accuracy the potassium chloride extracted by water. The former would lead to a higher result for gold than should be obtained; the latter might either give too low a result in consequence of imperfect drying, or too high if there were partial loss by volatilization, either during the ignition in hydrogen or in subsequently recovering the potassium chloride from solution. The quantities of material used were smaller than is probably desirable.

C. *Experiments of* LEVOL, 1850.§—A weighed quantity of gold was dissolved as auric chloride, the metal reduced from the solution by means of sulphur dioxide, and the sulphuric acid formed was determined as barium sulphate. In two experiments, reported as giving exactly the same result, 1 grm. of gold gave 1·782 grm. of barium sulphate. Hence, if Ba be taken = 136·86, S = 31·98, and O = 15·96, we have for the atomic weight of gold the number 195·86.

Of the sources of error to which this method is liable probably the most important are atmospheric oxidation of sulphurous to sulphuric acid and imperfect washing out of soluble compounds of barium from the barium sulphate. Both would tend to give too low a result for gold.

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* BERZELIUS, 'Lehrbuch,' 5. Aufl., vol. 3, p. 1188.

† "The Constants of Nature," 'Smithsonian Miscellaneous Collections,' Washington, D.C.

‡ BERZELIUS, 'Lehrbuch,' 5. Aufl., vol. 3, p. 1212.

§ 'Annales de Chimie,' [3], vol. 30, p. 355.

For all these earlier experiments details are wanting as to the exact mode of purification of the gold and other materials used, and in the weighings there appears to have been no correction introduced for atmospheric buoyancy; the results doubtless represent apparent, not absolute, weights.

[There is also to be quoted the statement of JULIUS THOMSEN,[*] that he found in hydrogen brom-aurate ($AuBr_3.HBr.5H_2O$) 32·11 per cent. of gold and 52·00 per cent. of bromine, from which he concluded that Au = probably about 197. Taking Br = 79·76, and calculating from the ratio of $Br_4 : Au$, the number is 197·01.]

Recent Careful Determinations of the Atomic Weight of Gold.

A. *Experiments of* GERHARD KRÜSS, 1886.[†]—The author has described in detail the means resorted to for the preparation of pure metallic gold, and especially for its separation from silver and the metals of the platinum group, with an account of the spectroscopic examination of the gold employed. He has then given a full account of:—*a.* His determinations of the gold and chlorine (the former reduced by a stream of sulphur dioxide; the latter precipitated and weighed as silver chloride) in a neutral solution of auric chloride, prepared by the action of water on the so-called auro-auric chloride (Au_2Cl_4),[‡] itself prepared by the direct action of chlorine on metallic gold; *b.* Like determinations of gold and chlorine in sublimed auric chloride, made by direct action of the elements on each other with careful regulation of the temperature; *c.* Determinations of the gold in a weighed quantity of potassium auri-bromide ($KAuBr_4$), the metal in some experiments reduced from a solution of the salt by sulphurous acid, in others reduced from the dry salt by heating in a stream of hydrogen; *d.* Determinations of the gold and bromine (the former thrown down by sulphurous acid; the latter precipitated as silver bromide) in the same salt, potassium auri-bromide; *e.* Determinations of the loss of weight (representing 3 atoms of bromine for 1 of gold) undergone by heating potassium auri-bromide gradually to 320° C., towards the end in a stream of hydrogen; *f.* Determinations of the quantity of potassium bromide recovered from the residue left in the experiments of *e* by treatment of this residue with water, separation of the metallic gold, careful evaporation of the liquid, and final cautious heating of the potassium bromide over a free flame. In the experiments of series *a* account was taken of the somewhat different processes of purification of the gold used, but, no corresponding differences being observable in the results obtained, no further record was made in the remaining series of the history of the gold used in these.

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[*] 'Journ. Prakt. Chem.,' vol. 13, 1876, p. 345.

[†] 'LIEBIG'S Annalen,' vol. 238, p. 30; and separate publication, G. KRÜSS, 'Untersuchungen über das Atomgewicht des Goldes,' München, 1886.

[‡] KRÜSS has in a later paper ('Berichte Deutsch. Chem. Gesell.,' vol. 20, p. 2634) denied the existence of auro-auric chloride as a definite compound, but admits that the substance so described by JULIUS THOMSEN yields on treatment with warm water a solution of pure neutral auric chloride, with separation of metallic gold.

After correction of the weighings for atmospheric buoyancy in such cases as seemed to the author to involve a correction worth noticing, the following results were calculated from the figures obtained, these figures agreeing in general closely with each other in each series :—

Series *a*.	Mean of	8	experiments.	Atomic weight of	gold	= 196·622
,, *b*.	,,	4	,,	,,	,,	= 196·143
,, *c*.	,,	9	,,	,,	,,	= 196·741
,, *d*.	,,	5	,,	,,	,,	= 196·743
,, *e*.	,,	4	,,	,,	,,	= 196·619
,, *f*.	,,	4	,,	,,	,,	= 196·620

Leaving out the results of series *b* on the ground of the very small quantity of sublimed auric chloride available, and the considerable discrepancy of one of the results (that in which most material was used) from the rest, the author calculates from the remaining 30 experiments the general mean 196·669; but, taking into account the greater or less closeness of agreement of the figures obtained by the several methods, he comes to the conclusion that 196·64 may better be assumed as the true atomic weight of gold. In these calculations Ag was assumed = 107·660, Cl = 35·368, Br = 79·750, and K = 39·040.

As regards possible sources of constant error in KRÜSS's experiments, it may be observed that—

1. In series *b* very small quantities of sublimed auric chloride were used—the whole amount available for all four experiments being only about seven-tenths of a gramme—and it is probable that a little free chlorine may have been physically retained by the chloride in spite of the long-continued passage over it of dry air. The experiment in which the largest quantity of material was used gave the atomic weight = but 194·79. On these grounds the author himself excludes the series from consideration in calculating his general mean.

2. In series *c*, *d*, and *e* the evidence is pretty strong, but perhaps not conclusive, to show that potassium auri-bromide can be rendered absolutely dry by exposure to air in a vessel containing phosphorus pentoxide, either at ordinary or higher temperatures, without, at the same time, undergoing any loss of bromine. The attainment of constant weight by the salt does not positively prove the entire removal of water. If moisture were retained the atomic weight of gold found would be brought out lower than it should be.

3. KRÜSS himself observed that in all cases in which he dissolved potassium auri-bromide in water a small residue of metallic gold was left, and, determining in a single experiment the amount of this (about ·05 per cent.), he used it as a corection for all

his results. As pointed out by THORPE and LAURIE[*] this partial decomposition of the salt was probably due to the action of dust from the air. If the results obtained from the solution were used, without any correction, to establish the atomic weight of gold, the tendency would of course be to a value lower than the truth. Although the correction introduced is small, it can hardly be supposed that it should be taken as constant in amount in all the experiments.

4. In series *e* it may be questioned whether traces of potassium bromide may not have been volatilized at the highest temperature used, or the residual potassium bromide may not have, to a small extent, exchanged bromine for oxygen while heated in air (before the use of the stream of hydrogen), the latter change being one to be guarded against whenever haloid salts are strongly heated in the presence of free oxygen. The tendency in both cases would be to a lower atomic weight for gold.

5. In series *f* there was risk of slight loss of potassium bromide during filtration and evaporation of its solution, and during exposure of the salt to the heat of a free flame, when there might possibly have been again slight replacement of bromine by oxygen, thus causing the atomic weight sought to come out too high, or else, on the other hand, risk of imperfect drying, which would give too low a value for the atomic weight in question.

On the whole it seems probable that the tendency of most of the constant errors to be suspected in connection with KRÜSS'S experiments—experiments carried out with remarkable patience, skill, and apparent freedom from merely "fortuitous" errors—was in the direction of an atomic weight for gold somewhat below, rather than above, the true value.

B. *Experiments of* THORPE *and* LAURIE, 1887.[†]—In these experiments potassium auri-bromide was used, and determinations were made :—*a*. Of the weight of the residue left on heating the salt over a Bunsen flame till bromine ceased to be given off (this residue consisting of metallic gold and potassium bromide), and the weight of the gold left by such residue after all potassium bromide had been washed out of it by water; *b*. Of the weight of silver necessary to be added as nitrate to the solution of potassium bromide obtained in *a* in order to just precipitate the bromine present; *c*. Of the weight of the silver bromide so precipitated. All suitable experimental precautions seem to have been taken, and the weighings were corrected for atmospheric buoyancy. The individual results in each series agreed with each other even more closely than in KRÜSS'S research.

The results obtained were as follows, using in calculation the numbers Ag = 107·66, Br = 79·75, and K = 39·03 :—

Series *a*. Mean of 8 experiments. Atomic weight of gold = 196·876
,, *b*. ,, 9 ,, ,, ,, ,, = 196·837
,, *c*. ,, 8 ,, ,, ,, ,, = 196·842

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[*] 'Chem. Soc. Journ.,' Dec., 1887, p. 866.

[†] 'Chem. Soc. Journ.,' June, 1887, p. 565, and Dec., 1887, p. 866.

The general mean of these values, giving equal weight to the different series, is 196·852.

As regards possible sources of constant error specially belonging to these experiments, it is to be noticed—

1. There is an advantage, as observed by the authors themselves, over the greater part of the experiments of KRÜSS in the nature of the relations employed not requiring that the potassium auri-bromide should be perfectly dry, the exact quantity of the original salt not needing in fact to be known.

2. In series *a* it is conceivable that there might have been slight volatilization of potassium bromide, or interchange in it to a small extent of bromine for oxygen, during the heating of the original salt, or retention of traces of potassium bromide by the metallic gold when washed—the latter but little probable. Any of these defects, if existing, would cause the method to give a higher value for Au than the true one.

3. In series *b* the probability seems to be in favour of not quite the whole of the original potassium bromide being actually used, and minute loss of silver solution having perhaps occurred, so that rather more of this solution was counted as used than the true quantity. If so, the former defect would tend to raise, the latter to lower, the atomic weight of gold.

4. In series *c*, in view of the evidence adduced to prove complete drying of the silver bromide, it is more likely that its weight as obtained was below, rather than above, the truth. Hence we should suspect, if any constant error exist, that it would rather tend towards an unduly high value for Au.

On the whole, there seems to be less reason to fear sources of constant error of any considerable amount in connection with the experiments of THORPE and LAURIE than with those of KRÜSS, and the drift is in the opposite direction, tending rather to give too high than too low a value for the atomic weight to be determined.

It should be mentioned that KRÜSS[*] has claimed that in the potassium auri-bromide used by THORPE and LAURIE there was probably as much free gold as he considered to exist in the salt used by himself, and on this assumption has calculated that the three series of experiments by the English chemists should, if corrected on this account, lead to the numbers 196·616, 196·559, and 196·575 respectively for Au. From this conclusion the latter chemists altogether dissent,[†] and express their confidence that in none of the preparations used by them was there free gold sufficient to account for the difference between their own results and those of KRÜSS.

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[*] 'Berichte Deutsch. Chem. Gesell.,' vol. 20, p. 2365.

[†] 'Chem. Soc. Journ.,' Dec., 1887, p. 868.

General Results of Former Determinations most deserving Confidence.

These recent researches, unquestionably by far the most valuable up to the present time, give us, when taken separately and together, the following values for the atomic weight of gold :—

1.	General mean of 5 series by Krüss, as calculated by himself .	196·640
2.	,, ,, 3 series by Thorpe and Laurie, as calculated by themselves	196·852
3.	,, ,, 1 and 2, giving equal value to each	196·746

Difficulties to be overcome in Determining the Atomic Weight of Gold.

Besides the special difficulties connected with each method which may be adopted, the determination of any high atomic weight with a degree of accuracy which enables the result to be accepted to a given decimal place is clearly a much less easy matter than would be the attainment of an apparently equal degree of precision for an atomic weight represented by a small number. In obtaining the atomic weight of lithium, the first with which, many years ago, I had any personal experience, a difference of unity in the first decimal place corresponded to about $\frac{1}{70}$th of the whole value considered to be correct. In getting the atomic weight of aluminum, worked on later, a like difference represented approximately $\frac{1}{270}$th of the whole value. But, in the case now considered, of the atomic weight of gold, unity in the first decimal place means but about $\frac{1}{1970}$th of the whole value. So that, looking at the matter in this light, it may be said that a degree of precision is demanded more than seven times as great as in the case of aluminum, and twenty-eight times as great as in the case of lithium.

There is also to be noticed, as the most obvious *general* difficulty to which all methods for determining the atomic weight of gold are more or less exposed, the instability of compounds of this metal; not merely the ease with which complete decomposition occurs, with separation of free gold, but the much more insidious and less easily detected trouble arising from the comparative ease with which aurous pass into auric compounds, and the reverse.

New Experiments by the Author.

The general difficulties just alluded to, and the special points to be investigated in regard to each method of determination tried, have demanded much time and work, and I cannot feel even now that all has been done that is desirable and possible; but the experiments projected have been so far completed as to seem to justify publication, and I am not likely soon to be able materially to extend them.

General Principles kept in View.

The improvements made of late years in manipulative methods and apparatus have tended to reduce very much the magnitude of what are commonly called "fortuitous" errors in our quantitative determinations of matter, and to increase greatly the accuracy of such determinations. Probably no modern work has had more influence in this direction than the classic researches of STAS on certain atomic weights—the precautions taken by him, and his remarkable manipulative skill, causing his results to bear almost the same relation to those of his immediate predecessors as did those of BERZELIUS to the work of the chemists of his earlier day. No one nowadays would undertake the determination of an atomic weight of one of the better known elements without taking such elaborate precautions as practically ensure pretty close concordance of results, when obtained by the same method, applied in the same hands. In the present state of the question of atomic weights and improvements in their determination, advances in mere delicacy of manipulation and success in merely securing close agreement of results by the same method are not alone sufficient. It cannot be too much insisted upon that we need, besides, well-directed and laborious investigation of possible sources of *constant errors*, and the adoption of means to guard against them. Careful preliminary study is required, in a general way, of the precise nature of each reaction employed, and how it may be influenced by the conditions of the experiment. We learn more and more of late that many of the reactions—perhaps it should rather be said all of the reactions—which have been generally supposed to be of the simplest nature are in reality complex.

The following are among the general principles which seem to be most important, as tending to greater accuracy and trustworthiness in atomic weight determinations; they have been in part stated in the author's earlier paper on the atomic weight of aluminum :—

1. In purifying the materials used, both the element of which the atomic weight is to be investigated (or any special compound containing it) and all substances used to react thereupon, resort should in all cases be had to "fractional" methods, assuming materials to be pure only when earlier and later fractions give no signs of any constant difference in the results which they yield.

2. Different and independent processes should be applied to the determination of the same atomic weight, and the results used to check each other. It is desirable that as many such different processes be applied as can be devised, provided each be reasonably free from apparent sources of error, even though it be usually impossible to properly assume that all are equally advantageous in this respect, and therefore of equal value. In the comparison of results obtained it should be noticed whether a given method tends on the whole to yield results probably higher or lower than the truth, though it may be gravely doubted whether the practice is commendable of

attempting any numerical estimate of relative value, by so-called "weighting" of the results in calculation.

3. In connection with each process there should be careful study of the reactions depended upon for the final determination of an atomic weight, looking especially to the possibility of the occurrence of secondary or subsidiary reactions.

4. Each process adopted should be as simple as possible, both in the nature of the chemical reaction or reactions and in the known liability to merely manipulative errors.

5. Each process should be carried out with, in some experiments larger, and in others smaller, quantities of material. But, on the whole, the quantities used should be kept within such limits as are most likely to admit of most accurate determinations being had under the conditions of the special process.

6. In the reactions depended upon only such other elements should be concerned as may be counted among those of which the atomic weights are already known with the nearest approach to exactness.

7. It is particularly desirable that, if possible, the atomic weight to be investigated shall be, by at least one process, compared directly with that of hydrogen, now almost universally taken as the basis for the whole list of the elements. It is remarkable for how very few of the elementary substances—not more than three or four—this direct comparison has been accurately made.

8. In the greater number of the processes available for atomic weight determinations the comparison with hydrogen must perforce be made indirectly. When this is the case, it is desirable that as few other elements as possible, the assumed atomic weights of which will have to be taken into account, shall be involved in each single reaction depended upon.

9. In selecting *different* processes to be applied to the determination of the atomic weight of a given element, in order that the results may check each other, it is desirable that, not the same, but as many different other elements as possible, shall be concerned in the several reactions, provided all such elements count amongst those of which the atomic weights may be considered in the first rank as to the accuracy with which they are known.

Means and Methods of Weighing Employed.

These were in the main the same as those which I had in former years used in determining the atomic weight of aluminum.

The balance chiefly used, made by Becker, was carefully cleaned, and all its parts adjusted, especially as to the position of the centre of gravity for each load to be used. A second balance by the same maker, of larger size, capable of taking a load of a kilogramme in each pan, was employed in weighing certain of the solutions experimented on, and was in like manner carefully adjusted and tested. All weighings were made by observation of the oscillations of the index on either side of the position of rest. A difference of weight of ·0001 grm. with the smaller balance, and ·0002 grm. with the larger instrument, was easily and distinctly observable with any load which the research required.

The same kilogramme weight was made the basis of a comparison with all my other weights which had been before used in the same way. This had been compared at Washington with the "star kilogramme" of the United States Coast Survey, the value of which is known in terms of the original "kilogramme of the Archives" at Paris. All the smaller weights were carefully rechecked against this and against each other, and their real values ascertained as referred to a vacuum. The necessary determinations were made of the specific gravity of all materials and vessels which were to be weighed, and the barometer and thermometer were read at the time of each weighing, so that all weights recorded in this paper represent real values *in vacuo*. In order to reduce to a minimum errors due to varying deposition of hygroscopic moisture, vessels of like material, shape, and size with those used to contain substances to be weighed were used as tare.

History and Mode of Purification of the Gold used in this Research.

Most of the metal needed was prepared by myself, with precautions presently to be mentioned; a part was obtained, as "proof gold," from the United States Mint at Philadelphia; another part from the United States Assay Office at New York; and a single specimen of English "trial plate" gold from the Royal Mint in London.

1. *Purification of Gold by the Author.*—It may fairly be concluded from the general history of the gold of commerce that the impurities most to be suspected, and most requiring special precautions for their removal, are silver and the metals of the platinum group. My preliminary experiments led me to believe that the greatest difficulty in the way of obtaining perfectly pure gold consists in getting rid of the last traces of silver, the chloride of this metal not being quite insoluble in a solution of auric chloride. For the removal of silver, I have chiefly depended upon evaporation of the gold solution with a little hydrobromic acid, followed by large dilution with water, and long continued clearing by subsidence. As regards the platinum metals, my results agreed substantially with those of HOFFMANN and KRÜSS,[*] but I have been inclined to lay some stress on reduction of the gold from its solution *with exclusion of light*, and on *fractional* reduction, using only the middle portion thrown down. I avoided altogether the use of ferrous salts as reducing agents, in view of the difficulty of preparing them in large quantity with assurance of their purity, and the trouble of thoroughly washing the precipitated gold. For the final precipitation of the gold formic acid seemed to offer real advantages; its volatility admits of easily getting it

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[*] 'LIEBIG'S Annalen,' vol. 238, p. 66.

free from any metallic contamination, and the reduction is more easily effected than with oxalic acid.

Starting with United States gold coin, it was first heated to bright redness in a muffle, as a precaution against the presence of any traces of mercury, and to remove any grease, &c., from the surface, and then dissolved in a mixture of pure hydrochloric and nitric acids in the right proportions. The solution was evaporated with excess of hydrochloric acid nearly to dryness, the auric chloride redissolved in a considerable quantity of water, and the solution allowed to settle for four or five days. The greater part of the clear liquid, drawn off with a syphon, and filtered through very fine siliceous sand,* was again evaporated nearly to dryness, adding towards the end a few drops of pure sulphuric acid, in case of the conceivable, though unlikely, presence of such traces of lead as this might reveal; much pure water was added, the solution again cleared by subsidence for several days, and the greater part of the clear liquid again drawn off and filtered. This solution was now rendered pretty strongly acid with hydrochloric acid, and fractionally precipitated by sulphurous acid (SO_2 was evolved from sodium sulphite), at as low a temperature as possible, and *in the dark*, putting aside the first and last portions of the metal thrown down, and reserving for further treatment the (largest) middle portion. The gold thus obtained was well washed with water, boiled with nitric acid alone, again washed, boiled with hydrochloric acid alone, again washed, dried, and heated strongly with fused acid sulphate of potassium in a porcelain crucible, boiled with dilute hydrochloric acid, and then with water. The metal was redissolved in aqua regia, the solution evaporated nearly to dryness, with addition of pure hydrobromic acid towards the end, very largely diluted with water, and allowed to stand for two days, well protected from dust, before again syphoning off as much of the clear portion as could be safely removed without risk of disturbing the remainder at the bottom, using a conical precipitating jar with greatest diameter below, and filtering the liquid through siliceous sand as before. The evaporation with hydrobromic acid was repeated twice more, and the clear solution—allowed the last time to stand a month before being syphoned off and filtered—was then reduced, once with oxalic acid (neutralizing the liquid with pure sodium hydroxide from the metal), once (after re-solution) with sulphurous acid, and once with formic acid, washing the reduced metal well each time before redissolving in aqua regia. In the first and second of these reductions a little of the metal first and last thrown down was rejected, and in the final reduction with formic acid the first portion precipitated, about one-fifth of the whole, was reserved for use, labelled A, *a*, the middle portion, about three-fifths, was labelled A, *b*, and the last portion, the remaining one-fifth, was also preserved for use, marked A, *c*, so that it might be seen whether any difference in the character of the metal could be detected in the atomic weight determinations. All of these fractions received a very thorough final washing with water.

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* The sand was carefully purified beforehand by boiling with nitric and hydrochloric acid, thorough washing with water, and heating to redness in the air.

Such part of the purified metal as was to be used in the preparation of gold compounds was not fused, but was heated in a glazed porcelain tube to moderate redness in a Sprengel vacuum. A small part of the metal used in the free state, and desired in compact form, was fused in a perfectly clean Beaufaye crucible with a little acid sulphate of potassium and borax, the button flattened, boiled with strong nitric and then strong hydrochloric acid, thoroughly washed with water, and, finally, heated in the Sprengel vacuum. Throughout the long process of purification, and especially towards its close, the most scrupulous care was taken to exclude dust, and to prevent grains of sand from the bottoms of beakers or any other impurities getting into the precipitated gold, upon which the acids used would not act, so as to obviate the risk of merely mechanical contamination, which, if overlooked, might lead to that being weighed as part of the gold which was, in fact, foreign to it.

2. *Purification of "Proof Gold" obtained from the United States Mint at Philadelphia.*—I owed to the kindness of Mr. J. B. **Eckfeldt**, Chief Assayer to the Philadelphia Mint, a liberal supply of the "proof gold" used in checking the gold assays there made, and he furnished me the following statement of the manner in which this purest metal is prepared, under his directions:—"The best cornets from the gold assays selected and dissolved in aqua regia. Solution evaporated, with additions of HCl, to nearly crystallization, diluted largely with water, and allowed to stand for three or four weeks. About seven-eighths of the solution drawn off from the silver chloride, and passed through several thicknesses of various filters. Solution somewhat concentrated, and alcohol and potassium chloride added, allowed to stand for some time (precip. traces of platinum*), and carefully filtered. Gold precipitated by addition of pure ferrous sulphate. Reduced gold washed repeatedly in boiling HCl, until washings show no iron, then well washed in pure water. Gold dissolved, and solution evaporated to crystallization, with repeated additions of hydrobromic acid,† diluted, and again allowed to stand for some time; filtered. Through the solution was passed pure SO_2 until all the gold was reduced; washed. Gold again dissolved, evaporated with HCl, diluted, and oxalic acid added, and heated, until all gold is down. Melted in white clay crucible with potassium chlorate and nitrate, afterwards with pure sodium carbonate and borax." Mr. **Eckfeldt** also informed me verbally that the proof gold thus purified is cast into a small bar in a perfectly clean and bright cast-iron mould; the bar is boiled in nitric acid, washed and dried, rolled between fine

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* The platinum of South American native gold, and of scrap gold from dentists, is at the Philadelphia Mint separated solely by alloying with enough silver, and dissolving out the latter metal with nitric acid. The platinum dissolves with the silver.

† I had, many months before, independently adopted and used hydrobromic acid to remove traces of silver more effectually than by hydrochloric acid, when I learned from Mr. **Eckfeldt** that he had thus habitually employed it.

steel rolls quite free from grease, and the strip finally cleaned for use with hot hydrochloric and then nitric acid.

In a letter of later date he wrote, "In preparing the 'proof' I seldom make over 10 oz. in one lot; from 8 to 10 oz. is the usual amount. There is comparatively little trouble in making 999·9 fine, but beyond that it is rather troublesome; and it seems that, with all the care, the final result is sometimes a little in doubt."

The fine gold received from the Philadelphia Mint is designated as B in this paper, in connection with the experiments in which it was used.

3. *Purification of "Proof Gold" obtained from the United States Assay Office at New York.*—Dr. H. G. TORREY, Chief Assayer in this office, was obliging enough to let me have several samples of his finest proof gold, used in checking the regular assays in his department. He informed me that this proof gold was independently prepared at New York, but was occasionally compared with that of the Philadelphia Mint. He furnished the following brief statement as to its preparation—"The process used in preparing the gold is to dissolve 'cornets' (or gold from assays) in nitro-hydrochloric acid, and after filtration precipitating by oxalic acid, and after thorough washing melting under borax. The operation is conducted with the utmost care throughout."

The gold from this source is designated as C in this paper.

4. *Gold from the "Trial Plate" of Fine Gold of the English Mint.*—Professor ROBERTS-AUSTEN, Chemist to the Royal Mint, was so kind as to let me have a specimen of a few grammes of gold cut from the trial plate of the pure metal prepared by him in 1873. In its preparation use was made of potassium chloride and alcohol to separate any platinum present in the original material, a long period of subsidence was allowed for the deposit of any silver chloride from the solution, and the whole process was applied on a large scale, resulting in the purification of some 70 ounces of fine gold, of which Professor ROBERTS-AUSTEN himself has said: "I have not been able to prepare, or to obtain from any source, gold of greater purity, even in small quantities." It seems, however, that the apparent standard of this gold was slightly reduced in rolling, the finished plate being counted as 999·95 fine in comparison with the same gold before rolling. A memorandum given me by Professor ROBERTS-AUSTEN states that this trial plate gold is 999·98 fine as compared with the purest gold obtained by STAS for the Belgian Mint.

This specimen of English trial plate gold is designated as D in the present paper.

All the samples of gold received from others—B, C, and D—were, before using them, carefully boiled in nitric acid to remove any possible traces of silver or other metal derived from the shears used in cutting the plates. They were also previously well washed with ether, to remove any grease, and afterwards with pure water, and were finally heated to redness in the Sprengel vacuum.

It may be remarked, in advance, that I have not been able to trace any probable connection between the history of the several samples of gold used and the values obtained for the atomic weight of the metal. Within the limits of accuracy attained, the results appear to have been sensibly the same by each method for all the gold used. Nor is there apparent in the results of KRÜSS, or those of THORPE and LAURIE, any evidence of a difference fairly traceable to the nature of the metal employed by them.

A considerable part of the gold prepared by myself was, after having once served for a determination of the atomic weight, redissolved and reprecipitated, and was afterwards more than once used in subsequent determinations, and yet no sign was obtained of any resulting influence upon the later values of the atomic weight as obtained, evidence being thus furnished of the purity, not only of the gold itself, but of the reagents used to act upon it, so far as any contamination of the metal was concerned. It may, therefore, be concluded with reason that the gold used in these experiments was of uniform character, and uniformly free from any known impurities, to such an extent, at any rate, as to sensibly change the results obtained.

It is to be noted that the only known elements having higher atomic weights than that of gold are mercury, thallium, lead, bismuth, thorium, and uranium. The presence of any of these in the gold experimented on, even in traces too minute to weigh, is in a very high degree unlikely. The presence of any other element or elements than these would, for analogous compounds, tend to lower the value obtained for the atomic weight of gold; so that, in considering the chances of error due to the nature of the metal used as gold, we should be inclined to say that the risk was rather in the direction of too low than too high a result being reached. But, if the possibility be observed of compounds not analogous being erroneously compared, the contrary error will be seen to be possible. Thus, in case the composition of an auric haloid salt obtained from a given amount of metallic gold should be examined, if any unsuspected silver were present there would be required for the same amount of the halogen three atoms of silver instead of one atom of gold, and, therefore, the apparent weight of gold as compared with that of the halogen would be increased instead of diminished, and a higher value obtained for the atomic weight sought.

General Precautions Observed in the Experiments for Determination of the Atomic Weight.

All the reagents used were prepared or purified by myself, and most carefully tested for any traces of such impurities as might reasonably be suspected, and as could affect their application to the purpose in view. Particular care was bestowed upon the examination of the distilled water, acids, and other materials used in large quantity. To remove organic matter from the water required, it was distilled from a small amount of potassium permanganate and sulphuric acid.

Scrupulous care to exclude atmospheric dust was observed. In the evaporation of some of the gold solutions the process was carried out in a glass bottle of considerably larger capacity than the volume of liquid to be treated, furnished with a well ground-glass stopper of special construction, as shown in figs. 1 and 2, the latter representing the stopper in place. Air, purified by passing through a red-hot tube, then through a solution of potassium permanganate and sulphuric acid, and dried by passing through concentrated sulphuric acid and over solid potash, was introduced by the tube *a*, which went down to near the level of the liquid to be evaporated, while this air, charged with vapour of water from the liquid, was withdrawn through the tube *b* by means of a water-jet pump; the bottle was moderately heated by immersion to the greater part of its height in a water bath.

Fig. 1.

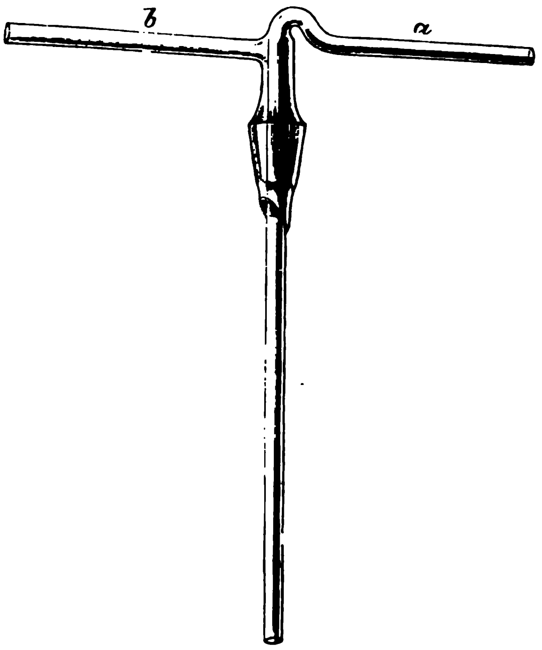
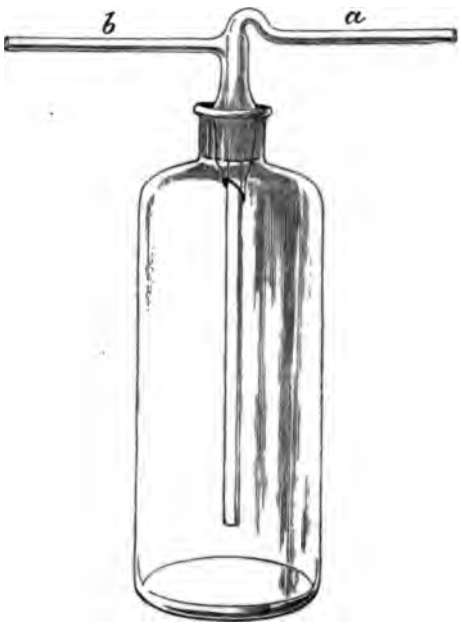


Fig. 2.



In filtering the gold solutions no paper or other organic material was used, but fine white siliceous sand, previously boiled in nitric and hydrochloric acids, washed with water, and well ignited to burn off any organic matter, was substituted, supporting it on coarser sand and larger fragments of quartz, similarly purified, and the whole arranged so as to prevent the possibility of any sand grains being mechanically carried into the filtered liquid. Vessels of hard glass and Berlin porcelain were employed. Care was taken to work in a clean laboratory atmosphere, free from gases or vapours which might affect the materials dealt with.

First Series of Experiments.

A neutral solution of auric chloride was prepared by cautiously heating auric chloride, made, as suggested by JULIUS THOMSEN, by the direct action of pure chlorine upon finely divided metallic gold, until such an amount of chlorine had been given off that on treating the residual material with moderately

Experiment.	Character of gold used.	Gold.	Silver required to precipitate Cl.
		grm.	grm.
I.	A, *a*	7·6075	12·4875
II.	A, *b*	8·4212	13·8280
III.	B	6·9407	11·3973
IV.	A. *c*	3·3682	5·5286
V.	C	2·8244	4·6371

In regard to conceivable sources of error connected with this method, it is to be observed that, in preparing the original auric chloride solution, if there should be any reaction between this gold salt and the water, leading to the formation of traces of hydrogen auri-chloride and precipitation of a little auric oxide or hydroxide, which might escape observation in admixture with the metallic gold left undissolved, the tendency would be to lower the atomic weight found for gold. If, by reaction between this residual metallic gold and the auric chloride solution, any traces of aurous chloride were produced and taken up by the solution of the higher chloride, the effect would be to raise the apparent value of the atomic weight.*

If, in the reaction of the silver solution upon that of auric chloride, *partial* withdrawal of chlorine should lead to the formation of any traces of aurous chloride, precipitated along with the chloride of silver, and not afterwards decomposed during the digestion of the precipitate with the remaining solution, the resulting error would also be in the direction of too high an atomic weight. The probability of the last supposition is diminished by an excess of silver for the whole amount of chlorine present having been added at once. It is not very likely that any one of these defects actually belongs to the method and affects its results to a sensible extent. Of the three I should be more inclined to suspect the possibility of the second than either of the two others.

Second Series of Experiments.

A neutral solution of auric bromide was prepared by a like process to that used in making the auric chloride of the first series: acting upon pure metallic gold with pure bromine (prepared with the precautions recommended by STAS), evaporating the solution to dryness out of reach of dust, cautious heating of the residue, re-solution of auric bromide, and filtration from undissolved metallic gold.

Two nearly equal portions of the solution were accurately weighed off, and treated as described above: in one reducing the gold to the metallic state and determining its weight; treating the other with a small excess of silver in solution as nitrate, filtering off the precipitate, concentrating the filtrate with the precautions already described, and determining in it the excess of silver by means of hydrobromic acid.

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* These two remarks apply, of course, also to KRÜSS's first series of experiments.

Reducing the amounts of silver actually used to the corresponding quantities for portions of auric bromide solution exactly equal to those from which in each case the gold was obtained, the results in six experiments stood as follows :—

Experiment.	Character of gold used.	Gold	Silver required to precipitate Br.
		grm.	grm.
I.	A. *b*	8·2345	13·5149
II.	A. *c*	7·6901	12·6251
III.	B	10·5233	17·2666
IV.	A. *a*	2·7498	4·5141
V.	C	3·5620	5·8471
VI.	A. *b*	3·9061	6·4129

In these experiments the sources of constant errors which suggest themselves as possible are essentially similar to those for the first series; but, if any such really exist, there is, of course, the likelihood of some difference being introduced by the substitution of bromine for chlorine. Hence the desirability of multiplying experiments in this modified form.

Third Series of Experiments.

For these experiments potassium auri-bromide was prepared with great care from an excess of metallic gold treated with bromine and potassium bromide, purified in accordance with STAS's suggestions, and the double salt five times recrystallized The last crystallization was conducted fractionally, in closed vessels, with special care to exclude dust, by gradual but pretty rapid cooling with agitation, and the earlier and later portions separated out were kept apart in after use.

For each atomic weight determination an unweighed quantity of this potassium auri-bromide was dissolved in water, the solution rendered uniform by agitation, and divided into two nearly equal parts, which were severally weighed with accuracy, and in one the gold reduced to metal as in the experiments of the first and second series, and in the other the total bromine precipitated by silver solution as before, the comparison being made once more between the weight of the gold and that of the silver equivalent to the bromine (in this case representing 4 atoms) existing in the double bromide.

Again stating the quantities of silver corresponding to portions of the auri-bromide solution exactly equal to those used in determining the gold, the following were the results obtained :—

Experiment.	Character of gold used.	Fraction of crystallized auri-bromide used.	Gold.	Silver required to precipitate Br.
			grm.	grm.
I.	A, *b*	First	5·7048	12·4851
II.	A, *b*	Second	7·9612	17·4193
III.	B	First	2·4455	5·3513
IV.	B	Second	4·1632	9·1153

Of the tendencies to constant error which may be imagined in connection with the experiments of the first two series, and which have been noticed above, the first may probably be considered as not applying to the method pursued in this third series, while the second and third might still be applicable. But the superior stability of the double salt constitutes an advantage in its favour, and, as it formed the chief material for the experiments of **Krüss** and of **Thorpe** and **Laurie**, a comparison with their results is desirable, the mode of treatment pursued by me in ascertaining the composition of the salt not having been quite the same as that used by these chemists.

Fourth Series of Experiments.

A weighed quantity of trimethyl-ammonium auri-chloride [$N(CH_3)_3HAuCl_4$] was decomposed by heating in the air, and the weight of the residual metallic gold determined. This trimethylamine salt was selected because the base is of simple and well established constitution, and may with reasonable probability be counted upon as obtainable in a state closely approaching purity, and because the gold salt is easily crystallized, possesses a considerable degree of stability, and contains approximately half its weight of gold, so as to offer the most favourable chance of determining with accuracy the ratio between the metal left behind and the sum of the remaining constituents driven off on ignition. Although its use in fixing the atomic weight of gold involves the atomic weights of three other elements—carbon, nitrogen, and chlorine—all three of these constants deserve to be ranked amongst those already known with the nearest approach to precision at present attainable.

In order to obtain pure trimethyl-ammonium chloride, the impure commercial salt, derived from the *vinasse* of beet-root sugar making, was used, first setting free and distilling off a considerable quantity of trimethylamine and condensing at about the right temperature, and subsequently purifying the product by **Hofmann's** method of treatment with ethyl oxalate and renewed distillation. The purified trimethylamine was several times fractionally distilled, and the portion of correct and most constant boiling-point finally neutralised with pure hydrochloric acid. The concentrated solution of trimethyl-ammonium chloride was now precipitated by a strong solution of auric chloride, the mother liquor decanted off, and the gold salt redissolved in hot water, and recrystallized several times. The bright yellow crystalline powder was dried, first over sulphuric acid and afterwards over phosphorus pentoxide, until it ceased to lose weight; towards the end of the drying the temperature of the vessel was raised to about 50° C. Preliminary experiments seemed to indicate the probable existence of this salt crystallized with a single molecule of water, but most of that prepared contained no constituent water, and it appeared easy to attain complete drying without any decomposition of the salt itself. Throughout its treatment the salt, which was not in any high degree hygroscopic, was well guarded from dust and from any possible decomposing effect of light.

The portion of the salt to be used in each experiment was contained in a small glass-stoppered weighing flask, which was removed just before it was needed from the phosphorus pentoxide desiccator, the stopper having been inserted; the flask was weighed, the greater part of its contents transferred quickly to a weighed porcelain crucible, the stopper at once replaced, and, the flask being again weighed, the quantity of gold salt taken from it was found by difference.

In order to avoid mechanical loss by spattering on igniting the crucible and its contents, the auri-chloride lying together at the bottom of the crucible was covered by a layer, nearly a centimetre deep, of clean, carefully purified, and just previously well-ignited siliceous sand, the weight of this sand being known by taking it from a weighing flask in which it had been cooled over phosphorus pentoxide, and noting the loss of weight of this flask. In applying heat to the crucible and its contents it was found necessary to heat gently for a long time, raising the temperature slowly, in order to prevent extensive charring at the bottom. Then, before the temperature had become too high, but after a considerable part of the volatile matter had been driven off, the sand was carefully stirred in with the remaining material so as to produce pretty uniform mixture, in order that the gold might not undergo partial welding together at a higher temperature, which might have led to wrapping up particles of carbon and their protection from combustion. In this operation a very small porcelain stirrer was used, as a platinum wire would have welded on and taken up some of the metallic gold; the weight of this stirrer was determined in advance, and checked after use. Finally, the contents of the crucible were submitted to very careful and prolonged heating to moderate redness, with free access of air and occasional cautious stirring, so as to burn away every trace of carbon. After cooling in a desiccator, the crucible and its remaining contents were weighed, giving the weight of the residual gold by subtraction of the weights of the crucible itself and the siliceous sand. As an additional safeguard against any particles of carbon left unburned escaping detection, the gold was afterwards dissolved out with aqua regia, and the white sand carefully looked over with a lens.

The results of five experiments thus conducted were as follows:—

Experiment.	Character of gold used.	Character of gold salt used.	Salt ignited.	Residual gold.	Loss by ignition.
			grm.	grm.	grm.
I.	A, *b*	Earlier crop of crystals .	14·9072	− 7·3754	= 7·5318
II.	A, *b*	Middle ,, ,, ,, .	15·5263	− 7·6831	= 7·8432
III.	A, *b*	Last ,, ,, ,, .	10·4523	− 5·1712	= 5·2811
IV.	C	Middle ,, ,, ,, .	6·5912	− 3·2603	= 3·3309
V.	C	Last ,, ,, ,, .	5·5744	− 2·7579	= 2·8165

In these experiments the most probable source of error may be fairly taken as arising from the presence of traces of methyl-ammonium or dimethyl-ammonium aurichloride with the trimethyl-ammonium salt. I know of no direct evidence that any such impurity was present, and the absence of any such evidence in the results from the earlier as compared with the later crops of crystals rather tells against the supposition of its presence, but one cannot feel certain of its entire absence. If present, its effect would be to raise the atomic weight obtained for gold. It is also conceivable that there may have occurred volatilization of gold to a minute extent as auric chloride, in accordance with the observation of KRÜSS that this salt may be sublimed in small quantity at moderate temperatures in a stream of chlorine; but, there being no such stream of chlorine in these experiments, and on the contrary the decomposing action of the hydrogen of the trimethylamine salt, this does not seem likely; the effect would, of course, be to raise the atomic weight obtained for gold. Another possible cause of error might consist in imperfect drying of the gold salt used, but the constancy of weight attained on drying renders it unlikely that any other than extremely minute error should come of this, though not altogether excluding the possibility of its occurrence; its tendency would, of course, be to lower the atomic weight obtained. Any trouble from hygroscopic moisture on the surface of the porcelain crucible and sand was, I think, satisfactorily guarded against by the use of a corresponding tare crucible, and by more than one weighing after a near approach to the true figures had been obtained, the crucibles having meanwhile been restored to the desiccator and kept therein for some time. The precautions taken seemed to afford sufficient protection against any merely mechanical loss during the ignition.

Fifth Series of Experiments.

In these experiments an attempt was made to determine the ratio between the weights of metallic gold and metallic silver deposited by the passage of one and the same electric current successively through solutions of the two metals. The simplicity and accuracy with which the direct weighings may be made seemed to present decided advantage, but various difficulties were encountered, and, after the expenditure of a very large amount of time and labour upon the method, it cannot be said, on the whole, to have satisfied me with its results.

While taking due note of the recent literature on the subject of the quantitative electro-deposition of metals from their solutions, especially the reports of work by A. CLASSEN,* Lord RAYLEIGH and Mrs. SIDGWICK,† Dr. GORE,‡ THOS. GRAY,§ and W. N. SHAW,‖ the author of the present paper made for himself a somewhat extended preliminary examination of the effect of varying conditions on such depositions, so far at least as seemed to be required for his immediate purpose.

The general arrangement of apparatus adopted consisted of a horizontal strip, 4 mm. thick, of vulcanite, or hard vulcanized india-rubber, about 26 cm. long by 3 cm. wide, near each end of which and in the middle of the width were two small holes, through which passed short bits of brass rod, each having attached to it above a binding screw, and below a forceps-like clip, which could be opened by pressure on two little outside studs, but closed firmly, on release of this pressure, by the elasticity of the metal. In these clips were supported the plates of metal to be immersed in the electrolysed solutions, and to serve as anode and cathode terminals respectively, there being two pairs of such plates, one pair near each end of the vulcanite strip, with four corresponding binding screws. The electric current passed from the first binding screw through one of two metallic solutions—as, for instance, that containing gold—between the first pair of plates, consisting of the same metal as that in this solution, then from the second binding screw to the third (at the other end of the vulcanite strip) by a stout copper wire above, and then through the second of the two solutions—as, for instance, that of silver—between the second pair of plates, consisting again of the same metal as that in the solution in which they were immersed, thus reaching the fourth and last binding screw, the first and last binding screws being of course, connected by wires with the terminals of the galvanic cells used to generate the current. Fig. 3 shows the disposition in question. The source of the electric current was for the most part galvanic cells of the Meidinger pattern, but in some of the experiments small Daniell cells, and also a Clamond thermo-electric battery, were used. The lower parts of the clips were heavily electroplated with the same metal as that in the solution to which they respectively belonged, in order to avoid any risk of contamination of the solution, in case there should be splashing or accidental immersion even for a moment of any part of the clip.

It was decided to place the plates vertically in the liquids, but to make the vertical height small in proportion to width, so as to preserve as far as possible a uniform condition of the solution in depth. The form adopted for the plates was that of fig. 4, the shaded part of the surface being coated with hard paraffine, with a view to preventing the strip by which the anode plate was suspended from its clip being cut across by solvent action at the surface of the liquid. This coating of paraffine was put on after the plates were first weighed, and carefully removed before the second weighing. The four plates for each experiment were of equal size as to length and breadth; in most of the experiments the immersed surface (of one side) measured about 25 square centimetres, though in some cases plates of double this size were used. The thickness was the same for plates of the same metal, but those of the different metals to be compared were made to differ in thickness to such an extent as to allow for the different rate of solution to be expected of the anode plate. I was indebted to the kindness of Mr. ECKFELDT, of the Philadelphia Mint, for having plates of "proof" gold and silver specially rolled for me, with all necessary precautions as to perfect cleanliness of the rolls, &c., so as to obtain the determinate thicknesses desired.*

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* A. CLASSEN, 'Quantitative Chemische Analyse durch Electrolyse,' 2te Aufl. Berlin, [illegible]

† 'Phil. Trans.,' [illegible]

‡ 'Nature,' [illegible]

§ 'Phil. Mag.,' [illegible]

‖ 'Phil. Mag.,' [illegible]

Fig. 3.

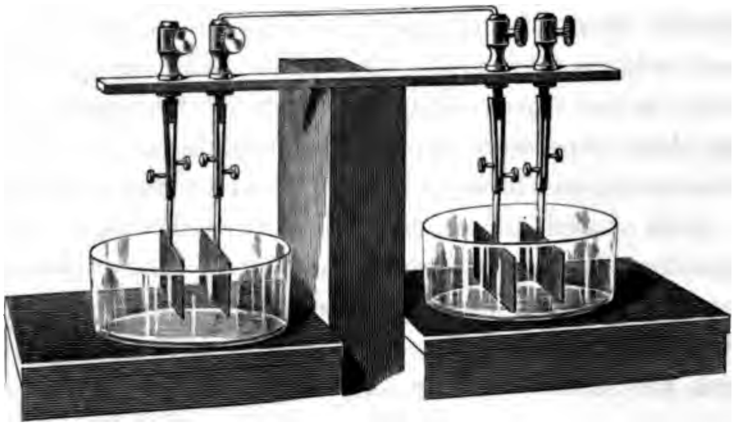
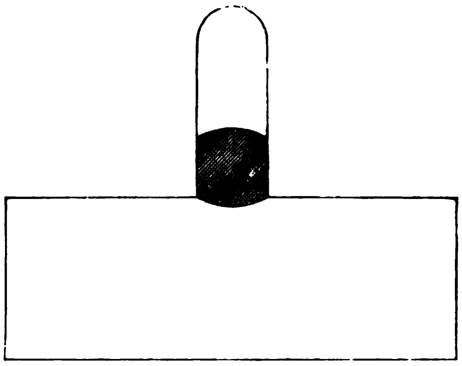


Fig. 4.



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* Mr. ECKFELDT informed me that his method of preparing the proof silver used for these plates was as follows:—" Nitrate of silver from the gold assay parting is, after careful filtering, precipitated with

By heating in a Sprengel vacuum I found traces of oxygen in the rolled silver plates, and extremely minute traces of gas, apparently also oxygen, were likewise obtained from the gold plates, before either had been used.

The middle of the vulcanite strip was supported at a suitable height, so as to allow of equal immersion of the two pairs of plates in their respective solutions, which were contained in small vessels of good hard glass, free from lead. Care was taken to keep the vulcanite strip dry, so that there should be no practical defect of insulation between the two plates of each pair; the necessity for this precaution having been shown in some of the very early preliminary experiments with copper plates, using a wooden supporting strip; some puzzling results being traced back to a little accidental moistening with sulphate of copper solution of the part of the strip between one pair of plates, while those of the other pair were well insulated as to the strip from which they hung.

In all the experiments the two pairs of plates, previously ignited in the Sprengel vacuum, cooled, and weighed, were placed in position in the clips, the distance between the parallel surfaces of the plates of each pair being the same, and in most of the experiments measuring about 2·5 cm., and connection was made with the terminals of the galvanic cell or cells used before immersion of the plates in the metallic solutions. All four plates were immersed at the same moment, and at the end of the experiment were in like manner lifted out of the solutions at the same moment, before the current had been broken. They were immediately introduced into one after another of several portions of distilled water, before removal from the clips, thorough washing, heating in the Sprengel vacuum, and final weighing.

A preliminary course of experiments was carried out with plates of pure electrotype copper (both pairs) in solutions of cupric sulphate, in order to test the effects, if any, of the following differences in the conditions of the two electrolysis cells compared.

1. *Effect of Difference in the Degree of Concentration of the Two Solutions.*—The solution in one of the two vessels in which the plates were immersed being made to contain but one-tenth the proportion of cupric sulphate existing in the other, acidification and all other conditions being the same for both, only a very minute difference was found between the quantities of copper deposited in the same time on the two cathode plates, and the difference was not invariably in the same direction. The tendency however, seemed on the whole to be toward a slightly larger amount thrown down

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hydrochloric acid, and the chloride of silver, after a thorough washing with pure water, is dried and reduced in the melting pot with pure carbonates of soda and potash and carbon in the shape of wheat flour, the melting being done in a clay crucible. The resulting silver bar is then dissolved in dilute nitric acid, and after standing some time filtered, precipitated, and reduced as before; then remelted with the addition of pure nitrate of potash and borax. This generally gives a bar somewhat brittle (crystalline in fracture). It is then remelted, and stirred with a pine stick, and chloride of ammonium added; when the chloride has disappeared the metal is poured. I find this method more satisfactory than any other I have tried."

from the stronger than from the weaker solution. In every case there was decidedly more copper dissolved from the anode than was deposited on the cathode plate.

2. *Effect of Difference in Acidity of Two Solutions, otherwise of the Same Strength.*—With the same proportion of cupric sulphate in both solutions, one was made to contain but one-tenth as much free sulphuric acid as the other; all other conditions remained the same for both. As before, the difference of result was insignificant, and somewhat variable in direction, with an apparent tendency towards a very slightly greater deposit on the cathode plate in the less acid as compared with the more acid solution. As before, there was in every case a distinctly greater loss of copper from the anode than gain on the cathode plate, especially in the more acid solution.

3. *Effect of Difference in Temperature of the Two Solutions.*—The proportion of cupric sulphate and of free acid being the same for both solutions, and all other conditions the same, one of the two, however, being maintained at 72°, 47°, or 37° C., while the other was at 2° C., thus establishing a difference in temperature of 70°, 45°, or 35° respectively, there was distinctly in every instance rather more copper thrown down on the cathode plate in the colder than in the warmer solution. The loss of weight of the anode plate was always greater than the gain at the cathode, and the difference in this respect was greater in the warmer than in the colder solution.

4. *Effect of Difference in the Size of the Plates.*—All other conditions being the same in both the electrolysis cells, the plates in one were made to present but one-fourth the surface of those in the other, so that the "current density" was proportionally increased in the former. Under these circumstances there was a constant, though but small, difference in the amount of copper deposited on the two cathodes, the quantity being greater on the cathode plate with smaller surface. The tendency seemed to be towards a greater excess of metal removed from the anode over that deposited on the cathode plate in the case of the larger plates, as compared with the smaller.

5. *Effect of Difference in the Distance between the Plates.*—The plates of both pairs being equal in size, and all other conditions being uniform, the plates in one of the two electrolysis cells were placed at a distance apart only one-fifth that intervening between those of the other pair. It was not clear that any constant difference of result could be detected, but the tendency seemed to be rather toward a very slightly greater deposit on the cathode plate in the case in which the plates were farther apart as compared with that in which they were nearer together. There was no recognisable difference in the proportion of metal dissolved off from the anode plate.

Similar experiments were made with two pairs of plates of pure silver, thus checking the results obtained with copper, and contrasting the behaviour of one at least of the less chemically alterable metals with that of the more easily alterable copper. It was intended to make a set of similar experiments also with gold plates only, but the available supply of pure gold in the form of rolled plates was not sufficient for the numerous experiments required. The silver solution used was one of potassium argento-cyanide, and the substitute for the free sulphuric acid of the copper experiments was an excess of potassium cyanide. The results obtained were essentially similar to those of the copper experiments, the effect of difference in temperature between the two solutions being, however, less decided, and the slight effect of difference in the size of the plates ("density of current") less constant and distinct.

In all the preceding experiments it was found that the most constant results under otherwise similar conditions were obtained by using feeble currents rather than those of greater strength, especially in the case of the silver solutions. There seemed, however, to be a limit to this. On the whole, the most satisfactory results were obtained (both in these preliminary experiments, and in those aiming at the atomic weight determination) with currents not exceeding $\frac{1}{100}$th of an ampère per square centimetre of surface of (one side of) the opposed plates, and in some cases a current but one-fifth of this maximum was used.

Having in view the indications afforded by the preliminary experiments, it was determined to use tolerably strong solutions of the metals to be deposited, with not more than a moderate excess of free acid, or, in the case of the double cyanide solutions, excess of potassium cyanide, to maintain the same temperature in both the electrolysis cells and to have this temperature as low as possible (about 2° C.), and to have the plates of the two metals to be compared equal in size, and at equal distances apart, using a weak electric current, and keeping watch over its strength by means of an ordinary hydrogen voltameter in the circuit.

In the actual experiments on the deposition of gold as compared with silver it was originally proposed to use a solution of potassium auri-cyanide against one of potassium argento-cyanide, with the expectation that 3 atoms of silver would be thrown down for 1 atom of gold. But the first attempts made showed clearly that this reaction could not be obtained. The comparison as to gain in weight of the gold and silver cathode plates gave results leading to an atomic weight for gold impossibly high if the silver deposited were taken to represent 3 atoms, and much too low if it were taken to represent but 1 atom. Hence it appeared that the potassium auri-cyanide had been partially, but not completely, reduced to auro-cyanide by the action of the current, and an intermediate result obtained as to the equivalent quantity of silver between that due to the one or the other gold salt if exclusively present.

A change was therefore made to the auro-cyanide in the preparation of the solution to be electrolysed. A pure form of potassium cyanide was prepared with the aid of alcohol, and carefully tested as to the absence of any metal capable of deposition from the watery solution on electrolysis. Auric chloride was precipitated by ammonia, the fulminating gold, after washing, dissolved in a strong solution of this potassium cyanide with the aid of heat, and the auro-cyanide crystallized out by cooling. The crystals were washed, redissolved in water, aurous cyanide separated from the solution by evaporation with hydrochloric acid, and the crystalline powder after cautious washing again dissolved in potassium cyanide solution, using for the purpose the barely necessary amount of the solvent liquid, but afterwards adding a further quantity, so as to have potassium cyanide in excess. Potassium argento-cyanide was prepared by precipitation of a solution of pure silver in nitric acid with the purified potassium cyanide, washing the precipitate, and re-solution with the aid of the necessary quantity of potassium cyanide, of which finally a moderate excess was added. The solutions of the gold and silver salts were made of equivalent strength, for the most part at the rate of 7 grm. of metallic gold for each 100 c.cm. of solution, and an approximately corresponding amount of silver, taken atom for atom. Both solutions received the same excess of potassium cyanide, generally equal to one-half of that already present in the double salt, but in some of the experiments it was found necessary to add yet more during the electrolysis in order to preserve the purely metallic character of the surface of the plates. As an additional security against admixture of auri-cyanide with the auro-cyanide of the gold solution, it was subjected for some time to electrolysis with unweighed gold plates immersed, these being reversed two or three times in position, just before the introduction of the weighed plates for a quantitative experiment. A number of attempts were made to substitute for the solution of potassium auro-cyanide one of sodium auro-thio-sulphate, of potassium or sodium auri-chloride, and of simple auric chloride, in the last two cases employing at the same time a solution of silver nitrate, but these efforts led to no success.

In many of the experiments made with the double cyanide solutions the cathode plates, both of gold and silver, after removal from the electrolysis cells and thorough washing, were found to curl up on being heated, the deposit, which in these cases was rather hard and brittle, swelling up in a remarkable way, with formation and bursting of little blebs or minute bubbles of the metallic surface, and parting off to some extent of the deposit from the original plate underneath. When the heating was carried out in the Sprengel vacuum small but quite appreciable amounts of hydrogen were found to be given off, having been occluded in the metal deposited. It seemed necessary to throw aside the results in all cases in which this condition of the deposit was well marked. Other experiments were vitiated by the gold deposit not being thoroughly compact, and still others by the surface not being clearly metallic, aurous cyanide making its appearance from the solution. It was hoped that in the experiments, free from apparent defect, any irregular behaviour of the gold solution at first might be got rid of by continued electrolysis, with reversal of the anode and cathode plates when necessary, until the ratio of gold to silver deposited should become constant; but confidence in this was greatly shaken when an instance occurred, followed afterwards by others, of sudden change in this ratio, attended with much less loss from the anode gold plate than the gain of the opposed cathode plate, pointing to deposition of gold from the auro-cyanide with simultaneous formation of auri-cyanide in the solution.*

* HITTORF ('POGGENDORFF, Annalen,' [4], vol. 16, p. 523), in the simultaneous electrolysis of gold and

Altogether but five experiments made in this way yielded results which seemed worthy of being used to determine the atomic weight of gold, and it is of course unsatisfactory to know that these were selected out of a much larger number, mainly because, while not known to be in any way vitiated by apparent defects, they lead to values for the atomic weight in question close to those obtained by other methods and other experimenters. It is possible that this near approach to agreement may merely result from a balance of errors in opposite directions, which taken separately would have caused the experiments to be rejected. Some other experiments, under apparently similar conditions, gave figures for the atomic weight differing from those reported by one or two whole units.

These only admissible results are the following :—

Experiment.	Character of gold in solution.	Character of gold in plates.	Gold deposited.	Silver deposited.
			grm.	grm.
I.	A, *b*.	B	5·2721	2·8849
II.	,,	,,	6·3088	3·4487
III.	,,	,,	4·2770	2·3393
IV.	,,	,,	3·5123	1·9223
V.	,,	,,	3·6804	2·0132

Aside from other difficulties liable to be encountered in carrying out this electrolytic method, the two most important sources of possible inherent error which suggest themselves are the occlusion of hydrogen by the metallic deposit and the instability of the atomicity of gold in the solution electrolysed.

The separation of hydrogen on the cathode plate, whether in bubbles (which may be avoided by proper regulation of the current) or occluded by the metal (which does not seem to be completely avoidable with any current, although the amount of occluded gas was extremely small in a number of my experiments), must be ascribed to decomposition, simultaneous with that of the cyanide of gold, either of water or, more probably, of cyanide of potassium, with secondary action of the potassium on the water. In either case, it is by no means clear that the proportion of current giving rise to this liberation of hydrogen can be counted upon as the same in the gold solution and in that of silver; and hence, even though it be fairly assumed that FARADAY's principle of equivalent electrolysis by the same current is strictly correct for the *ensemble* of chemical actions in the two cells, the portion of current actually concerned in depositing gold or silver only in each of the respective cells may conceivably not be

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silver solutions, the gold as potassium auri-chloride, obtained results which showed that this metal was deposited at the rate of 1 atom for 3 of silver. Calculating on this basis from his two experiments, the atomic weight of gold comes out = 196·311 and 194·197; for silver = 107·66.

In one experiment of my own, using sodium auri-chloride, the result showed that the gold was thrown down for the most part as a triad, but partly as a monad, element.

quite the same, so that the weights of the two metals thrown down may not be strictly equivalent.* It was, therefore, deemed important to work with feeble currents, and, while heating all the plates in a Sprengel vacuum before weighing, to reject the results of all those experiments in which the quantity of gas thus discharged amounted to more than the merest trace. But, if the source of error in question still exist at all, it might affect the atomic weight of gold in comparison with that of silver, either by making the former appear higher or lower than the truth.

The source of error most to be feared, however, in connection with the application of this electrolytic method to the determination of the atomic weight of gold, is the uncertainty of having all the gold throughout the process in the form of potassium auro-cyanide in the solution, in view of the transition observed to auri-cyanide during electrolysis, although change in the opposite direction occurs with even greater ease. Each of the two salts appears to admit of electrolytic decomposition, and the presence of any traces of the auri-cyanide, in which the gold has triad character, while the calculation is based on the supposed presence of monad gold only would, of course, tend to make the atomic weight of the metal appear lower than the truth.

Sixth Series of Experiments.

These experiments consisted merely in the further application of electrolysis to the deposition of metallic gold from a solution of potassium auro-cyanide, comparing the weight of the metal thrown down, however, not with the weight of silver, but with the volume of hydrogen gas liberated by the action of the same current, the object being to thus secure, with an assumed knowledge of the density of hydrogen, a direct comparison of the atomic weight of gold with that of the element most generally taken as the basis of the numerical constants in question.

A cell containing the same solution of potassium auro-cyanide as was used in the

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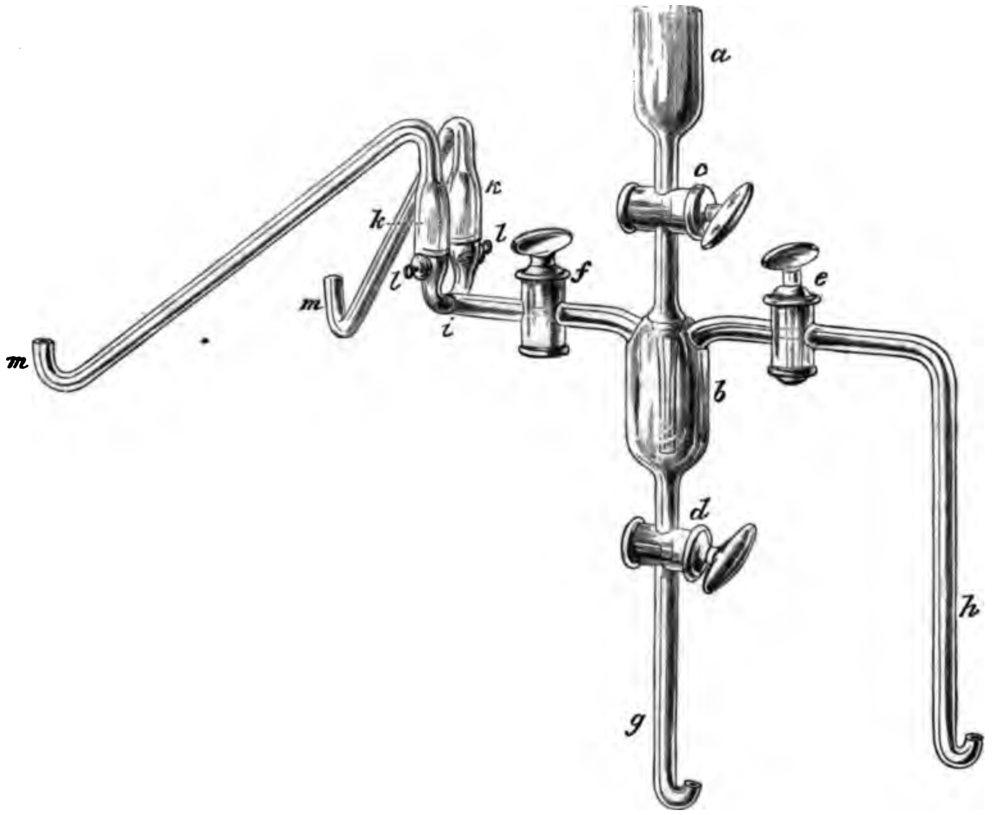
* As bearing on the question of the simultaneous decomposition of two electrolytes in the same solution, the following results may be recorded of an experiment made with a solution of mixed zinc and copper sulphates, with excess of potassium cyanide, the anode plate being of brass and the cathode plate of platinum, and an analysis made of the proportions of the two metals in the anode plate, in the solution as first taken, and in the alloy deposited on the cathode plate and subsequently dissolved off from it by means of nitric acid.

Proportion of copper to zinc.	In the brass anode plate.	In the solution electrolysed.	In the alloy deposited on the cathode plate.
Copper	68·74	13·81	73·34
Zinc	31·26	86·19	26·66
	100·00	100·00	100·00

Different results would undoubtedly have been obtained by substituting some other metal for one of those taken.

fifth series of experiments, and having immersed in it a pair of plates of "proof" gold, as already described, was employed for the deposition of the gold. The same current which traversed this cell was passed through a hydrogen voltameter of special construction,* made of glass, in a single piece, the general character of which will be seen from fig. 5.

Fig. 5.



When this instrument was to be prepared for use, it was cautiously heated pretty strongly in an air-bath to remove the film of moisture and air from the internal surface, drawing dry air through by means of an aspirator. Clean mercury, previously heated, was then poured in through the funnel *a*, going down to nearly the bottom of the cylindrical vessel *b*, until this vessel—about 30 mm. in diameter and 60 mm. in height—was completely filled, and also the tubes and stop-cocks *c*, *d*, *e*, and *f*, each of these in succession being opened to allow escape of air, and afterwards closed; *f* was a three-way stop-cock, which could either be made to open communication between the parts of the tube on either side of it, or to simply close this tube, or to close this tube and establish communication between the vessel *b* and the outside air through the base of the stop-cock; it was in this last-named way that air and surplus mercury were allowed to escape, filling the tube between *b* and *f* with mercury, but not allowing of any of the metal going further along the tube towards *i*. The stopcock *c* was closed, with the tube on which it was situated completely full of mercury, and leaving surplus mercury in the funnel *a*. In filling *b* and its connected tubes care was taken to leave no visible bubbles of air. Pure water mixed with one-twelfth its weight of pure sulphuric acid was boiled for some time in a small flask to expel all dissolved air, keeping up the volume by additions from time to time of water kept boiling in a second flask; the lower turned-up end of the tube *h* was then immersed in the dilute acid, and the lower end of *g* in a cup of mercury; on opening the stopcocks *e* and *d* mercury ran out from *g*, and the dilute acid came in through *h*, filling about half full the cylinder *b*. Closing *d* and *e*, opening *c*, and keeping up a supply of mercury in the funnel *a*, *f* was now turned so as to force out through the base of this stopcock the little mercury in the tube behind it, and fill this tube with the acidulated water. Then *f* was turned so as to allow of this acidulated water being forced on to the bend *i* and into the two little voltameter tubes *k* and *k*, filling these about one-third full. While these tubes were being thus filled the extremities of the delivery tubes *m* and *m* were in communication with a Sprengel pump, so that they were very nearly exhausted of air. The stopcock *f* having been closed, *e* was opened, and by suitable tilting of the apparatus, and running in of mercury from the funnel *a*, nearly all of the acidulated water from *f* backwards was expelled through the tube *h*. A repetition of the procedure by which the cylinder *b* had been partially filled with acidulated water now served to partially fill it with well-boiled and still hot distilled water to which no acid had been added. The two delivery tubes *m* and *m* were severally detached from the Sprengel pump, after allowing (by a special separate arrangement of tubes with stopcocks) hydrogen to enter one of the two and oxygen the other, and when thus filled the ends of these two tubes were dipped under mercury, and the two platinum wires, *l* and *l*, sealed into the voltameter tubes were connected by the little rings on their outer ends with the terminals of the galvanic cells whence the electric current was to be derived, taking care, of course, to connect to the negative pole the wire of the tube already filled in its upper part with hydrogen, and to the positive pole the wire of the oxygen tube. Viewed from the front, the two voltameter and delivery tubes presented the appearance shown in fig. 6. The little voltameter tubes *k* and *k* had an external diameter of about 12 mm. and a length of 40 mm. The platinum wires, *l* and *l*, serving as electrodes were 1 mm. in diameter, and extended beyond the interior surface of the glass (into which they were sealed) for only 3 mm. in length. They could be well covered, and the voltameter tubes filled to one-third their capacity, with only about 2 c.cm. of the acidulated water. By careful tilting of the apparatus laterally it was found to be possible to so regulate the pressure of mercury at the ends of the delivery tubes, and therefore the gaseous tension in the two voltameter tubes, that the acidulated water was not forced over from the one to the other, which, had it occurred, would have allowed admixture of the two gases; this required constant watching, however, and there was needed from time to time a little tapping of the apparatus to get rid of the effect of irregular adhesion of the liquid to the walls of the voltameter tubes.

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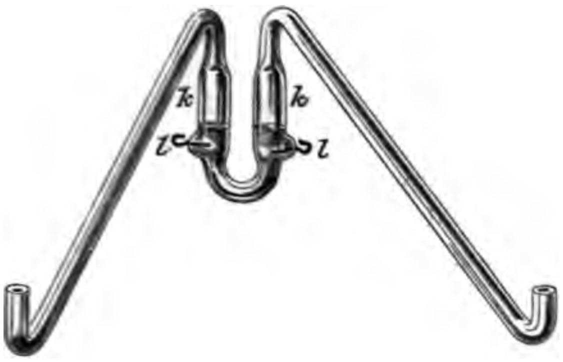
* This piece of apparatus—an excellent specimen of skilful glass-blowing—was made, from drawings furnished by me, by Mr. EMIL GREINER, of 63, Maiden Lane, New York.

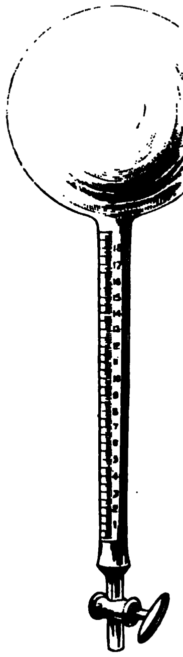
Fig. 6.



It will be seen that, with the arrangement described, the electrolysis could be effected of acidulated water, thoroughly deprived in advance of dissolved air, and in quantity so small as to be capable of retaining in solution but infinitesimal quantities of the hydrogen and oxygen electrolytically separated. As the decomposition proceeded, the quantity of liquid in the voltameter tubes could be maintained constant by opening the stopcock *c*, with a supply of mercury in the funnel *a*, and then cautiously opening *f*, so as to feed forward a little of the air-free water from the cylinder *b*, thus leaving the proportion of acid unaltered. The surface presented by the platinum wire electrodes was so small as to allow of occlusion of the gases to only an extremely minute extent, and both the hydrogen and oxygen were allowed to escape for some time before any was collected for measurement.

The hydrogen only was collected and measured. I had hoped to apply this form of voltameter to a more exact determination of the relative volumes of hydrogen and oxygen derived from water by electrolysis than is possible with the voltameters of more common construction. But I have not yet seen my way to getting over the difficulties connected with the presence of ozone, hydrogen dioxide, BERTHELOT's persulphuric acid, or other by-products in the oxygen gas evolved at the positive pole. If this could be accomplished, a useful contribution might possibly be made to the question, revived and worked upon of late by several chemists, of the exact atomic weight of oxygen. The vessel for collecting and measuring the hydrogen, shown in fig. 7, consisted of a spherical globe of tolerably stout glass, with a capacity of about 250 c.cm., having a neck of about 1 cm. internal diameter, and 22 cm. long. This neck had etched upon it a simple linear scale of millimetres. At the mouth it was fitted with a well ground perforated glass stopper, forming part of a glass stopcock with an outer orifice of about 1 mm. bore. The exact capacity of the whole globe and neck was ascertained by heating it in an air-bath to remove air and moisture condensed on the interior surface, drawing dry air through with an aspirator, then filling the globe with heated mercury, allowing it to cool to an accurately noted temperature, immersing the body of the globe in an outer vessel of mercury so as to prevent extension or flexure of the glass by the weight of the contained metal, filling up to the very mouth with mercury, inserting the stopper with the stopcock open, thus forcing out through its orifice the last of the air, closing the stopcock, removing from the orifice tube, by an iron wire, the drop or two of mercury remaining in it, and then emptying the flask, and carefully weighing in successive portions the mercury which it had held. The hydrogen from the voltameter was collected in this flask, without its stopper, the flask having been previously filled with mercury, with the needful precautions for removal of all air, and inverted over a mercury trough. In each experiment the process of electrolysis was arrested when the hydrogen had filled the body of the globe and reached to a point rather more than half way down the length of the neck, the gold plates being of course withdrawn at the same moment from their cell of gold solution, set away to soak in distilled water, and afterwards thoroughly washed, dried, heated in the Sprengel vacuum, cooled, and weighed. The portion of hydrogen collected was dried by successive balls of fused potash introduced and withdrawn by means of platinum wire. The neck of the flask having, in advance of the collection of hydrogen, been passed through a cork, this was used to close the mouth, placed downwards, of a vessel through which a stream of water was caused to flow rapidly from the pipes supplying the University buildings. The atmospheric temperature of the day on which the electrolysis experiment was made having been such as not to differ too much from the temperature of the water from the pipes, the gas occupied such a volume after effectual exposure to this latter temperature that

Fig. 7.



the mercury marked a point somewhere within the length of the neck, which point was noted by the millimetre scale, the thermometer immersed in the flowing water, and the barometer and its attached thermometer being read at the same time. It remained only to insert the stopcock stopper under the mercury of the little mercury trough, close the stopcock, withdraw the flask from the trough, reject the drop or two of mercury from the stopcock orifice by means of a wire, remove the portion of mercury left in the neck of the flask, and weigh it carefully. Its weight, with consideration of its temperature when the stopcock was closed, gave the volume of the portion of the flask not occupied by hydrogen, and this, subtracted from the whole volume of the interior of the flask, as found by the original calibration, gave the volume, under known conditions of temperature and pressure, of the hydrogen which had been collected. From two calibrations at different temperatures a correction was obtained for the expansion of the glass of the flask, but it was hardly necessary to take this into account, in view of the small limits within which temperature varied in all the experiments made.

But three experiments carried out by this method led to results which seemed worthy of confidence. These results were as follows:—

Experiment.	Character of gold in solution.	Character of gold in plates.	Gold deposited.	Hydrogen liberated.	
				Vol. at 0° C. and 760 mm.	Weight.
			grm.	c.cm.	grm.
I.	A. i	C	4·1472	226·64	·02035
II.	A. i	C	4·0226	227·13	·02038
III.	A. i	C	4·0855	231·55	·02078

In calculating the weight of hydrogen from its observed volume, REGNAULT's value for the weight of a litre of this gas at 0° C. and 760 mm. was taken as the basis. The correction of which Lord RAYLEIGH not long since pointed out the need—namely, for the compression of the vacuous glass flask by atmospheric pressure—was adopted from the experiments of J. M. CRAFTS ('Comptes Rendus,' vol. 106, p. 1662), and the corrected value, ·08988 grm., was still further corrected for the difference in the force of gravity at Paris and at the University of Virginia (in C. G. S. units, 980·94 : 979·95), giving as the value to be used ·08979 grm.

The electrolysis of the water was carried on very slowly, so as to keep the density of the current low with such small electrodes as were used. Nevertheless, as the hydrogen voltameter required constant watching, it became necessary to bring the whole time of an experiment within moderate limits, and hence a considerably stronger current was used than in the simultaneous deposition of gold and silver in the fifth series, this circumstance being less favourable to the satisfactory deposition of the gold. It would have been desirable to use a larger flask, and to collect a greater volume of hydrogen ; but this, on account of the time required, would have made an experiment exceedingly troublesome and difficult.

In the work of this series the same unsatisfactory need for selecting only such results as came fairly close to the figures expected, and rejecting several others on the ground of very considerable departure therefrom, and the same sources of possible constant error in regard to the gold deposit present themselves which have already been noticed under the head of the fifth series. As regards the hydrogen, one is led to consider possible diffusion of hydrogen and oxygen between the two little voltameter tubes, and slight imperfection in the drying of the hydrogen obtained. The former would, on the whole, probably tend to diminish the volume of gas collected, and hence to raise the apparent value of the atomic weight of gold. The latter would have the opposite tendency. That neither can have had more than an extremely minute influence was fairly proved by testing a part of the hydrogen obtained, on the one hand by passing it through a red-hot glass tube, and on the other by submitting it to more extended drying by contact with phosphorus pentoxide both before and after such heating; in neither case was there appreciable change of volume.

Notwithstanding the desirability of comparing the atomic weight of any other element directly with that of hydrogen, the difficulty is not to be overlooked of doing this for an element having so high an atomic weight as that of gold. There is a manifest objection to the necessity of dealing with such minute quantities of hydrogen as those concerned in these experiments. A very small error in the determination of the hydrogen greatly affects the value found for an atomic weight nearly two hundred times as large. It is true that the measurement of the volume of the hydrogen admits of being made with such precision as to leave room for but an extremely minute error in the corresponding weight, yet this measurement is not one of *limitless* delicacy, particularly if the difficulty be properly appreciated of ascertaining with certainty the precise temperature of the gas at the time its volume is read. Moreover, in measuring the volume of the gas, and thence deducing its weight, there is need not merely for a knowledge of changes of temperature and pressure, but for *absolutely* correct readings of the barometer and thermometer, so that there must usually be a degree of hesitation in accepting the readings of even fairly standard instruments, when temperature and pressure come to be placed in comparison with these conditions as affecting the results of REGNAULT for gaseous density. Nor can the results of that great physicist be assumed as themselves free from all possible need of further correction.

The error of direct comparison with so small an atomic weight as that of hydrogen is, however, after all only *masked* by substituting an *indirect* comparison through some larger atomic weight, since the assumed value of the latter is uncertain within limits which depend upon *its* comparison with the atomic weight of hydrogen.

Seventh Series of Experiments.

In pursuance of the attempt to connect directly the atomic weight of gold with that of hydrogen, metallic zinc was prepared as nearly as possible in a state of purity, and, a known quantity of the metal having been dissolved in dilute sulphuric acid, the amount of hydrogen evolved was determined by volume. A solution of pure auric chloride or bromide was then treated with a known quantity of the same zinc, more than sufficient for the complete precipitation of all the gold present; the excess of zinc was dissolved by dilute sulphuric acid, and the volume of hydrogen given off was determined. The precipitated gold was carefully collected, washed, dried, ignited, and weighed. The difference between the volume of hydrogen which the zinc gave when thus partly used to replace a known quantity of gold and the volume which it would have given if replacing hydrogen alone represented, of course, the volume of a quantity of hydrogen equivalent to the gold precipitated and weighed. From this volume, under known conditions of temperature and pressure, the weight of the hydrogen was calculated on the basis of REGNAULT's results for the density of the gas, after application of the needful corrections, as in the sixth series of these experiments.

In a preliminary notice of my work read before the Chemical Section of the British Association at the Manchester meeting of 1887, it was pointed out that the method just described has certain advantages in principle. It does not require that the weight of the gold salt in solution be known, so that all difficulties in regard to drying such salt without decomposition are disposed of. It does not depend upon a knowledge of the atomic weight of the halogen in combination with gold, or upon a knowledge of the atomic weight of zinc. It does not even require that the zinc be of assured purity, provided only it be uniform in character, so that a given weight of it can be trusted to yield always the same quantity of hydrogen, and there be no impurities present capable of interfering with the collection of the whole of the precipitated metallic gold in a state of purity. The chief difficulty consists in the accurate ascertainment of the total volume of hydrogen evolved from the solution of a satisfactorily large quantity of zinc; when the gold solution comes to be used, as the volume of hydrogen given off on solution of the *surplus* zinc may be made quite small, its measurement becomes both easy and exact.

The pure zinc required was obtained by fractionally distilling in a Sprengel vacuum some very nearly pure metal from the Bertha Zinc Works, in South-western Virginia, using a long combustion-tube of hard Bohemian glass, and substantially the same arrangement of apparatus as that described by MORSE and BURTON* in connection with their work on the atomic weight of zinc. The original metal was found, by an analysis in the laboratory of the University of Virginia, to contain less than ·04 per cent. of foreign matter, almost solely consisting of lead and iron. It was four or five times redistilled *in vacuo*, rejecting each time about one-third of the quantity treated. The process is easily carried out, and in the final product, completely soluble in dilute sulphuric acid without visible residue, no trace of detectable impurity could be found.

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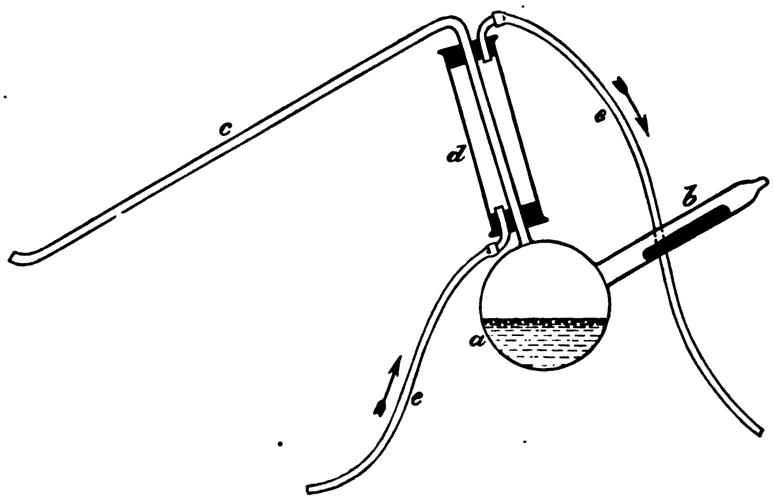
* 'American Chemical Journal,' vol. 10, p. 312. Tubes of glazed porcelain, closed at one end, had been specially procured for use in thus distilling zinc, but it was found that they were quite unnecessary.

Fig. 8.



For the evolution of hydrogen on solution of this zinc in acid the little piece of apparatus represented in fig. 8 was used, the same that I had used in my work of several years ago on the atomic weight of aluminum.[*] The description formerly given of the details of an experiment with this apparatus may be repeated with but trifling change of language. A rather more than sufficient quantity of diluted sulphuric acid, its volume accurately measured, having been introduced into the bulb *a* by means of a little tube-funnel passed through the tube *b*, the outer end of which was originally open, taking care to leave the surface of *b* clean, the metallic zinc, in a single piece of elongated shape, and having a little bit of slender platinum wire wrapped round it, was passed into *b*, held nearly horizontal, so that the metal did not slip down into the bulb, but rested 40 or 50 mm. from it; *b* was now drawn off in the lamp flame, and sealed with a well-rounded end. The bulb was touched for a moment or two with the hand, so as to expel a very little air, and the outer end of the small tube *c* was introduced into the mercury of the trough, taking care that *b* was still kept in such a position as to prevent the zinc coming in contact with the dilute acid. After a sufficient lapse of time for the apparatus to have acquired the temperature of the room, the barometer and thermometer and the difference of level of the mercury in the trough and in *c* were read off;[†] so that, knowing the volume of dilute acid introduced and of metallic zinc (the latter from its weight), calibration of the bulb and tubes after the experiment was over completed the data necessary to determine the volume of air which the apparatus contained at the beginning. The piece of zinc was now made to slide down into the bulb, the end of the gas delivery-tube *c* having been brought under the mouth of the measuring flask. Over-rapid evolution of hydrogen and any considerable rise of temperature were prevented, partly by tilting the bulb so that the little piece of zinc rested against one side and exposed but a part of its surface to the action of the liquid, and partly by cooling the outside of the bulb with water. To guard against more than traces of aqueous vapour being carried away with the hydrogen, a rapid current of ice-water was kept up through *d*.

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[*] 'Phil. Trans.,' 1880, p. 1026.

[†] All readings were, of course, made from a distance with the aid of a small telescope.

As soon as the last of the zinc had disappeared, leaving the liquid quite clear, *c* was brought up into a nearly vertical position, and the apparatus left to itself until the temperature of the room had been attained. The barometer and thermometer and the height of the mercury in *c* above that in the trough were now read and recorded.

Lifting *c* straight up from the trough, the mercury in this tube was got out by running a wire up and down in it, and, inverting it, the whole of the remaining space in *a*, *b*, and *c* was filled up with solution of zinc sulphate and free acid of the same strength with that already contained, this liquid being run in from a graduated burette through a slender tube-funnel, and the volume used noted, so as to show how much liquid had been already present.

The apparatus having been now emptied, washed out, and calibrated (with water, instead of mercury, on account of the difficulty of getting the interior quite dry), the volume of gas remaining in it at the close of the experiment was had from the difference between the total capacity (to the level of the mercury in *c*) and the volume of liquid which the bulb had contained at the close of the experiment, these taken together with the data for pressure and temperature.

The dilute acid was saturated with pure hydrogen just before being used (and in the experiments with auric chloride or bromide the main portion of water holding this salt in solution was similarly treated), and a preliminary experiment showed that there was but an extremely minute difference between the amount of gas removable from such liquid by heating in a Sprengel vacuum and from that containing zinc sulphate after the solution of the metal; so that, practically, the question of retention of gas in solution by the liquid might be neglected.

The sulphuric acid was diluted to 25 per cent. by weight, only a small bit of platinum wire was wrapped round the zinc, and the temperature of the bulb was not allowed to rise beyond about 20° C. Thus the risk of evolving other gaseous products than hydrogen*—as hydrogen sulphide or sulphur dioxide—was avoided, and on testing for these impurities the hydrogen collected no traces of them were found.

The measuring flask used to collect the hydrogen was of the same character as that used for the experiments of the sixth series, but of much larger size, holding about a litre. The quantity of zinc taken for each experiment was calculated to give a volume of gas which, under the conditions of temperature and pressure of the day, would bring the mercury to somewhere near the middle of the neck, and the gas, previously dried by balls of fused potash, was measured after the temperature had been rendered as nearly as possible fixed by the circulation round the outside of the flask of an active stream of water from the laboratory supply pipes. On account of slight rise of temperature during the solution of the metal, the volume of hydrogen left in the bulb and tubes was always less than that of the air in the same at the beginning; and, after reduction to normal temperature and pressure, the difference had to be subtracted from the gas collected in the flask.

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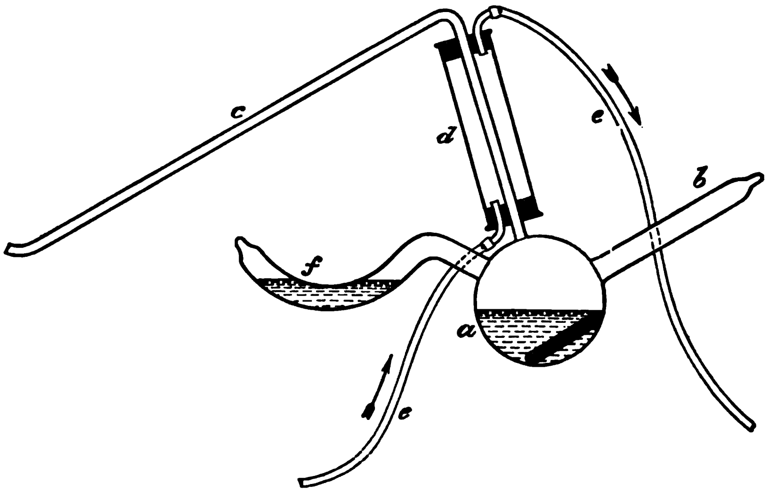
* MUIR and ADIE: "On the Interaction of Zinc and Sulphuric Acid," 'Chem. Soc. Journ.,' Jan., 1888, p. 47.

In the experiments with auric chloride or bromide the quantity of hydrogen given off on solution of the *surplus* zinc was so small that it could be easily measured in a little gas tube, the same method of double calibration with mercury being used as for the larger volumes. In these experiments the bulb used had a second side tube, *f*, as shown in fig. 9, to hold the sulphuric acid, while *a* contained the aqueous solution of the gold salt; this acid was already somewhat diluted, and was introduced into *a*, after complete precipitation of the gold, very gradually, so as to avoid any considerable rise of temperature. The quantity of water used was such as to make the whole volume of liquid very nearly the same in the experiments with zinc alone and in those with zinc and the auric salt. Care was taken to ascertain, after measurement of the hydrogen, that it had been effectually freed by the potash balls not only from moisture, but from any traces of hydrochloric acid formed and carried over.

Fig. 9.



In order to connect the weight of the zinc with that of the hydrogen produced by its solution, it was necessary that the weight of the metal should be *absolute*, or in terms of equal value with those used in REGNAULT's researches on the density of hydrogen; hence, as has been already stated, the weights used were such as had had their real values determined, and the precaution of double weighing was applied.

The quantities of metal used being small, the centre of gravity of the balance beam was so adjusted as to give great sensitiveness. In calculating the weight of the hydrogen from its volume, the same value for the weight of a litre of the gas was assumed as has been already stated, viz., ·08979 grm., being the result of REGNAULT'S determinations, with the correction pointed out by Lord RAYLEIGH and numerically estimated by CRAFTS, and further corrected for the force of gravity at the University of Virginia.

The haloid salts of gold were prepared as for the experiments of the first and second series, and the careful filtration of their solutions was followed by long continued standing at rest before the portions required were gently drawn off for use. Great care was taken in removing the last traces of precipitated gold from the bulb—to facilitate which the connected tubes were all cut off short—and in repeatedly washing the metal, first with dilute sulphuric acid, then with pure hydrochloric acid, and, finally, with water, before drying, heating (in the Sprengel vacuum), cooling, and weighing.

The results obtained by this method were much freer from irregularity, and much more satisfactory, than those of the electrolytic experiments. All are reported, except one or two cases obviously vitiated by mechanical defects of manipulation, and, in consequence, not carried out to the end.

Experiments with Zinc alone.

Experiment.	Zinc dissolved.	Hydrogen obtained, at 0° C. and 760 mm.		Hydrogen, at 0° C. and 760 mm., for 1 grm. of zinc.
	grm.	c.cm.		c.cm.
I.	2·6990	922·64	Equivalent to	341·85
II.	2·6771	915·33		341·91
III.	2·7029	924·20		341·93
IV.	2·7117	927·51		342·04

Or a total amount of 10·7907 grm. of zinc gave 3689·68 c.cm. of gas,[*] equivalent to 341·93 c.cm. of hydrogen for 1 grm. of zinc. This value was adopted in calculating the fifth column of the following table.

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[*] These figures represent an atomic weight for zinc = 65·142, taking the weight of a litre of hydrogen at 0° C. and 760 mm. as ·08979 grm., and assuming the zinc used to have been absolutely pure, and the quantity of hydrogen collected to have been strictly equivalent to it; neither of the two latter assumptions is essential to the use made in this paper of the experiments. REYNOLDS and RAMSAY in their recent paper ('Chem. Soc. Journ.,' Dec., 1887, p. 854) on the atomic weight of zinc arrive at a somewhat higher value, on the basis of a like comparison of the weight of the metal with the volume of hydrogen liberated by it, but they assume the weight of the litre of hydrogen under normal temperature and pressure as ·0896 grm., which must be considered too low in view of the recently applied correction of Lord RAYLEIGH.

Experiments with Gold Salt and Zinc.

Experiment.	Character of gold used.	Character of gold salt.	Gold precipitated	Hydrogen, at 0° C. and 760 mm.		Hydrogen equivalent to gold.	
				Corresponding to total zinc.	Obtained from residual zinc.	Vol. at 0° C. and 760 mm.	Weight.
			grm.	c.cm.	c.cm.	c.cm.	grm.
I.	A, *b*	$AuCl_3$	10·3512	1779·44	−23·34	=1756·10	=·15768
II.	A, *b*	$AuBr_3$	8·2525	1428·99	−28·61	=1400·38	=·12574
III.	A, *b*	$AuCl_3$	8·1004	1393·43	−18·56	=1374·87	=·12345
IV.	C	$AuCl_3$	3·2913	582·82	−24·18	= 558·64	=·05016
V.	C	$AuBr_3$	3·4835	606·20	−15·27	= 590·93	=·05306
VI.	D	$AuBr_3$	3·6421	643·31	−25·20	= 618·11	=·05550

In considering possible causes of constant error in the experiments of this last series it seems most likely that they would affect the exact determination of the weight of the precipitated gold, either by mechanical loss of some minute particles of the metal, tending to lower the atomic weight, or by incomplete washing out of the zinc salt, with an influence in the opposite direction. Any failure to remove the last traces of moisture from the hydrogen was, I think, effectually guarded against, at any rate within such limits as would have sensibly affected the resulting atomic weight; and any error due to retention of hydrogen in solution by the liquid must also have been inappreciably small, in view of the precautions taken and the close similarity of conditions in the experiments with zinc alone and with zinc and the auric salt.

Calculation of Results.

In calculating the atomic weight of gold from the data furnished by the experiments which have been described, I have thought it best to conform to the most general usage of those who have been working on questions of this sort of late years, so as to facilitate comparisons with the results of others. Hence, although the atomic weight has been calculated separately from the figures of each experiment reported, the value deduced from each series has not been taken as the arithmetical mean of the separate results, nor has the probable error of these or of the mean been calculated by the method of least squares, as was done in my paper on the atomic weight of aluminum, but, instead, the general result for each series has been obtained, as in the calculations of MEYER and SEUBERT, from the aggregate quantities of the materials employed, though I am by no means convinced that this mode of reckoning is in all cases sound in principle, giving, as it does, weight to each experiment in proportion to the quantity of material employed.

The atomic weights assumed for the other elements involved are those which have been most generally accepted in calculations of this kind, based for the most part on the experiments of STAS, and representing, with greatest probability, the values as at present known to us. They are as follows :—

H = 1.
Ag = 107·66.
Cl = 35·37.
N = 14·01.
C = 11·97.

Calculated Results.

The following are the values obtained for the atomic weight of gold from the different series of experiments :—

FIRST Series.

($Ag_3 : Au :: 322{\cdot}98 : x.$)

Experiment.	Silver.	Gold.	Atomic weight of gold.	
	grm.	grm.		
I.	12·4875	7·6075	196·762	
II.	13·8280	8·4212	196·694	
III.	11·3973	6·9407	196·688	Lowest value
IV.	5·5286	3·3682	196·770	Highest value
V.	4·6371	2·8244	196·723	
	47·8785	29·1620	196·722	

SECOND Series.

($Ag_3 : Au :: 322{\cdot}98 : x.$)

Experiment.	Silver.	Gold.	Atomic weight of gold.	
	grm.	grm.		
I.	13·5149	8·2345	196·789	
II.	12·6251	7·6901	196·731	Lowest value
III.	17·2666	10·5233	196·843	Highest value
IV.	4·5141	2·7498	196·746	
V.	5·8471	3·5620	196·756	
VI.	6·4129	3·9081	196·828	
	60·1807	36·6678	196·790	

THIRD Series.

(Ag_4 : Au : : 430·64 : x.)

Experiment.	Silver.	Gold.	Atomic weight of gold.
	grm.	grm.	
I.	12·4851	5·7048	196·772
II.	17·4193	7·9612	196·817 Highest value
III.	5·3513	2·4455	196·799
IV.	9·1153	4·1632	196·685 Lowest value
	44·3710	20·2747	196·775

FOURTH Series.

($\mathrm{N(CH_3)_3HCl_4}$: Au : : 201·40 : x.)

Experiment.	Loss by ignition of trimethyl-ammonium auri-chloride.	Gold.	Atomic weight of gold.
	grm.	grm.	
I.	7·5318	7·3754	197·218
II.	7·8432	7·6831	197·289 Highest value
III.	5·2811	5·1712	197·209
IV.	3·3309	3·2603	197·131 Lowest value
V.	2·8165	2·7579	197·210
	26·8035	26·2479	197·225

FIFTH Series.

(Ag : Au : : 107·66 : x.)

Experiment.	Silver.	Gold.	Atomic weight of gold.
	grm.	grm.	
I.	2·8849	5·2721	196·747
II.	3·4487	6·3088	196·945 Highest value
III.	2·3393	4·2770	196·837
IV.	1·9223	3·5123	196·709 Lowest value
V.	2·0132	3·6804	196·817
	12·6084	23·0506	196·823

SIXTH Series

($H : Au :: 1 : x.$)

Experiment.	Hydrogen.	Gold.	Atomic weight of gold.	
	grm.	grm.		
I.	·02053	4·0472	197·136	
II.	·02039	4·0226	197·283	Highest value
III.	·02079	4·0955	196 994	Lowest value
	·06171	12·1653	197·137	

SEVENTH Series.

($H_3 : Au :: 3 : x.$)

Experiment.	Hydrogen.	Gold.	Atomic weight of gold.	
	grm.	grm.		
I.	·15768	10·3512	196·941	
II.	·12574	8·2525	196 894	
III.	·12345	8·1004	196·851	
IV.	·05016	3·2913	196·848	Lowest value
V.	·05306	3·4835	196·956	Highest value
VI.	·05550	3·6421	196·865	
	·56559	37·1210	196·897	

General Mean of Results.

If each of the foregoing series of experiments be represented by the result calculated from the aggregates of material used, and if equal weight be attached to the results of all the methods, the general mean derived from the whole of the 34 experiments will be as follows :—

First series	196·722	Lowest value.
Second series	196·790	
Third series	196·775	
Fourth series	197·225	Highest value.
Fifth series	196·823	
Sixth series	197·137	
Seventh series	196·897	
General mean.	196·910	

The results of the fifth and sixth series, obtained by electrolysis, are, I am convinced, much less entitled to confidence than any of the others. If these two be excluded, the general mean of the remaining series will be **196·882,** a number differing but little from the mean of all.

The highest value is that derived from the fourth series—ignition of trimethylammonium auri-chloride. It has been seen that the individual results of this series agree fairly well with one another, and, when examined in connection with the facts as to the different crops of crystals of the salt used, do not seem to present any evidence of want of uniformity in the material. But, as it may still be suspected that traces of dimethyl- or of monomethyl-ammonium auri-chloride may have been present, and have caused the apparent value of the atomic weight of gold to come out higher than the truth, if we exclude also this series, the general mean of the remaining four will be **196·796.**

Finally, if for the sake of comparison with the results of KRÜSS and of THORPE and LAURIE the general mean be taken for the first three series only, in which auric chloride and bromide were examined, the result is **196·762**—intermediate between the general means of the two previous researches, but rather nearer to that derived from the work of THORPE and LAURIE than of KRÜSS.

It will be observed that, although there is pretty close agreement among the means of results obtained by altogether different methods, this agreement is not so close as that presented by the results of the nearly similar methods pursued in the first three series. This cannot but suggest the probability of there being still sources of minute errors inherent in the methods themselves, and not dependent upon mere imperfections of manipulation in carrying these methods out. Although there is thus to be noticed a slight tendency on the part of each method to yield high or low figures severally, with the exception of the results of the fourth series there does not appear to be any considerable reason to see in the values obtained confirmation of the special suspicions in connection with each method which have been stated. There is no clear evidence of any difference in the results which can be traced to the history of the particular samples of gold used; a larger number of somewhat low results seem to have been yielded by the metal designated as (C)—*i.e.*, obtained from the United States Assay Office at New York—than by the others, but the difference is not marked or constant enough to warrant any trustworthy conclusions as to the character of this material.

Concluding Remarks.

The atomic weight of gold as deduced from the experiments reported in this paper is entirely in accord with the place occupied by the metal in MENDELEJEFF'S "periodic" classification of the elements, but this is equally true of the slightly different values obtained by KRÜSS, and by THORPE and LAURIE, and the only difficulty at one time apparent as to this point—namely, the relative positions of gold on the one hand and of platinum, iridium, and osmium on the other—has been reached, not by any change in the atomic weight of gold, but by changes affecting the values for the atomic weights of the three other metals, as these values have been determined by the more [illegible] it is very desirable that, in order to a fuller and more exact examination of the Mendeléeff table of the elements, there be accomplished as soon as possible a general revision of the atomic weights of all the elements of well-determined [illegible] many of which are still very imperfectly known.

As to any bearing of the results of the present paper on the so-called hypothesis of Prout,* the general mean of all my results, or even the general mean with exclusion of the values obtained by electrolysis, approaches the integer number 197 rather more nearly than does the final number arrived at by Thorpe and Laurie, and still more nearly than does that considered by Krüss to express the final result of his experiments. If the results of the fourth series be alone regarded, the general mean will be nearer the integer than is the Krüss number, but not quite so near as that of Thorpe and Laurie. I feel that somewhat greater confidence may be placed in my own work simply on the ground of its involving the use of more completely different and independent methods—a principle which I believe to be of the first importance in any attempts at increased accuracy in the determination of atomic weights.

At the same time, as has already been pointed out, this work seems to me to furnish some probable evidence that not all inherent defects of method have been eliminated. Whether or not such defects may exist to an extent sufficient to account for the remaining difference between the value obtained and the integer multiple of the atomic weight of hydrogen there does not seem to be ground on which to express a positive opinion. But this research does not supply any clear evidence contradictory of such a possibility.

On this point, and generally as to the attainment of what is sometimes rather too easily spoken of as the greatest possible accuracy in the determination of an atomic weight—particularly of an element for which the value is as high as that for gold—any one who has actually worked at such a determination will be pretty sure to feel more strongly the difficulty of the task, and to be [illegible] with more caution than to [illegible] results in [illegible] that the last word has been spoken.

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[*] [illegible]

[†] [illegible]

XIV. *Magnetic and other Physical Properties of Iron at a High Temperature.*

By JOHN HOPKINSON, *M.A.*, *D.Sc.*, *F.R.S.*

Received April 16,—Read May 9, 1889.

[PLATES 12–20.]

IT is well known that for small magnetising forces the magnetisation of iron, nickel, and cobalt increases with increase of temperature, but that it diminishes for large magnetising forces.* BAUER† has also shown that iron ceases to be magnetic somewhat suddenly, and that the increase of magnetisation for small forces continues to near the point at which the magnetism disappears. His experiments were made upon a bar which was heated in a furnace and then suspended within a magnetising coil and allowed to cool, the observations being made at intervals during cooling. This method is inconvenient for the calculation of the magnetising forces, and the temperature must have been far from uniform through the bar. In my own experiments‡ on an impure sample of nickel the curve of magnetisation is determined at temperatures just below the temperature at which the magnetism disappears, which we may appropriately call the critical temperature.

AUERBACH§ and CALLENDAR|| have shown that the electrical resistance of iron increases notably more rapidly than does that of other pure metals. BARRETT,¶ in announcing his discovery of recalescence, remarked that the phenomenon probably occurred at the critical temperature. TAIT** investigated the thermo-electric properties of iron, and found that a notable change occurred at a red heat, and thought it probable that this change occurred at the critical temperature.

It appeared to be very desirable to examine the behaviour of iron with regard to magnetism near the critical temperature, and to ascertain the critical temperatures

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* ROWLAND, 'Phil. Mag.,' Nov., 1874.

† 'WIEDEMANN, Annalen,' vol. 11, 1880.

‡ 'Roy. Soc. Proc.,' June, 1888.

§ 'WIEDEMANN, Annalen,' vol. 5, 1878.

|| 'Phil. Trans.,' A, 1887.

¶ 'Phil. Mag.,' Jan., 1874.

** 'Edinburgh Roy. Soc. Trans.,' Dec., 1873.

for different samples. It also appeared to be desirable to trace the resistance of iron wire up to and through the critical temperature, and to examine more particularly the phenomenon of recalescence, and determine the temperature at which it occurred.

The most interesting results at which I have arrived may be shortly stated as follows:—

For small magnetising forces the magnetisation of iron steadily increases with rise of temperature till it approaches the critical temperature, when it increases very rapidly, till the permeability in some cases attains a value of about 11,000. The magnetisation then very suddenly almost entirely disappears.

The critical temperatures for various samples of iron and steel range from 690° C. to 870° C.

Heating iron a little above the critical temperature does not entirely wipe out all effects of previous magnetisation.

The temperature coefficient of electrical resistance is greater for iron than for other metals; it increases greatly with increase of temperature till the temperature reaches the critical temperature, when it suddenly changes to a value more nearly approaching to other metals. Recalescence does occur at the critical temperature. The quantity of heat liberated in recalescence has been measured and is found to be quite comparable with the heat required to melt bodies.

Since making the experiments and writing the preliminary notes which have already appeared in the 'Proceedings of the Royal Society,' my attention has been called to two papers which deal in part with some of the matters on which I have been experimenting. PIONCHON[*] has shown that the specific heat of iron is very much greater at a red heat than at ordinary temperatures. W. KOHLRAUSCH,[†] in an interesting paper, shows that, whereas the temperature coefficient of resistance of iron is much greater than usual for temperatures below the critical temperature, it suddenly diminishes on passing that temperature. He also identifies the temperature of recalescence with the critical temperature. So far as resistance of iron is concerned, W. KOHLRAUSCH has anticipated my results, which I give, however, for the sake of completeness.

Magnetic Experiments.

The method of performing the magnetic experiments was the same as that used by ROWLAND. The copper wire was, however, insulated carefully with asbestos paper laid over the wire, and with layers of asbestos paper between the successive layers of the wire. The insulation resistance between the primary and the secondary coils was always tested, both at the ordinary temperature and at the maximum temperature used. At the ordinary temperature this resistance always exceeded a megohm; at the maximum temperature it exceeded 10,000 ohms, and generally lay between 10,000 and 20,000 ohms. The ring to be examined, with its coils of copper wire, was placed in a cylindrical cast-iron box, and this in a Fletcher gas furnace, the temperature of which was regulated by the supply of gas. The temperatures were estimated by the resistance of the secondary coil. It was observed that the resistance of this coil at the ordinary temperature increased slightly after being raised to a high temperature; this I attribute to oxidation of the wire where it leaves the cast-iron box. However, it introduced an element of uncertainty into the determination of the actual temperatures, amounting, perhaps, to 20° C. at the highest temperature. This error will not affect the differences between neighbouring temperatures, with which we are more particularly concerned.

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[*] 'Comptes Rendus,' vol. 103, p. 1122.

[†] 'WIEDEMANN, Annalen,' vol. 33, 1888.

The resistance of the ballistic galvanometer is 0·43 ohm; to this additional resistances were added to give the necessary degree of sensibility. The ratio of two successive elongations of the galvanometer is $(1+r)/1 = 1{\cdot}12/1$. The time of oscillation T and the sensibility varied a little during the experiments, but so little, that the correction would fall within the limits of errors of observation in these experiments.

The total induction $= \left\{\left(1+\frac{r}{2}\right)\frac{C}{\alpha}\frac{T}{2\pi}\right\}\frac{1}{2n}\mathrm{R.A.}10^8$, where C is the current which gives the deflection α, n is the number of turns in the secondary coil, R the resistance of the secondary circuit, A the mean of the first and second elongations on reversal of the current in the primary.

The magnetising force $= 4\pi mc/l$, where m is the number of turns in the primary, l the mean length of lines of force in the ring, c the current in absolute measure in the primary.

With my galvanometer as adjusted, a Grove's cell, the E.M.F. of which was at the time determined to be 1·800 volt, gave a deflection of 158·5 divisions through a resistance of 50,170 ohms, whence

$$\frac{C}{\alpha} = \frac{1{\cdot}800}{158{\cdot}5 \times 50{,}170} = 0{\cdot}0000002264,$$

$$T = 13{\cdot}3.$$

Hence

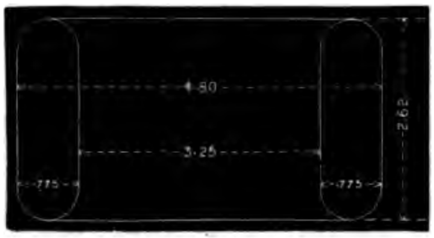
$$\left(1+\frac{r}{2}\right)\frac{C}{\alpha}\frac{T}{2\pi} = 5{\cdot}09 \times 10^{-7}.$$

The ring method of experiment is open to the objection that the magnetising force is less in the outer than in the inner portions of the ring. The results, in fact, give the average results of forces which vary between limits.

Wrought Iron.—The sample of wrought iron was supplied to me by Messrs.

MATHER and PLATT. I have no analysis of its composition. I asked for the softest iron they could supply.*

The dimensions of the ring were as shown in the accompanying sketch :—



The area of section is 1·905 sq. cm. The area of the middle line of the secondary coil is estimated to be 2·58 sq. cms. This estimate is, of course, less accurate than the area of section of the ring itself.

The secondary coil had 48 convolutions, the primary 100 convolutions.

At the beginning of the experiments the insulation resistance of the secondary from the primary was in excess of 1 megohm; the resistance of the secondary and the leads was 0·692, the temperature being 8°·3 C.

The resistance of the leads to the secondary and of the part of the secondary external to the furnace was estimated to be 0·04.

A curve of magnetisation was determined at the ordinary temperature on the virgin sample with the following results, shown graphically in Curve I.; in each case the observation was repeated twice with reversed direction of magnetising currents, and the kicks in the galvanometer were found to agree very closely together :—

Magnetising force .	0·15	0·3	0·6	1·2	2·2	4·4	8·2	14·7	24·7	37·2	69·2
Induction per sq. cm.	39·5	116	329	1,560	6,041	10,144	12,633	14,059	14,702	15,149	15,959

The ring was next heated and observations were made with a magnetising force of 8·0 to ascertain roughly the point at which the magnetism disappeared. After the magnetism had practically disappeared and the temperature was roughly constant, as indicated by the resistance, being 2·92 before the experiment and 2·85 after the experiment, corresponding with temperatures of 838° C. and 812° C., the induction was determined for varying magnetising forces.

Magnetising force . . .	2·4	4·2	8·0	21·0	49·8
Total induction	small	12·3	22·7	58·2	143

This shows that the induction is, so far as the experiment goes, proportional to the inducing force.

Taking the total induction as 143, corresponding to a force of 49·8, we have

* [*Added July* 2, 1889.—Sir JOSEPH WHITWORTH and Co. have since kindly analysed this sample for me with the following result :—

	C	Mn	S	Si	P	Slag†
Per cent. . . .	·010	·143	·012	Nil	·271	·436

---

† Containing 74 per cent. SiO_2 (Silica).]

induction in the iron 109, or 57 per sq. cm., giving permeability equal to 1·14, showing that the material has suddenly become non-magnetic.

The ring was now allowed to cool, some rough experiments being made during cooling. When cold the resistance of the secondary and the leads was found to be 0·697 ohm. The ring was again heated till the resistance of the secondary reached 2·845 and the magnetism had disappeared. It was next allowed to cool exceedingly slowly, and the following observations were made with a magnetising force of 0·075 C.G.S. unit:—

Resistance of secondary . .	2·81	2·80	2·79	2·78	2·765
Temperature	796°	792°	788°	785°	781°
Induction per sq. cm. . . .	0	0	0	126·8	

showing that magnetisation returns at a temperature corresponding to resistance between 2·78 and 2·765.

Systematic observations then began. The results are given in the following tables and the curves to which reference is made. The curves are in each case set out to two scales of abscissæ, the better to bring out their peculiarities.

TABLES 1–4.

Table 1, Curve II.		Table 2, Curve III.		Table 3, Curve IV.		Table 4, Curve V.	
Resistance of secondary before experiment	2·76	2·75		2·72		2·67	
Temperature of secondary before experiment	778° C.	775° C.		763° C.		744° C.	
Resistance of secondary after experiment	2·75	2·73		2·695		2·66	
Temperature of secondary after experiment	775° C.	767° C.		754° C.		741° C.	
Magnetising force.	Induction per sq. cm.	Magnetising force.	Induction per sq. cm.	Magnetising force.	Induction per sq. cm.	Magnetising force.	Induction per sq. cm.
0·075	511·8	0·075	494	0·075	328	0·075	227
0·15	1313·9	0·15	1033		260		180
0·3	2482·6	0·30	3286	0·15	710	0·15	473
0·6	3257·4	0·6	4520		635		425
1·2	3659·2	1·2	5367	0·3	2304	0·3	1281
2·4	4104·0	2·4	5668	0·6	5281		1172
4·4	4520·0	4·2	6056	1·2	6544	0·6	5377
		7·8	6228	2·2	7318	2·2	8165
		12·8	6587	7·6	8036	7·6	9295
		45·2	6945	13·0	8323	47·2	9781
				46·6	8581		

TABLES 5–8.

	Table 5, Curve VI.	Table 6, Curve VII.	Table 7, Curve VIII.	Table 8, Curve IX.
Resistance of secondary before experiment	2·61	2·47	2·29	2·0
Temperature of secondary before experiment	722° C.	670° C.	603° C.	494° C.
Resistance of secondary after experiment	2·61	2·47	2·21	1·94
Temperature of secondary after experiment	722° C.	670° C.	573° C.	472° C.

Magnetising force.	Induction per sq. cm.	Magnetising force.	Induction per sq. cm.	Magnetising force.	Induction per sq. cm.	Magnetising force.	Induction per sq. cm.
0·075	163 125	0·075	77	0·075	68 50	0·075	54·7 35·8
0·15	305 278	0·15	162	0·15	128 108	0·15	98 75
0·30	762 726	0·3	427	0·30	307 275	0·3	245 195
0·6	4,004	0·6	1,516	0·60	908 834	0·6	742 590
2·2	8,952 8,895	2·2	9,381	2·2	9,604	2·2	9,433
7·6	10,410	7·6	11,562	7·6	11,992	7·6	12,273
47·2	11,224 11,111	47·8	12,859	50·6	14,470	53·5	15,201

At this stage the ring was allowed to cool down, and on the following day a determination was made of the curve at ordinary temperature of 9°·6 C. (Curve X.)

Magnetising force .	0·075	0·15	0·3	0·6	1·2	2·2	4·0	6·8	11·4	17·3	57·0
Induction per sq. cm.	21·6 13·0	41·1 32·0	116 93	308 273	1,482	6,912	10,341	12,410	13,640	14,255	15,623

The ring was next heated till the resistance reached about 2·4, was allowed to cool somewhat, and a curve was determined (Curve XI.) at a resistance of 1·69 to 1·64. Temperature 378° C. to 354° C.

Magnetising force .	0·075	0·15	0·3	0·6	1·2	2·2	4·0	7·6	13·1	51·7
Induction per sq. cm.	38 44	93 101	263	874	4,288	8,818	11,296	12,589	13,404	15,174

In addition to the variation of magnetisability depending on the temperature, these numbers show one or two interesting facts. Where two observations are given these are the results of successive reversals in opposite directions. After each experiment the ring was demagnetised by reversals of current ; thus currents successively diminishing in amount were passed through the primary, each current being reversed ten times. The last currents gave magnetising forces 1·2, 0·6, 0·3, 0·15, 0·075, 0·05. The inequality of successive observations is due to the residual effect of the current last applied; it is remarkable to observe how greatly this small force affects the result. In Curve XI. the first deflection was caused by a reversal of a current opposite to the last demagnetising current.

Comparing Curves X. and I. we see that the effect of working with the sample is to diminish its magnetisability for small forces, a fact which will be better brought out later.

Referring now to the temperature effects, we see that as the temperature rises the steepness of the initial part of the curve increases, but the maximum magnetisation diminishes. The coercive force, that is, the force required to completely demagnetise the material after it has been exposed to a great magnetising force, also, judging from the form of the ascending curves, diminishes greatly.

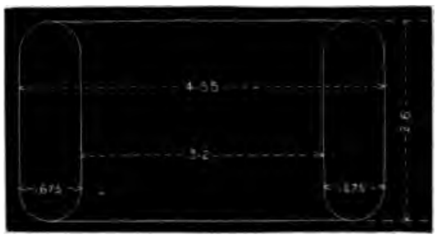
In Curves XII., XIII., and XIV. the abscissæ are temperatures, and the ordinates are induction-magnetising force, called by Sir WILLIAM THOMSON the permeability, and usually denoted by μ. These curves correspond to constant magnetising forces of 0·3, 4·0, 45·0. They best illustrate the facts which follow from these experiments. Looking at the curve for 0·3, we see that the permeability at the ordinary temperature is 367; that as the temperature rises the permeability rises slowly, but with an accelerated rate of increase; above 681° C. it increases with very great rapidity, until it attains a maximum of 11,000 at a temperature of 775° C. Above this point it diminishes with extreme rapidity, and is practically unity at a temperature of 786° C.

Regarding the iron as made up of permanently magnetic molecules, the axes of which are more or less directed to parallelism by magnetising force, we may state the facts shown by the curve by saying that rise of temperature diminishes the magnetic moment of the molecules gradually at first, but more and more rapidly as the critical temperature at which the magnetism disappears is approached, but that the facility with which the molecules have their axes directed increases with rise of temperature at first slowly, but very rapidly indeed as the critical temperature is approached.

Whitworth's Mild Steel.—This sample was supplied to me by Sir JOSEPH WHITWORTH and Co., who also supplied me with the following analysis of its composition:—

	C	Mn	S	Si	P
Per cent. . .	·126	·244	·014	·038	·047

The dimensions of the ring were as shown in the accompanying sketch.



The area of section of the ring is 1·65 sq. cm. The area of the middle line of the secondary coil is estimated to be 2·32 sq. cms.

The secondary coil had 56, the primary 98, convolutions.

The resistance of the secondary and leads was 0·81 at 12° C.

The ring was at once raised to a temperature at which it ceased to be magnetic; with a magnetising force of 32·0, the total induction was observed to be 80·8, giving the value of the permeability 1·12.

The insulation resistance between the primary and the secondary was observed to be 12,000 ohms.

The ring was now allowed to cool very slowly; at resistance of 3·00, corresponding to a temperature of 723° C., the ring was non-magnetic; at 2·99, corresponding to 720° C., it was distinctly magnetic.

The following five series of observations were made at descending temperatures, the means of two observations being in each case given; the sample was demagnetised by reversals after each experiment:—

TABLES 9–13.

Table 9, Curve XV.		Table 10, Curve XVI.		Table 11, Curve XVII.		Table 12, Curve XVIII.		Table 13, Curve XIX.	
Resistance at beginning of experiment	2·99	2·71		2·31		1·80		0·812	
Temperature at beginning of experiment	721° C.	630° C.		500° C.		333° C.		12° C.	
Resistance at end of experiment	2·95	2·75		2·245		1·80		0·812	
Temperature at end of experiment	708° C.	645° C.		478° C.		333° C.		12° C.	
Magnetising force.	**Induction per sq. cm.**	**Magnetising force.**	**Induction per sq. cm.**	**Magnetising force.**	**Induction per sq. cm.**	**Magnetising force.**	**Induction per sq. cm.**	**Magnetising force.**	**Induction per sq. cm.**
0·075	607	0·075	140	0·075	77	0·075	73	0·075	19
0·15	1214	0·15	295	0·15	161	0·15	125	0·15	48
0·3	2031	0·3	1,098	0·30	396	0·30	293	0·3	119
0·6	2698	0·6	4,175	0·60	1,847	0·6	813	0·6	312
1·2	3181	1·2	6,163	1·2	5,217	1·2	4,552	0·9	884
2·2	3607	2·1	8,122	2·1	7,642	2·1	7,840	1·7	5,087
7·6	4118	7·5	10,900	7·7	11,586	7·4	12,232	3·3	9,535
36·9	4800	38·0	12,074	40·4	14,816	42·6	15,180	6·1	12,387
								10·7	13,991
								45·0	16,313

The following experiment is instructive, as showing a phenomenon which constantly recurs, namely, that after not quite perfect demagnetisation, as above described, the first kick of the galvanometer being in the same direction as the last magnetising force, the first kick is very materially greater than the reverse kick for small magnetising forces, is somewhat less for medium forces, and about the same for great forces. I have no explanation of this to offer.

The ring was heated until the resistance of the secondary coil was about [illegible], corresponding to a temperature of 529° C. Currents successively diminishing in amount were then passed through the primary, each current being reversed [illegible] times. The last currents gave magnetising forces 1·2, 0·6, 0·3, 0·15, [illegible], and [illegible], the intention being to demagnetise the sample. The ring was allowed to cool till the resistance of secondary was [illegible], corresponding to a temperature of [illegible] C. The following series of observations was made, the first kick was in each case produced by a reversal of current from the direction of the last demagnetising current, the second kick by a reversal in the opposite sense.

Table [illegible]

Magnetising force.	[illegible]	[illegible]
[illegible]	[illegible] [illegible]	[illegible]
[illegible]	[illegible] [illegible]	
[illegible]	[illegible] [illegible]	
[illegible]	[illegible] [illegible]	
[illegible]	[illegible] [illegible]	[illegible]
[illegible]	[illegible] [illegible]	
[illegible]	[illegible] [illegible]	
[illegible]	[illegible] [illegible]	[illegible]
[illegible]	[illegible] [illegible]	
[illegible]	[illegible] [illegible]	

The [illegible]

The [illegible]

given, as they illustrate further the point last mentioned. In the first two series only one kick was taken, to save time.

TABLE 15.

Magnetising force.	Galvanometer kick.	Resistance in circuit.	Induction per sq. cm.	Resistance of coil.	Temperature. ° C.
				3·025	733
0·075	64·5	3·455	61		
0·15	287·0	3·454	273		
0·3	244·0	13·453	903		
0·6	199·0	23·452	1286		
1·2	241·0	23·451	1554		
2·0	290·0	23·450	1870		
				3·019	731

TABLE 16.

Magnetising force.	Galvanometer kick.	Resistance in circuit.	Induction per sq. cm.	Resistance of coil.	Temperature. ° C.
				3·018	730
0·075	133	13·448	492		
0·15	305	13·448	1128		
0·3	302	23·448	1948		
0·6	91	103·449	2584		
1·2	95	103·449	2698		
37·4	137	103·449	2891		
				3·019	731

TABLE 17.

Magnetising force.	Galvanometer kick.	Resistance in circuit.	Induction per sq. cm.	Resistance of coil.	Temperature. ° C.
				3·018	730
0·075	214	13·448	792		
0·075	149	13·447	551		
0·075	145	13·445	536		
0·6	102	103·444	2897		
38·4	150	103·442	4260		
				3·012	729

TABLE 18.

Magnetising force.	Galvanometer kick.	Resistance in circuit.	Induction per sq. cm.	Resistance of coil.	Temperature. ° C.
0·075	229	13·44	847	3·01	728
0·075	155	13·44	573		
0·3	89	103·44	2528		
0·075	154	13·43	570		
0·3	96	103·43	2726		
1·2	132	103·43	3749		
7·3	156	103·43	4430		
37·2	181	103·43	5155	3·0	725

The sample was again heated until it became non-magnetic. A magnetising force of 0·075 was applied by a current in the primary during heating, and was taken off entirely by breaking the primary circuit when the sample was non-magnetic. The sample was allowed to cool to the ordinary temperature of the room, 12° C., and the following series of observations was made, the first reversal being from the direction of the force of 0·075 which had been applied when the ring was heated.

TABLE 19.

Magnetising force.	Galvanometer kick.	Resistance in circuit.	Induction per sq. cm.
0·075	120	1·244	41
,,	87	,,	30
0·15	249	,,	85
,,	210	,,	72
0·3	62	11·244	193
,,	58	,,	179
0·6	178	,,	550
,,	154	,,	476
1·2	59	101·244	1,590
,,	55	,,	
2·2	227	,,	6,300
,,	223	,,	
4·0	357	,,	10,080
,,	363	,,	
7·3	226	201·24	12,553
,,	228	,,	
12·1	252	,,	13,991
,,	254	,,	
18·8	268	,,	14,876
,,	270	,,	
25·9	275	,,	15,318
,,	278	,,	
42·4	293	,,	16,148
,,	291	,,	

In addition to the fact that the first kick is largest for small forces, this shows, I think, that heating a sample above the critical temperature does not destroy its remembrance of magnetic force applied before and during heating. It would seem that the molecules of iron lie as they were placed by the magnetising force even after their magnetisation has disappeared by heating, and that when they become again capable of magnetisation by cooling the effect of the position of their axes is again apparent.

The ring was now demagnetised by reversed currents, but these were successively reduced to a force of 0·0075, instead of 0·05 as heretofore, and the following series of observations was made :—

TABLE 20.

Magnetising force.	Galvanometer kick.	Resistance in circuit.	Induction per sq. cm.
0·075	77·0	1·24	27
,,	79·0		
0·15	180·0	,,	62
,,	183·0		
0·3	52·0	11·24	161
,,	52·5		
0·6	126·0	,,	389
,,	125·0		
1·2	47·5	101·24	1,314
,,	47·0		
2·1	222·0	,,	6,172
,,	223·0		
4·0	361·0	,,	10,119
,,	366·0		
7·5	228·0	201·24	12,636
,,	228·0		
12·3	253·0	,,	13,991
,,	252·0		
18·8	270·0	,,	14,903
,,	269·0		
25·1	276·5	,,	15,277
,,	276·0		
42·2	291·0	,,	16,037
,,	289·5		

This series shows two things : first, when the demagnetising force is taken low enough there is no asymmetry in the galvanometer kicks ; second, the effect of demagnetising by reverse currents is to reduce the amount of induction for low forces.

The ring was now heated to a resistance of secondary of 3·18, temperature 783° C., the ring becoming non-magnetic at 3·03, temperature 734° C. or thereabouts, a magnetising force of about 12 C.G.S. units being constantly applied. The magnetising force was then taken off and the ring was allowed to cool, and the following series was made ; the first kick being in all cases produced by reversal from the direction of the current applied during heating.

TABLE 21.

Magnetising force.	Galvanometer kick.	Resistance in circuit.	Induction per sq. cm.
0·15	28·5	11·26	87
"	28·0		
"	218·0	1·26	75
"	214·0		
0·3	66·5	11·26	205
"	66·0		
0·6	182·0	"	565
"	167·0	"	518
"	162·0	"	502
"	157·0	"	488
1·2	329·0	21·26	1,925
"	293·0	"	1,714
2·2	230·0	101 26	6,398
"	227·0		
4·0	181·0	201·26	10,000
"	179·0		
7·3	225·0	"	12,410
"	223·0		
11·6	250·0	"	13,850
"	249·0		
18·0	264·0	"	14,626
"	263·0		
28·3	274·0	"	15,180
"	274·0		
46·2	288·0	"	15,900
"	286·0		

From this table it will be observed that the induction for low forces has again increased; that the ring still recollects its state previous to heating.

The ring was again demagnetised, with currents ranging down to 0·0075, and the following series of experiments was made :—

TABLE 22.

Magnetising force.	Galvanometer kick.	Resistance in circuit.	Induction per sq. cm.
0·075	74·5	1·26	26
"	76·5		
0·15	175·0	"	62
"	180·0		
0·3	51·5	11·26	161
"	52·5		
0·6	125·0	"	389
"	125·0		
1·2	231·0	21·26	1,331
"	224·0		
2·2	223·0	101·26	6,272
"	224·0		
4·0	361·0	"	10,192
"	365·0		
7·7	224·0	201·26	12,576
"	229·0		
13·1	252·0	"	14,016
"	254·0		
20·4	266·0	"	14,847
"	269·0		
28·8	277·0	"	15,346
"	276·0		
51·7	292·0	"	16,455
"	292·0		

It will be seen that this series agrees very closely with Table 20, evidence of the general accuracy of the results.

The ring was lastly demagnetised and heated to a resistance of secondary of 3·19, temperature 787° C., under a magnetising force ·075, which was removed when the ring was at its highest temperature; the ring was cooled, and the following observations made. In this case, however, the first kick was due to a reversal from a current opposed to the current which was applied during heating.

TABLE 23.

Magnetising force.	Galvanometer kick.	Resistance in circuit.	Induction per sq. cm.
0·075	84·0	1·43	33
"	84·5		
0·15	192·0	1·43	75
"	195·0		
0·3	60·0	11·43	192
"	62·0		
0·6	153·0	"	480
"	154·0		
1·2	321·0	21·43	1,891
"	? 302·5		
2·2	239·0	101·43	6,678
"	238·0		
4·0	367·0	"	10,262
"	366·0		
7·3	227·0	201·43	12,576
"	226·0		

This shows doubtfully the effects of magnetisation previous to heating, but, comparing it with Table 10, it completes the proof that the asymmetry was in that case due to the magnetising force, which had been stopped when the ring was non-magnetic.

I have dwelt at length on these experiments because they show two things: first, that heating until the ring becomes non-magnetic does not clear the material of the magnetism when it is afterwards cooled; second, that demagnetisation by reversal does not bring back the material to its virgin state, but leaves it in a state in which the induction is much less for small forces and greater for medium forces than a perfectly demagnetised ring would show.

To return to the effects of temperature, Curves XX., XXI., and XXII. show the relation of permeability to temperature for magnetising forces 0·3, 4, and 30.

It will be seen that they present the same general characteristics as the curves for wrought iron. The irregularities are due in part, no doubt, to the dependence of the observations on previous operations on the iron; in part, to uncertainty concerning the exact agreement of temperature of iron and temperature of secondary coil.

Whitworth's Hard Steel.—This sample was supplied to me with the following analysis of its composition:—

	C	Mn	S	Si	P
Per cent.	962	·212	·017	·164	·016

The dimensions of the ring were exactly the same as the mild steel.

The secondary coil had 56, the primary 101, convolutions.

The resistance of the secondary and leads was ·732 at 8° C.

Experiments were first made with the ring cold, partly to show the changes caused by annealing, and partly to examine the behaviour of the virgin steel.

The first series given in Table 24 was made on the virgin steel. The actual elongations on the galvanometer are given, as they afford a better idea of the probable errors of observation. These show that for very small forces the first and second elongations are practically equal, but that for forces between 1 C.G.S. unit and 14 C.G.S. units the first elongation is very materially greater than the later elongations.

The ring was now demagnetised, with magnetising forces ranging down to 0·0045, and the experiment was repeated, the results being shown in Table 25. Comparing them with Table 24, we see that the effect has been to reduce the inductions for low forces, as was the case with mild steel, and to render the kicks practically equal, whether they arise from the current first applied or subsequently applied.

The ring was not now demagnetised; the last current, giving a magnetising force 35·36, was removed, but not reversed, and a series of experiments made, the first reversal in each case being from the direction of the current of 35·36 last applied. The results are given in Table 26.

TABLES 24–26.

Table 24, Curves XXIII. and XXIV.

Magnetising force.	Galvanometer kick.	Resistance in circuit.	Induction per sq. cm.
0·065	27·0 28·0	1·164	9
0·13	57·5 57·5	”	18
0·26	116·0 117·5	”	37
0·52	234·0 236·0	”	75
1·04	56·5 55·5	11·164	172
2·08	123·5 117·5	”	379 361
	116·0 116·5	”	356
3·74	302·0	”	927
	276·0	”	847
	270·0	”	829
	262·0	”	804
	261·5 261·5	”	802
	258·5 257·0	”	792
6·66	93·5 89·5	101·16	2,543
	87·0 85·0	”	2,391
	85·5 83·5	”	2,349
10·61	250·5 247·0	”	6,922
	234·5 226·0	”	6,394
	225·0 230·0	”	6,338
15·18	168·0 171·0	201·16	9,346
	173·0 169·0	”	9,456
20·28	190·0 197·0 194·0 193·0	”	10,728
35·88	226·0 228·0 227·0 227·0	”	12,553

Table 25, Curve XXV.

Magnetising force.	Galvanometer kick.	Resistance in circuit.	Induction per sq. cm.
0·065	26·5 26·0	1·164	8
0·13	55·0 53·5	”	17
0·26	106·0 106·0	”	34
0·52	213·0 213·0	”	68
1·04	51·5 51·5	11·164	158
2·08	108·0 105·0	”	328
3·74	241·0 240·0	”	740
6·66	80·0 78·0	101·164	2,196
10·82	223·0 226·0	”	6,227
15·18	163·0 164·0	201·164	9,069
21·0	193·0 197·0	”	10,783
35·36	226·0 226·0	”	12,498

Table 26.

Magnetising force.	Galvanometer kick.	Resistance in circuit.	Induction per sq. cm.
0·065	25·0 15·0	1·164	8
0·26	111·5	”	36
	57·0	”	18
	59·5	”	19
	57·5	”	18
3·95	311·5	11·164	956
	140·5	”	431
	144·0 145·0	”	445
	136·0 132·0	”	411
	135·0 134·0	”	414
11·44	290·0	101·16	8,062
	257·0	”	7,145
	254·0 251·0	”	7,033
	252·0 250·0	”	6,950
16·43	175·0 173·0	201·16	9,622
	172·0 172·0	”	9,512

The ring was now thoroughly demagnetised and heated till it became non-magnetic. It was then cooled slowly, and the following observations were made :—

TABLES 27–33.

	Table 27.	Table 28, Curve XXVIII.	Table 29, Curve XXIX.	Table 30, Curve XXX.
Resistance at beginning of experiment	2·805	2·795	2·77	2·74
Temperature at beginning of experiment	687° C.	682° C.	674° C.	664° C.
Resistance at end of experiment	2·795	2·77	2·74	2·72
Temperature at end of experiment	682° C.	674° C.	664° C.	657° C.

Magnetising force.	Induction per sq. cm.	Magnetising force.	Induction per sq. cm.	Magnetising force.	Induction per sq. cm.	Magnetising force.	Induction per sq. cm.
0·065	9	0·065	24	0·065	45	0·065	43
0·13	21	0·13	53	0·26	197	0·26	184
0·26	61	0·26	123	1·04	873	1·04	..
		0·52	291	3·22	3578	..	1087
		1·04	821	8·32	4629	3·32	3621
		2·08	1595	18·2	5396	8·32	4771
		3·33	2215			18·51	5652
		5·51	2868				
		8·32	3301				

	Table 31, Curve XXXI.	Table 32, Curve XXXII.	Table 33, Curve XXXIII.
Resistance at beginning of experiment	2·72	2·43	2·35
Temperature at beginning of experiment	657° C.	561° C.	534° C.
Resistance at end of experiment	2·73	2·35	2·28
Temperature at end of experiment	661° C.	534° C.	511° C.

Magnetising force.	Induction per sq. cm.	Magnetising force.	Induction per sq. cm.	Magnetising force.	Induction per sq. cm.
0·065	42	0·065	27	0·065	24
0·26	171	0·26	112	0·26	103
1·04	1010	1·04	539	1·04	516
3·22	3706	3·22	..	2·08	1406
..	..	..	3396	3·43	3243
8·32	4885	8·53	5377	8·53	5414
19·8	5708	21·2	6707	21·53	6768

When cold, the resistance of the secondary coil and leads was 0·768 ; in calculating the temperatures, it is assumed that the cold resistance is 0·768. It is obvious that there is here considerable uncertainty concerning the actual temperatures, owing to the changes in the condition of the wire due to its oxidation.

The following series was next made, the mean results being given in

TABLE 34, Curve XXVI.

Magnetising force.	Galvanometer kick.	Resistance in circuit.	Induction per sq. cm.
0·065	29	1·198	10
0·13	58	1·198	19
0·26	120	1·198	40
0·52	251	1·198	83
1·04	66	11·198	203
3·74	170	21·198	991
6·03	159	101·2	4,420
9·78	283	,,	7,867
13·94	176	201·2	9,733
15·81	187	,,	10,341
22·67	211	,,	11,668

The ring was now demagnetised, and another series of determinations was made, the mean results being given in

TABLE 35, Curve XXVII.

Magnetising force.	Galvanometer kick.	Resistance in circuit.	Induction per sq. cm.
0·065	26	1·198	9
0·13	54	,,	18
0·26	111	,,	37
0·52	236	,,	78
1·04	60	11·198	185
2·08	132	,,	407
3·74	327	,,	1,007
6·24	130	101·2	3,614
9·78	265	,,	7,367
13·10	168	201·2	9,290
15·7	187	,,	10,341
22·67	211	,,	11,668

Comparing Curves XXV. and XXVII., we see the effect of annealing the iron to be to increase its permeability. Comparing Curves XXVI. and XXVII. we see the effect of demagnetising by reversed currents. Curve XXXIV. shows the relation of permeability to temperature for a force of 1·5.

Manganese Steel.—The sample of this steel was given to me by Mr. HADFIELD, who also supplied me with the following two analyses of the sample :—

TABLE 36.

Resistance of secondary and leads.	Temperature.	Total induction.	Permeability.
	° C.		
0·77	9·0 (room)	67·7	1·08
2·20	476·0	93·1	1·95
3·00	757·0	101·7	2·19
3·23	816·0	71·7	1·45
3·30	841·0	72·0	1·42
3·14	787·0	72·0	1·38
2·80	674·0	92·3	1·99
0·79	8·8 (room)	94·5	1·99

As the changes in the temperature were in this case made somewhat rapidly, the temperature of the ring lags behind the temperature of the copper.

These show: first, that at no temperature does this steel become at all strongly magnetic; second, that at a temperature of a little over 750° C. there is a substantial reduction of permeability; third, that above this temperature the substance remains slightly magnetic; fourth, that annealing somewhat increases the permeability of the material.

Resistance of Iron at High Temperatures.

These experiments were made in a perfectly simple way. Coils of very soft iron wire, pianoforte wire, manganese steel wire, and copper wire were insulated with asbestos, were bound together with copper wire so placed as to tend by its conductivity for heat to bring them to the same temperature, and were placed in an iron cylindrical box for heating in a furnace. They were heated with a slowly rising temperature, and the resistance of the wires was successively observed, and the time of each observation noted. By interpolation the resistance of any sample at any time intermediate between the actual observations could be very approximately determined. The points shown in Curves XXXV., XXXVI., XXXVII., were thus determined. In these curves the abscissæ represent the temperatures, and the ordinates the resistance of a wire having unit resistance at 0° C. Curve XXXVII. is manganese steel, which exhibits a fairly constant temperature coefficient of 0·00119; Dr. FLEMING gives 0·0012 as the temperature coefficient of this material. Curve XXXV. is soft iron; at 0° C. the coefficient is 0·0056; the coefficient gradually increases with rise of temperature to 0·019, a little below 855° C.; at 855° C. the coefficient suddenly, or at all events very rapidly, changes to 0·007. Curve XXXVI. is pianoforte wire; at 0° C. the coefficient is 0·0035; the coefficient increases with rise of temperature to 0·016, a little below 812° C.; at 812° C. the coefficient suddenly changes to 0·005. The actual values of the coefficients above the points of change must be regarded as somewhat uncertain, because the range of temperature is small, and because the accuracy of the results may be affected by the possible oxidation of the copper. The temperatures of change of coefficient, 855° C. and 812° C., are higher than any critical temperature I had observed. It was necessary to determine the critical temperatures for magnetisation for the particular samples. A ring was formed of the respective wires, and was wound with a primary and secondary coil, and the critical temperature was determined as in the preceding magnetic experiments: it was found to be for the soft iron 880° C., for the hard pianoforte wire 838° C. These temperatures agree with the temperatures of sudden change of resistance coefficient within the limits of errors of observation.*

Some interesting observations were made on the permanent change in the resistance at ordinary temperatures caused in the wires by heating to a high temperature. In the following table are given the actual resistances of wires at the temperature of the room :—

	Before heating.	After first heating.	Second heating.	Third heating.
Soft iron	0·629	0·624	0·72	0·735
Pianoforte wire	0·851	0·794	0·79	0·74
Manganese steel . . .	1·744	1·656	1·61	1·61

In a second experiment the resistances before heating were: soft iron 0·614, pianoforte wire 0·826; after heating, soft wire 0·643, pianoforte wire 0·72.

The effects are opposite in the cases of soft iron and pianoforte wire.

Recalescence of Iron.

Professor BARRETT has observed that, if an iron wire be heated to a bright redness and then allowed to cool, this cooling does not go on continuously, but after the wire has sunk to a very dull red it suddenly becomes brighter and then continues to cool down. He surmised that the temperature at which this occurs is the temperature at which the iron ceases to be magnetisable. In repeating Professor BARRETT's experiments, I found no difficulty in obtaining the phenomenon with hard steel wire, but I failed to observe it in the case of soft iron wire, or in the case of manganese steel wire. Although other explanations of the phenomenon have been offered, there can never, I think, have been much doubt that it was due to the liberation of heat owing

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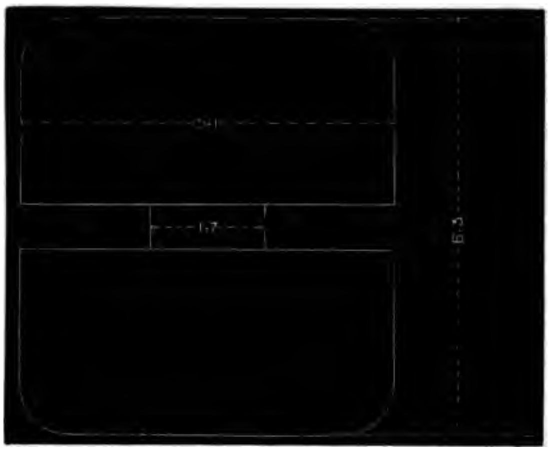
* [*Note added July* 2, 1889.—Sir JOSEPH WHITWORTH and Co. have kindly analysed these two wires for me, with the following results :—

	C	Mn	S	Si	P	
Soft iron wire	·006	·289	·015	·034	·141	per cent.
Pianoforte wire . . .	·724	·157	·010	·132	·030	„]

to some change in the material, and not due to any change in the conductivity or emissive power. This has indeed been satisfactorily proved by Mr. NEWALL.[*] My method of experiment was exceedingly simple. I took a cylinder of hard steel 6·3 cms. long and 5·1 cms. in diameter, cut a groove in it, and wrapped in the groove a copper wire insulated with asbestos.



The cylinder was wrapped in a large number of coverings of asbestos paper to retard its cooling; the whole was then heated to a bright redness in a gas furnace; was taken from the furnace and allowed to cool in the open air, the resistance of the copper wire being, from time to time, observed. The result is plotted in Curve XXXVIII., in which the ordinates are the logarithms of the increments of resistance above the resistance at the temperature of the room, and the abscissæ are the times. If the specific heat of the material were constant, and the rate of loss of heat were proportional to the excess of temperature, the curve would be a straight line. It will be observed that below a certain point this is very nearly the case, but that there is a remarkable wave in the curve. The temperature was observed to be falling rapidly, then to be suddenly retarded, next to increase, then again to fall. The temperature reached in the first descent was 680° C. The temperature to which the iron subsequently ascends is 712° C. The temperature at which another sample of hard steel ceased to be magnetic, determined in the same way by the resistance of a copper coil, was found to be 690° C. This shows that, within the limits of errors of observation, the temperature of recalescence is that at which the material ceases to be magnetic. This curve gives the material for determining the quantity of heat liberated. The dotted lines in the curve show the continuation of the first and second parts of the curve; the horizontal distance between these approximately represents the time during which the material was giving out heat without fall of temperature. After the bend in the curve, the temperature is falling at the rate of 0·21° C. per second. The distance between the two straight parts of the curve is 810 seconds. It follows that the heat liberated in recalescence of this sample is 173 times the heat liberated when the iron falls in temperature 1° C. With the same sample, I have also observed an ascending curve of temperature. There is, in this case, no reduction of temperature at the point of recalescence, but there is a very substantial reduction in the rate at which the temperature rises.*

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[*] 'Phil. Mag.,' June, 1888.

A similar experiment was made with a sample of wrought iron substantially the same as the wrought iron ring first experimented upon. The result is shown in Curve XXXIX. It will be seen that there is a great pause in the descent of this curve at a temperature of 820° C., but that the curve does not sensibly rise. This shows why soft iron apparently does not recalesce. Determining the heat liberated in the same way as before, we find the temperature falling after the bend in the curve at the rate of $0^{\circ}\cdot217$ C. per second. The distance between the two straight parts is 960 seconds. Hence, heat liberated in recalescence is 208 times the heat liberated when the iron falls 1° C. in temperature. The temperature at which a sample ordered at the same time and place ceased to be magnetic was 780° C. Comparing this result with that for hard steel, we see that the quantity of heat liberated is substantially the same, but that in the case of the soft iron there is no material rise of temperature.†

[* *Note added 2nd July*, 1889.—Some remarks of Mr. TOMLINSON'S suggested that it might be possible that there would be no recalescence if the iron were heated but little above the critical point. To test this, I repeated the experiment, heating the sample to 765° C., very little above the critical point. Curve XXXVIIIA. shows the result. From this it will be seen that the phenomenon is substantially the same whether the sample is heated to 988° C. or to 765° C.]

[† *Note added 2nd July*, 1889.—In order to complete the proof of the connexion of recalescence and the disappearance of magnetism, a block of manganese steel was tried in exactly the same way as the blocks of hard steel and of iron. The result is shown in Curve XL., from which it will be seen there is no more bend in the curve than would be accounted for by the presence of a small quantity of magnetic iron, such a quantity as one would expect from the magnetic results, supposing the true alloy of manganese and iron to be absolutely non-magnetic.]

XV. *The Diurnal Variation of Terrestrial Magnetism.*

By Arthur Schuster, *F.R.S., Professor of Physics in Owens College. With an Appendix by* H. Lamb, *F.R.S., Professor of Mathematics in Owens College.*

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I. *Introduction.*

In the year 1839 Gauss published his celebrated Memoir on Terrestrial Magnetism, in which the potential on the Earth's surface was calculated to 26 terms of a series of surface harmonics. It was proved in this Memoir that, if the horizontal components of magnetic force were known all over the Earth, the surface potential could be derived without the help of the vertical forces, and it is well known now how these latter can be used to separate the terms of the potential which depend on internal from those which depend on external sources. Nevertheless Gauss made use of the vertical forces in his calculations of the surface potential in order to ensure a greater degree of accuracy. He assumed for this purpose that magnetic matter was distributed through the interior of the Earth, and mentions the fair agreement between calculated and observed facts as a justification of his assumption. In the latter part of the Memoir it was suggested that the same method should be employed in the investigation of the regular and secular variations.

The use of harmonic analysis to separate internal from external causes has never been put to a practical test, but it seems to me to be especially well adapted to enquiries on the causes of the periodic oscillations of the magnetic needle.

If the magnetic effects can be fairly represented by a single term in the series of harmonics as far as the horizontal forces are concerned, there should be no doubt as to the location of the disturbing cause, for the vertical force should be in the opposite direction if the origin is outside from what it should be if the origin is inside the Earth. As the expression for the potential contains in one case the distance from the Earth's centre in the numerator, in the other case in the denominator, and as the vertical force depends on the differential coefficient with regard to distance from the Earth's centre, each single term in the series is of opposite sign according to the location of the cause; but what is true for each single term need not be true for the sum of the series. By a curious combination of terms the vertical forces might possibly be of the same sign, on whichever of the two hypotheses it is calculated. In any case, however, the differences between the two results will be of the same order of magnitude as the vertical force itself. If it is then a question simply of deciding whether the cause is outside or inside, without taking into account a possible combination of both causes, the result should not be doubtful, even if we have only an approximate knowledge of the vertical forces.

Two years ago I showed that the leading features of the horizontal components for diurnal variation could be approximately represented by the surface harmonic of the second degree and first type, and that the vertical variation agreed in direction and phase with the calculation on the assumption that the seat of the force is outside the Earth. The agreement seemed to me to be sufficiently good to justify the conclusion that the greater part of the variation is due to causes outside the Earth's surface. Nevertheless, it seemed advisable to enter more fully into the matter, as in the first approximate treatment of the subject a number of important questions had to be left untouched. I now publish the results of an investigation which has been carried as far as the observations at my disposal have allowed me to do. My original conclusions have been fully confirmed, and some further information has been obtained which I believe to be of importance. The results of the calculation point not only to an external source, but to an additional internal source, standing in fixed relationship to the external cause. This we might have expected. A varying potential due to external causes must be accompanied by currents induced in the Earth's body, which, in turn, must affect the magnetic needle. The phase of these currents and their magnitude lead us to form definite conclusions on the average conducting power of the Earth, and it will be seen that there is strong evidence that the average conductivity is very small near the surface, but must be greater further down. In this part of the investigation I had much assistance from my colleague, Professor LAMB.

I hope that the results obtained in this paper may induce the heads of magnetic observatories to consider the suggestions which I have made at the end of it, as their adoption would very materially assist further investigations.

I had, in the first place, to fix on a year for which we possess as complete magnetic records as possible. The phase of the variation of horizontal force changes sign in a latitude not far removed from that of Lisbon, and it seemed to me, therefore, essential that the excellent observations there made by Sen. JOS. CAPELLO should be made use of. The observations are published as far as 1872, and I had to take a year therefore anterior to this. It seemed also desirable to make use of the St. Petersburg observations, as it is the most northerly station for which we have records extending over a period of years, and as Mr. H. WILD's well-known skill gives special value to the observations made under his direction. The obseravtions (continued since 1878 in Paulowsk) were interrupted in 1871 and 1872, and we have to go back, therefore, as far as 1870 if we want to utilise the St. Petersburg and Lisbon observations simultaneously. As far as the horizontal components are concerned, we also possess good records of 1870 at Greenwich and Bombay. Four stations are sufficient to find to the necessary accuracy the potential on the surface of the Earth, but it would be of advantage if in future similar investigations a greater number of stations could be utilised.

The observations are published in very different form by the various observatories. The units of force at Bombay and St. Petersburg are the Gaussian unit millimeter-milligram-second. At Lisbon it is the foot-grain-second. At Greenwich the variations are given in terms of the whole vertical and horizontal force. At Bombay, moreover, the observations are given not for a certain hour, but for a time which varies with different instruments between 12 and 19 minutes past each hour. For this there was some reason originally, but at present it would be far better if Bombay would adopt the practice of other observatories. The daily variations had, in the first place, to be all reduced to C.G.S. units, and, further, instead of variations in declination and horizontal force, we had to find the components of the periodic force towards the geographical North and West. I am much indebted to Mr. Wm. Ellis, of the Greenwich Observatory, for the help he has given me in the reduction of the observations to a form which I could use in my calculations. A good many of the computations were done under his direct superintendence, and much time and trouble was saved me in consequence.

The daily variation of declination and horizontal force was expressed in the form

$$a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + a_3 \cos 3t + b_3 \sin 3t + a_4 \cos 4t + b_4 \cos 4t,$$

where t represents astronomical time. The summer months, April to September, were treated separately from the winter months, October to March. The unit of force, for convenience's sake, was taken as 10^{-6} C.G.S.

Tables I. and II. give the coefficients which were calculated according to a well-known method from the original observations.

Tables III. and IV. give the same coefficients reduced to forces directed to the geographical North and West, instead of to the magnetic North and West.

Before showing how, with the help of these coefficients, the surface potential can be calculated, I must deduce a few formulæ which will be used hereafter.

Table I.—Force to Magnetic West.

Coefficients in the expansion

$$a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + a_3 \cos 3t + b_3 \sin 3t + a_4 \cos 4t + b_4 \sin 4t.$$

The unit of force is 1 C.G.S. × 10^{-6}; t being astronomical time.

	Bombay.		Lisbon.		Greenwich.		St. Petersburg.	
	Summer.	Winter.	Summer.	Winter.	Summer.	Winter.	Summer.	Winter.
a_1	+ 64·3	+ 6·3	+ 125·5	+ 121·5	+ 168·0	+ 154·5	+ 109·0	+ 95·7
b_1	+ 116·5	+ 33·2	+ 213·7	+ 132·6	+ 192·3	+ 113·5	+ 219·4	+ 96·7
a_2	+ 134·3	+ 6·5	+ 126·1	+ 53·6	+ 129·1	+ 32·7	+ 104·2	− 12·8
b_2	+ 59·5	+ 23·2	+ 152·2	+ 111·1	+ 114·6	+ 84·0	+ 113·2	+ 69·6
a_3	+ 103·5	+ 40·4	+ 75·1	+ 51·0	+ 65·8	+ 38·4	+ 37·9	+ 21·8
b_3	+ 7·1	+ 7·6	+ 56·2	+ 51·0	+ 42·5	+ 30·6	+ 59·2	+ 28·9
a_4	+ 15·5	+ 33·1	+ 13·1	+ 30·7	+ 9·3	+ 21·3	+ 2·8	+ 6·2
b_4	+ 18·7	− 9·6	− 5·9	+ 20·3	+ 2·6	+ 17·6	+ 10·9	+ 13·7

TABLE II.—Force to Magnetic North.

Coefficients in the expansion

$$a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + a_3 \cos 3t + b_3 \sin 3t + a_4 \cos 4t + b_4 \sin 4t.$$

The unit of force is 1 C.G.S. × 10^{-6}; t being astronomical time.

	Bombay.		Lisbon.		Greenwich.		St. Petersburg.	
	Summer.	Winter.	Summer.	Winter.	Summer.	Winter.	Summer.	Winter.
a_1	+ 386·9	+ 332·5	− 124·2	− 78·6	− 213·7	− 117·4	− 257·9	− 111·8
b_1	+ 5·7	+ 38·2	+ 71·5	− 15·0	+ 154·0	+ 22·8	+ 150·2	− 5·8
a_2	+ 152·7	+ 125·0	− 28·3	− 54·8	− 112·6	− 87·5	− 149·1	− 69·0
b_2	− 11·8	+ 7·3	+ 43·0	− 0·9	+ 78·9	+ 24·6	+ 45·7	− 12·8
a_3	+ 41·1	+ 39·6	+ 23·1	− 2·3	− 2·3	− 16·4	− 15·8	− 18·4
b_3	− 41·8	− 31·4	+ 19·0	+ 19·0	+ 27·1	+ 27·1	+ 39·7	+ 22·9
a_4	− 4·3	+ 15·3	+ 12·2	+ 6·9	+ 12·1	+ 2·0	− 9·0	− 3·8
b_4	− 13·7	− 21·1	− 12·3	+ 18·8	+ 8·4	+ 16·6	+ 3·3	+ 6·4

TABLE III.—Force to Geographical West.

Coefficients in the expansion

$$a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + a_3 \cos 3t + b_3 \sin 3t + a_4 \cos 4t + b_4 \sin 4t.$$

The unit of force is 1 C.G.S. × 10^{-6}; t being astronomical time.

	Bombay.		Lisbon.		Greenwich.		St. Petersburg.	
	Summer.	Winter.	Summer.	Winter.	Summer.	Winter.	Summer.	Winter.
a_1	+ 57·5	+ 0·4	+ 74·3	+ 86·5	+ 85·4	+ 105·4	+ 99·6	+ 91·6
b_1	+ 116·4	+ 32·5	+ 225·3	+ 119·1	+ 233·3	+ 114·6	+ 224·6	+ 96·4
a_2	+ 131·6	+ 4·3	+ 108·3	+ 31·1	+ 83·1	+ 1·0	+ 98·7	− 15·3
b_2	+ 59·7	+ 23·1	+ 157·7	+ 103·8	+ 134·6	+ 87·4	+ 114·8	+ 69·1
a_3	+ 102·8	+ 39·7	+ 78·5	+ 47·0	+ 61·1	+ 30·5	+ 37·3	+ 21·2
b_3	+ 7·8	+ 8·1	+ 59·3	+ 54·4	+ 49·2	+ 38·0	+ 60·6	+ 29·7
a_4	+ 15·6	+ 32·8	+ 16·6	+ 31·2	+ 12·9	+ 20·7	+ 2·5	+ 6·1
b_4	− 18·5	− 9·2	− 9·8	+ 25·6	+ 5·3	+ 22·2	+ 11·0	+ 13·9

TABLE IV.—Force to Geographical North.

Coefficients in the expansion

$$a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + a_3 \cos 3t + b_3 \sin 3t + a_4 \cos 4t + b_4 \sin 4t.$$

The unit of force is 1 C.G.S. × 10^{-6}; t being astronomical time.

	Bombay.		Lisbon.		Greenwich.		St. Petersburg.	
	Summer.	Winter.	Summer.	Winter.	Summer.	Winter.	Summer.	Winter.
a_1	+ 388·0	+ 332·4	− 160·2	− 116·1	− 258·1	− 162·9	− 261·6	− 115·2
b_1	+ 7·8	+ 38·8	− 7·6	− 60·4	+ 79·5	− 17·2	+ 142·2	− 9·3
a_2	+ 155·1	+ 125·1	− 70·5	− 70·1	− 149·8	− 93·4	− 152·8	− 68·5
b_2	− 10·8	+ 7·7	− 12·8	− 39·6	+ 35·2	− 5·5	+ 41·6	− 15·3
a_3	+ 42·9	+ 40·3	− 4·5	− 20·0	− 24·6	− 28·5	− 17·2	− 19·2
b_3	− 41·7	− 31·3	− 1·8	0·0	+ 11·0	+ 15·1	+ 37·6	+ 21·9
a_4	− 4·0	+ 15·9	+ 6·8	− 4·2	+ 8·2	− 5·3	− 9·1	− 4·0
b_4	− 14·0	− 21·3	− 9·5	+ 10·5	+ 7·0	+ 9·6	+ 2·9	+ 5·9

II. *Some Formulæ useful in the Analysis by Spherical Harmonics.*

In the expansion of mathematical expressions into spherical harmonics, it will often occur that we have to express powers of the cosine of an angle, or a cosine of some multiple of an angle, in terms of the differential coefficients of zonal harmonics with respect to the cosine of the argument. The necessary equations for the powers of cosines can be easily obtained by differentiation of the well-known formulæ, giving the powers of a cosine in terms of zonal harmonics. But I think it will be useful here to give the general equations which I have had to use in expressing $\cos mu$ in terms of $d^p\mathrm{P}_i/d\mu^p$, where p is any given number, $\mu = \cos u$ and P_i the zonal harmonic of degree i.

We may start from the expression

$$\cos mu = \mathrm{A}_m\mathrm{P}_m + \mathrm{A}_{m-2}\mathrm{P}_{m-2} + \ldots \mathrm{A}_i\mathrm{P}_i + \ldots \quad \ldots \quad (1),$$

where

$$\mathrm{A}_i = -(2i+1)\frac{\{m-(i-2)\}\{m-(i-4)\}\ldots(m-2)\,m^2\,(m+2)\ldots\{m+(i-2)\}}{\{m-(i+1)\}\{m-(i-1)\}\ldots(m-1)(m+1)\ldots\{m+(i+1)\}},$$

if m and i be even, and

$$\mathrm{A}_i = -(2i+1)\frac{\{m-(i-2)\}\{m-(i-4)\}\ldots(m-1)(m+1)\ldots\{m+(i-2)\}}{\{m-(i+1)\}\{m-(i-1)\}\ldots(m-2)(m+2)\ldots\{m+(i+1)\}},$$

if m and i be odd.

Both expressions are included in the general one

$$A_i = -(2i+1)\,m\frac{\{m-(i-2)\}\{m-(i-4)\}\ldots\{m+(i-2)\}}{\{m-(i+1)\}\{m-(i-1)\}\ldots\{m+(i+1)\}},$$

where successive factors of both numerator and denominator increase by 2.*

If in formula (1) we substitute for any term P_i

$$\frac{dP_{i+1}}{d\mu} - \frac{dP_{i-1}}{d\mu} = (2i+1)\,P_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

we obtain an expression for $\cos m\theta$ in terms of the first differential coefficients of the zonal harmonics; and, if in the formula so obtained we successively apply the transformation

$$\frac{d^pP_{i+1}}{d\mu^p} - \frac{d^pP_{i-1}}{d\mu^p} = (2i+1)\frac{d^{p-1}P_i}{d\mu^{p-1}},$$

we finally obtain the required expression in terms of any required differential coefficient.

I have obtained in this way the equation

$$(-1)^{p+1}\frac{\cos mu}{1.3.5\ldots(2p+1)} = A_{m+p}\,d^pP_{m+p} + A_{m+p-2}\,d^pP_{m+p-2} + \ldots A_i\,d^pP_i + \ldots \quad (3),$$

where d^pP_i is written shortly for $d^pP_i/d\mu^p$, and

$$A_i = (2i+1)\,m\frac{\{m-(i-p-2)\}\{m-(i-p-4)\}\ldots\{m+(i-p-2)\}}{\{m-(i+p+1)\}\{m-(i+p-1)\}\ldots\{m+(i+p+1)\}} \quad . \quad (4),$$

except for the last term of the series, which will be given presently (5). If $p = 0$, this expression agrees with the one previously given, and, as I shall proceed to show, if it is true for any value p, it will also be true for the value $p+1$. Assume, then, the equation (3) to hold.

The relation

$$d^{p+1}P_{(i+1)} - d^{p+1}P_{(i-1)} = (2i+1)\,d^pP_i$$

shows that the factor multiplying $d^{p+1}P_{i+1}$ in the expression for $\cos m\theta$ will depend only on A_i and on $A_{(i+2)}$ in (3). If $B_{(i+1)}$ is this factor, we have

$$B_{(i+1)} = \frac{A_i}{2i+1} - \frac{A_{i+2}}{2i+5},$$

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* In the very useful book on 'Spherical Harmonics,' by **Ferrers**, the factor m does not occur in the general expression for A_i (page 33); but, from the deduction of the formula, it is clear that the factor m must be taken twice when it occurs in the numerator, and not at all when it occurs in the denominator. In using the equation I was at first led into error by the ambiguity, and hence I believe the expression given above to be clearer.

or, by substituting A_i from (4) and A_{i+2} from the corresponding equation,

$$B_{i+1} = m \frac{\{m-(i-p-2)\}\{m-(i-p-4)\}\ldots\{m+(i-p-2)\}}{\{m-(i+p+1)\}\{m-(i-p-1)\}\ldots\{m+(i+p+1)\}}$$
$$\left[1 - \frac{\{m-(i-p)\}\{m+(i-p)\}}{\{m-(i+p+3)\}\{m+(i+p+3)\}}\right].$$

The square bracket reduces to

$$\frac{(i-p)^2-(i+p+3)^2}{\{m-(i+p+3)\}\{m+(i+p+3)\}} = -\frac{(2i+3)(2p+3)}{\{m-(i+p+3)\}\{m+(i+p+3)\}},$$

so that

$$-(-1)^{p+1}\frac{\cos m\theta}{1.3.5\ldots(2p+3)} = A_{m+p+1}\, d^{p+1}P_{m+p+1} + \ldots A_{i+1}\, d^{p+1}P_{i+1}\ldots,$$

where

$$A_{i+1} = m(2i+3)\frac{\{m-(i-p-2)\}\{m-(i-p-4)\}\ldots\{m+(i-p-2)\}}{\{m-(i+p+3)\}\{m-(i+p+1)\}\ldots\{m+(i+p+3)\}},$$

which agrees with (4) if $p+1$ is written for p, and $i+1$ for i.

The end terms require special consideration.

As only the even or only the odd zonal harmonics enter into any one series, the difference between $m+p$ and i in the equation (3) must be even; hence, the numerators in the fractional expression of (4) consist always of even numbers, whatever values m, i, or p may have.

If $(m+p)$ be odd, the zonal harmonics will all be odd, and the last term of the series will depend on d^pP_{p+1} if p be even, and d^pP_p if p be odd, for the differential coefficients of the zonal harmonics of lower degree will vanish. The expression (4) in neither case gives the correct factor, and we must substitute for it in both cases

$$(2i+1)\frac{1}{\{m-(i+p+1)\}\{m-(i+p-1)\}\ldots\{m+(i+p+1)\}} \quad . \quad . \quad (5).$$

If in (4) i is put equal to $(p+2)$, the first and last term will be equal to m; in that case the m is only taken once.

The expression (5) is easily proved, and shown to hold also if $(m+p)$ be even.

The first term of the series is included in the general expression (4).

I have deduced with the help of the equation (3) the following relations, which will be used in this paper :—

$$\cos u = \tfrac{1}{3}\frac{dP_2}{d\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A),$$

$$\cos 2u = \tfrac{4}{15}\frac{dP_3}{d\mu} - \tfrac{9}{15}\frac{dP_1}{d\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B),$$

$$\cos 3u = \tfrac{8}{35}\frac{dP_4}{d\mu} - \tfrac{3}{7}\frac{dP_2}{d\mu} \quad \ldots \ldots \ldots \quad (C),$$

$$\cos 4u = \tfrac{64}{315}\frac{dP_5}{d\mu} - \tfrac{16}{45}\frac{dP_3}{d\mu} + \tfrac{3}{35}\frac{dP_1}{d\mu} \quad \ldots \ldots \ldots \quad (D),$$

$$\cos 5u = \tfrac{128}{693}\frac{dP_6}{d\mu} - \tfrac{24}{77}\frac{dP_4}{d\mu} + \tfrac{5}{63}\frac{dP_2}{d\mu} \quad \ldots \ldots \ldots \quad (E),$$

$$\cos 6u = \tfrac{512}{3003}\frac{dP_7}{d\mu} - \tfrac{128}{455}\frac{dP_5}{d\mu} + \tfrac{4}{55}\frac{dP_3}{d\mu} + \tfrac{1}{105}\frac{dP_1}{d\mu} \quad \ldots \ldots \quad (F);$$

$$\cos u = \tfrac{1}{105}\frac{d^3P_4}{d\mu^3} \quad \ldots \ldots \ldots \quad (G),$$

$$\cos 2u = \tfrac{4}{945}\frac{d^3P_5}{d\mu^3} - \tfrac{7}{135}\frac{d^3P_3}{d\mu^3} \quad \ldots \ldots \ldots \quad (H),$$

$$\cos 3u = \tfrac{8}{3465}\frac{d^3P_6}{d\mu^3} - \tfrac{1}{55}\frac{d^3P_4}{d\mu^3} \quad \ldots \ldots \ldots \quad (I),$$

$$\cos 4u = \tfrac{64}{45045}\frac{d^3P_7}{d\mu^3} - \tfrac{16}{1755}\frac{d^3P_5}{d\mu^3} + \tfrac{7}{297}\frac{d^3P_3}{d\mu^3} \quad \ldots \ldots \ldots \quad (K),$$

$$\cos 5u = \tfrac{128}{135135}\frac{d^3P_8}{d\mu^3} - \tfrac{8}{1485}\frac{d^3P_6}{d\mu^3} + \tfrac{5}{429}\frac{d^3P_4}{d\mu^3} \quad \ldots \ldots \ldots \quad (L),$$

$$\cos 6u = \tfrac{512}{765765}\frac{d^3P_9}{d\mu^3} - \tfrac{128}{36465}\frac{d^3P_7}{d\mu^3} + \tfrac{4}{585}\frac{d^3P_5}{d\mu^3} - \tfrac{7}{1287}\frac{d^3P_3}{d\mu^3} \quad \ldots \quad (M).$$

There is another formula useful in similar investigations, which may find a place here, although I have not used it in the final reductions. It is the expression of a zonal harmonic in terms of the n^{th} differential coefficient of other zonal harmonics.

$$\begin{aligned} P_i = {} & \frac{(2i+2n+1)}{(2i+2n+1)(2i+2n-1)\ldots(2i+1)}\, d^nP_{i+n} \\ & - \frac{n}{1}\,\frac{(2i+2n-3)}{(2i+2n-1)(2i+2n-3)\ldots(2i-1)}\, d^nP_{i+n-2} \\ & + \frac{n\,.\,n-1}{1\,.\,2}\cdot\frac{(2i+2n-7)}{(2i+2n-3)\ldots(2i-3)}\, d^nP_{i+n-4} \ldots \\ & \pm \frac{2i-(2n-1)}{(2i+1)(2i-1)\ldots(2i-(2n-1)}\, d^nP_{i-n}, \end{aligned}$$

the general p^{th} term being

$$(-1)^{p-1}\frac{n.n-1\ldots n-(p-2)}{1.2\ldots p-1}\;\frac{2i+2n+5-4p}{(2i+2n+3-2p)(2i+2n+1-2p)\ldots(2i+3-2p)}\, d^nP_{i+n-2p+4}.$$

The proof is conducted exactly in the same way as for equation (3). If i is equal to or greater than $2n$, the series has its full number of terms, viz., $n+1$, otherwise the series breaks off owing to the differential coefficients vanishing.

III. *Expansion of Potential in terms of a Series of Surface Harmonics.*

We know from observation that, excepting the Arctic regions, the daily variation of the West force is nearly the same along the same circle of latitude. It is this fact which renders the present investigation possible, as comparatively few places of observation will be necessary to give us a very fair idea of the nature of the oscillation over a considerable area of the Earth's surface. If at any place we find that the daily oscillation of any one element can be expressed in the form

$$a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + \ldots,$$

when t is reckoned by local time, we may get the variation at any point of the same latitude circle by writing $t + \lambda$ for t, where λ is the longitude towards the East from some standard meridian and t now is the time of the standard meridian. At the time $t = 0$ we have then the variation of the force to geographical West in different longitudes expressed by

$$Y = a_1 \cos \lambda + b_1 \sin \lambda + a_2 \cos 2\lambda + b_2 \sin 2\lambda.$$

The coefficients will be functions of the latitude, and by expressing these functions in a series of proper form we may at once obtain an expression for the potential.

If X is the force to geographical North, Y the force to geographical West, and Z the vertical force, reckoned positive upwards, we have, putting u for the colatitude, and λ for the longitude towards the East,

$$X = \frac{dV}{a\,du}, \qquad Y \sin u = \frac{dV}{a\,d\lambda}, \qquad Z = -\frac{dV}{dr},$$

a being the Earth's radius.

If we can expand Y sin u in terms of surface harmonics our task is accomplished, and for this purpose we need only express $a_1 \sin u$, $b_1 \sin u$, &c., in a series of tesseral harmonics.

The expansion of a function of an angle in terms of the trigonometrical functions of its multiples is so easily carried out by the method of least squares, if the function is given for a regular series of submultiples of 2π, that it seemed to me to be the easiest method of proceeding to obtain first by interpolation from the observed coefficients of Y its values for equidistant circles of latitude.

To obtain a curve for each of the values a_1, b_1, a_2, b_2, as depending on the latitude, we have the following data. The values are directly observed for four points in the Northern hemisphere. Taking the potential (as it is observed to be) symmetrical on the Northern and Southern hemispheres, and fixing the period to which we apply the calculation to be the one for which the mean value of the observed summer variations in the Northern hemisphere holds good, we must put, for the West force at the same period, in the Southern half of the globe, the observed *winter* values of the corresponding Northern latitudes, reversing, however, the sign to make our expression agree with observation. We have then eight values for the different terms of Y through which we might at once proceed to draw a curve; but, making use of the observed variations of Northern force, we can also calculate the direction of the tangent of the required curve for the same eight points. This has been done as follows. Let Y be expressed as

$$\Sigma(a_n \cos n\lambda + b_n \sin n\lambda),$$

and X as

$$\Sigma(\alpha_n \cos n\lambda + \beta_n \sin n\lambda).$$

The existence of a potential implies the relation

$$\frac{d\mathrm{Y} \sin u}{du} = \frac{d\mathrm{X}}{d\lambda}$$

or

$$\frac{d\mathrm{Y}}{du} = \frac{d\mathrm{X}}{d\lambda} \operatorname{cosec} u - \mathrm{Y} \cot u,$$

from which, by substitution,

$$\Sigma\left(\frac{da_n}{du} \cos n\lambda + \frac{db_n}{du} \sin n\lambda\right)$$

$$= \Sigma[(n\beta_n \operatorname{cosec} u - a_n \cot u) \cos n\lambda - (n\alpha_n \operatorname{cosec} u + b_n \cot u) \sin n\lambda];$$

and, therefore,

$$\frac{da_n}{du} = n\beta_n \operatorname{cosec} u - a_n \cot u,$$

$$\frac{db_n}{du} = - n\alpha_n \operatorname{cosec} u + b_n \cot u.$$

These equations give the rate of change per radian; to get the rate of change per degree of colatitude we have to multiply the differential coefficients with the circular measure of one degree. The following Table gives the rate of change of the West force per degree of colatitude, calculated as explained from the North force.

Table V.

The unit of force in this Table is 10^{-6} C.G.S., and the unit of latitude is 1 degree.

	Bombay.		Lisbon.		Greenwich.		St. Petersburg.	
	Summer.	Winter.	Summer.	Winter.	Summer.	Winter.	Summer.	Winter.
da_1/du . . .	− 0·201	+ 0·714	− 1·210	− 2·561	+ 0·356	− 2·793	+ 1·951	− 3·086
db_1/du . . .	− 7·853	− 6·326	+ 0·432	+ 0·931	+ 2·118	+ 2·053	+ 2·342	+ 1·107
da_2/du . . .	− 1·184	+ 0·258	− 2·087	− 2·207	+ 0·150	− 0·330	− 0·079	− 0·605
db_2/du . . .	− 6·079	− 4·754	+ 0·948	+ 1·684	+ 5·445	+ 3·318	+ 7·185	+ 2·689
da_3/du . . .	− 2·921	− 1·968	− 1·218	− 0·658	− 0·416	+ 0·600	+ 2·805	+ 1·650
db_3/du . . .	− 2·420	− 2·277	− 0·527	+ 0·580	+ 0·988	+ 1·564	− 0·030	+ 1·112
da_4/du . . .	− 1·125	− 1·768	− 1·084	+ 0·503	+ 0·501	+ 0·622	+ 0·329	+ 0 640
db_4/du . . .	+ 0·407	− 1·117	− 0·471	+ 0·018	− 1·036	+ 0·109	+ 0·936	+ 0·137

&c., we have an equation for that part of Y which depends on $\cos\lambda$ and $\sin\lambda$. After multiplication with $\sin u$, one side of the equation contains the colatitude in form of tesseral harmonics only, and hence we obtain at once the required expansion of $V^{(1)}$. The equations are

$$Y^{(1)} = \cos\lambda\left[24{\cdot}09\frac{dP_1}{d\mu} + 35{\cdot}59\frac{dP_2}{d\mu} - 8{\cdot}45\frac{dP_3}{d\mu} + 2{\cdot}39\frac{dP_4}{d\mu} - 2{\cdot}41\frac{dP_5}{d\mu} - 1{\cdot}64\frac{dP_6}{d\mu}\right]$$

$$+ \sin\lambda\left[48{\cdot}00\frac{dP_1}{d\mu} + 81{\cdot}52\frac{dP_2}{d\mu} + 5{\cdot}66\frac{dP_3}{d\mu} - 7{\cdot}62\frac{dP_4}{d\mu} - 1{\cdot}09\frac{dP_5}{d\mu}\right.$$

$$\left. - 5{\cdot}78\frac{dP_6}{d\mu} - 1{\cdot}57\frac{dP_7}{d\mu} - 0{\cdot}34\frac{dP_7}{d\mu}\right].$$

With the help of

$$\frac{dV}{d\lambda} = Ya\sin u,$$

we have, finally,

$$- V^{(1)}/a = \cos\lambda[48{\cdot}00T_1{}^1 + 81{\cdot}52T_2{}^1 + 5{\cdot}66T_3{}^1 - 7{\cdot}62T_4{}^1$$

$$- 1{\cdot}09T_5{}^1 - 5{\cdot}78T_6{}^1 - 1{\cdot}57T_7{}^1]$$

$$+ \sin\lambda[- 24{\cdot}09T_1{}^1 + 35{\cdot}59T_2{}^1 + 8{\cdot}45T_3{}^1 - 2{\cdot}39T_4{}^1$$

$$+ 2{\cdot}41T_5{}^1 + 1{\cdot}64T_6{}^1 + 0{\cdot}34T_7{}^1] \quad . \quad . \quad [A].$$

To obtain, similarly, an expression for $V^{(2)}$, the series of coefficients a_2, b_2 were expressed in terms of $\sin u$, $\sin 2u$, &c., giving the equations

$$a_2 = 67{\cdot}0\sin u + 61{\cdot}3\sin 2u + 7{\cdot}4\sin 3u - 11{\cdot}6\sin 4u + 15{\cdot}9\sin 5u + 18{\cdot}6\sin 6u,$$

$$b_2 = 21{\cdot}5\sin u + 113{\cdot}2\sin 2u + 7{\cdot}7\sin 3u - 12{\cdot}6\sin 4u - 10{\cdot}8\sin 5u - 28{\cdot}2\sin 6u.$$

From the known expansion of $\cos pu$, in terms of the first differential coefficient of the zonal harmonics, we can obtain by differentiation an equation which will at once give us $V^{(2)}$ in the required form.

Thus, for instance, we have by Equation D, page 474,

$$\cos 4u = \tfrac{64}{315}\frac{dP_5}{d\mu} - \tfrac{16}{45}\frac{dP_3}{d\mu} + \tfrac{3}{35}\frac{dP_1}{d\mu},$$

and differentiating with respect to μ,

$$4\sin 4u\,/\sin u = \tfrac{64}{315}\frac{d^2P_5}{d\mu^2} - \tfrac{16}{45}\frac{d^2P_3}{d\mu^2} + \tfrac{3}{35}\frac{d^2P_1}{d\mu^2}.$$

By substituting this and other similar expressions, we find sin $4u$, sin $6u$ in terms of $\sin u \, d^4 P_i / d\mu^4$. This gives, by substitution and integration with respect to λ,

$$\begin{aligned} -V^{(3)}/a = & (-0{\cdot}0401\, T_8^3 + 0{\cdot}1426\, T_6^3 + 0{\cdot}2523\, T_4^3) \cos 3\lambda \\ & + (0{\cdot}0319\, T_8^3 - 0{\cdot}0936\, T_6^3 - 0{\cdot}5577\, T_4^3) \sin 3\lambda \\ & + (0{\cdot}00033\, T_9^4 + 0{\cdot}00279 T_7^4 - 0{\cdot}00411\, T_5^4) \cos 4\lambda \\ & + (-0{\cdot}00075\, T_9^4 + 0{\cdot}00078\, T_7^4 - 0{\cdot}01978\, T_5^4) \sin 4\lambda. \end{aligned}$$

We have obtained thus finally, 38 coefficients in the expansion of V/a, which, for the sake of reference, I collect into Table VII.

In this Table C_n^m is the coefficient of $T_n^m \cos m\lambda$.

S_n^m „ „ $T_n^m \sin m\lambda$.

TABLE VII.

The unit of force is C.G.S. 10^{-6}.

C_1^1	− 48·00	C_2^2	− 2·948	C_4^3	− 0·252
C_2^1	− 81·52	C_3^2	− 7·936	C_6^3	− 0·143
C_3^1	− 5·66	C_4^2	− 0·630	C_8^3	+ 0·040
C_4^1	+ 7·62	C_5^2	− 0·341		
C_5^1	+ 1·09	C_6^2	+ 0·200	C_5^4	+ 0·00411
C_6^1	+ 5·78	C_7^2	+ 0·407	C_7^4	− 0·00279
C_7^1	+ 1·57			C_9^4	− 0·00033
S_1^1	+ 24·09	S_2^2	+ 10·764	S_4^3	+ 0·578
S_2^1	+ 35·59	S_3^2	+ 4·715	S_6^3	+ 0·094
S_3^1	− 8·45	S_4^2	− 0·214	S_8^3	− 0·032
S_4^1	+ 2·39	S_5^2	− 0·731		
S_5^1	− 2·41	S_6^2	+ 0·293	S_5^4	+ 0·01978
S_6^1	− 1·64	S_7^2	+ 0·264	S_7^4	− 0·00078
S_7^1	− 0·34			S_9^4	+ 0·00075

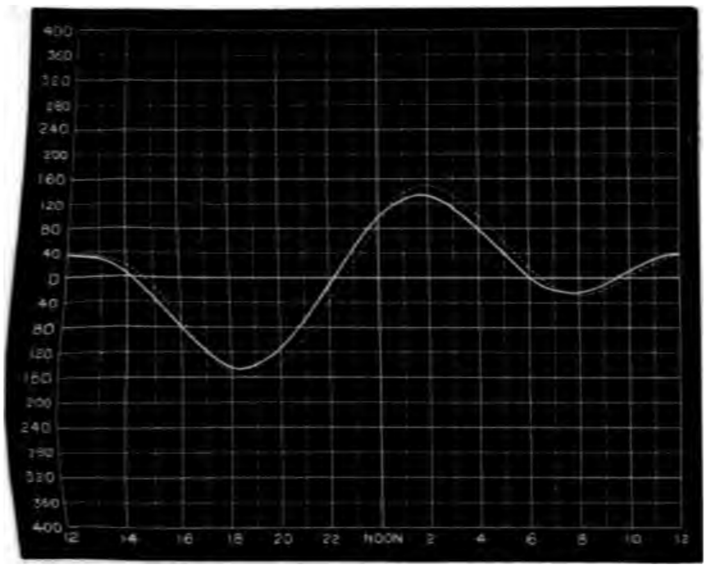
In order to show how far these numbers correctly represent the forces which have been made use of in their computation I have calculated backwards from the potential the force to geographical West. Table VIII. exhibits the results, showing by comparison the coefficients calculated from the formulæ for the potential with those obtained directly by observation. It will be seen that the agreement is satisfactory, except for the coefficients a_3 and b_3.

In figs. 1, 2, 3, and 4 the curves for the mean of the year are plotted. In these curves only the first four terms $a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t$ have been used. The numbers from which the curves are plotted are given in Table IX.

The curves are seen to be almost identical, and having, therefore, obtained an expression for the potential which correctly represents the observed West force, we may turn to the main object of this inquiry and calculate the vertical forces.

The calculation of the various components from the potential involves the knowledge of the differential coefficients of zonal harmonics and of the tesseral harmonics for the

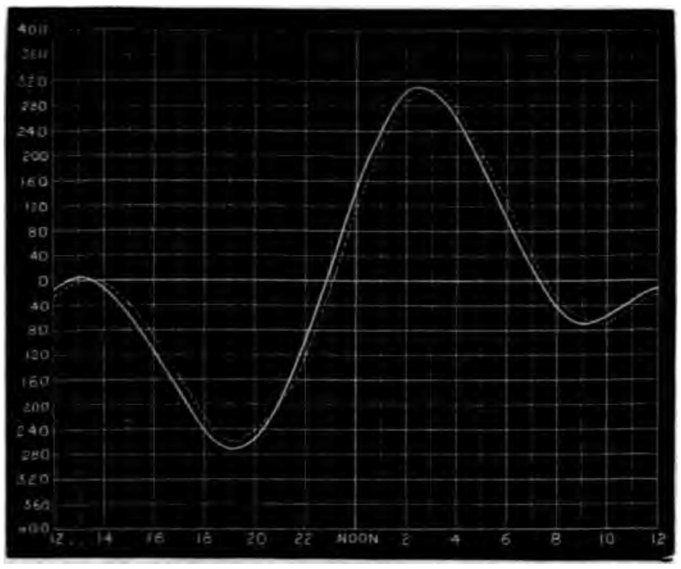
Fig. 1.



BOMBAY.

Comparison between calculated and observed curve of West force. The abscissæ denote astronomical time, the ordinates the force to geographical West, the unit of force being C.G.S. 10^{-6}.
Observed curve, white line. Calculated curve, dotted line.

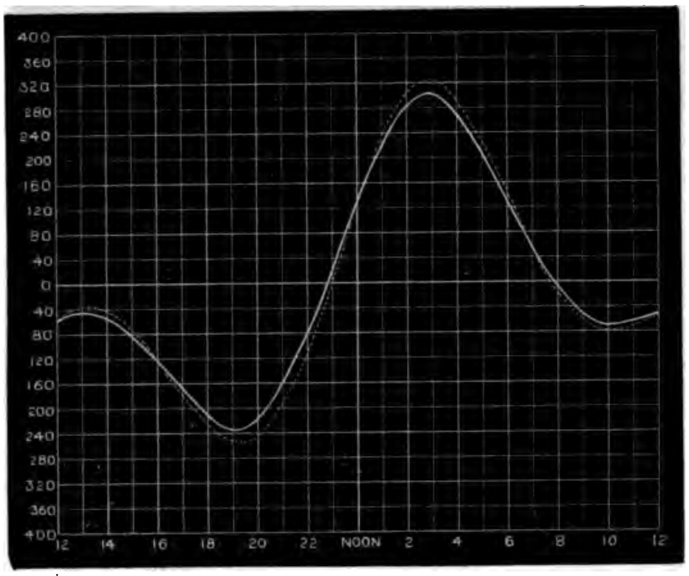
Fig. 2.



LISBON.

Comparison between calculated and observed curve of West force. The abscissæ denote astronomical time, the ordinates the force to geographical West, the unit of force being C.G.S. 10^{-6}.
Observed curve, white line. Calculated curve, dotted line.

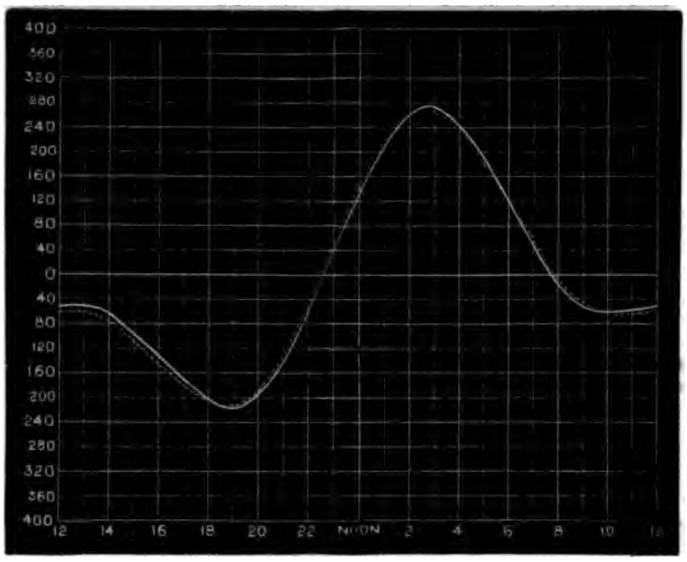
Fig. 3.



GREENWICH.

Comparison between calculated and observed curve of West force. The abscissæ denote astronomical time, the ordinates the force to geographical West, the unit of force being C.G.S. 10^{-6}. Observed curve, white line. Calculated curve, dotted line.

Fig. 4.



ST. PETERSBURG.

Comparison between calculated and observed curve of West force. The abscissæ denote astronomical time, the ordinates the force to geographical West, the unit of force being C.G.S. 10^{-6}. Observed curve, white line. Calculated curve, dotted line.

four observing stations. The numbers which I have computed for this purpose are given in Table X.

In that table T_p^q stands for

$$\frac{d^q P_p}{d\mu^q} \sin^q u,$$

where $\mu = \cos u$ and P_p is the zonal harmonic of degree p.

TABLE VIII.—Comparison between observed and calculated coefficients of West force.

		a_1.		b_1.		a_2.		b_2.	
		Observed.	Calculated.	Observed.	Calculated.	Observed.	Calculated.	Observed.	Calculated.
Bombay . .	Summer . .	+ 58	+ 58	+ 116	+ 124	+ 132	+ 124	+ 60	+ 73
	Winter . .	0	− 4	+ 32	+ 41	+ 4	+ 8	+ 23	+ 39
	Mean . . .	+ 29	+ 27	+ 74	+ 82	+ 68	+ 66	+ 41	+ 56
Lisbon . . .	Summer . .	+ 74	+ 88	+ 225	+ 232	+ 108	+ 90	+ 158	+ 172
	Winter . .	+ 87	+ 52	+ 119	+ 112	+ 31	+ 10	+ 104	+ 105
	Mean . . .	+ 80	+ 70	+ 172	+ 172	+ 70	+ 50	+ 131	+ 138
Greenwich . .	Summer . .	+ 85	+ 91	+ 233	+ 255	+ 83	+ 85	+ 135	+ 149
	Winter . .	+ 105	+ 95	+ 115	+ 113	+ 1	− 5	+ 87	+ 104
	Mean . . .	+ 95	+ 93	+ 174	+ 184	+ 42	+ 40	+ 111	+ 127
St. Petersburg	Summer . .	+ 100	+ 79	+ 225	+ 229	+ 99	+ 92	+ 115	+ 100
	Winter . .	+ 92	+ 121	+ 96	+ 93	− 15	− 6	+ 69	+ 72
	Mean . . .	+ 96	+ 100	+ 161	+ 161	+ 42	+ 43	+ 92	+ 87

		a_3.		b_3.		a_4.		b_4.	
		Observed.	Calculated.	Observed.	Calculated.	Observed.	Calculated.	Observed.	Calculated.
Bombay . . .	Mean of year	+ 71	+ 19	+ 8	− 24	+ 24	+ 20	− 14	− 12
Lisbon . . .	„	+ 63	+ 108	+ 57	+ 88	+ 24	+ 21	+ 8	+ 5
Greenwich . .	„	+ 41	+ 82	+ 44	+ 74	+ 17	+ 31	+ 14	+ 21
St. Petersburg .	„	− 12	− 7	− 8	+ 16	− 18	− 20	− 1	+ 27

TABLE IX.—Comparison between calculated and observed variation of force to geographical West. The unit of force is C.G.S. 10^{-6}. The first four terms only of the series $a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t$ have been taken into account.

Astronomical time.	Bombay.		Lisbon.		Greenwich		St. Petersburg.	
	Observed.	Calculated.	Observed.	Calculated.	Observed.	Calculated.	Observed.	Calculated.
	a_1 + 29	+ 27	+ 80	+ 70	+ 95	+ 93	+ 96	+ 100
	b_1 + 74	+ 82	+ 172	+ 172	+ 174	+ 184	+ 161	+ 161
	a_2 + 68	+ 66	+ 70	+ 50	+ 42	+ 40	+ 42	+ 43
	b_2 + 41	+ 56	+ 131	+ 138	+ 111	+ 127	+ 92	+ 87
Hour.								
0	+ 97·0	+ 93·0	+ 150·0	+ 120·0	+ 137·0	+ 133·0	+ 138·0	+ 143·0
1	+ 126·6	+ 132·5	+ 247·9	+ 224·4	+ 228·8	+ 235·6	+ 216·8	+ 219·0
2	+ 131·6	+ 145·9	+ 303·7	+ 291·1	+ 286·4	+ 302·5	+ 264·3	+ 263·9
3	+ 113·8	+ 133·1	+ 309·2	+ 309·1	+ 301·2	+ 322·9	+ 273·7	+ 271·5
4	+ 80·1	+ 100·0	+ 267·4	+ 278·5	+ 273·3	+ 295·8	+ 246·1	+ 243·2
5	+ 40·6	+ 57·0	+ 191·8	+ 210·0	+ 211·8	+ 230·7	+ 190·0	+ 187·7
6	+ 6·0	+ 16·0	+ 102·0	+ 122·0	+ 132·0	+ 144·0	+ 119·0	+ 118·0
7	− 15·4	− 13·0	+ 19·4	+ 35·8	+ 51·6	+ 55·5	+ 48·2	+ 48·9
8	− 19·9	− 24·0	− 39·4	− 30·5	− 13·9	− 17·2	− 9·3	− 7·4
9	− 9·2	− 17·1	− 66·0	− 65·9	− 55·2	− 62·7	− 46·1	− 43·9
10	+ 10·4	+ 2·1	− 61·7	− 69·1	− 70·4	− 78·5	− 61·3	− 59·9
11	+ 29·6	+ 24·3	− 37·7	− 48·8	− 65·8	− 71·0	− 60·6	− 61·2
12	+ 39·0	+ 39·0	− 10·0	− 20·0	− 53·0	− 53·0	− 54·0	− 57·0
13	+ 32·2	+ 37·9	+ 4·3	+ 0·2	− 45·0	− 39·4	− 52·0	− 57·6
14	+ 7·4	+ 17·1	− 6·9	− 2·1	− 52·2	− 42·5	− 62·9	− 70·3
15	− 31·8	− 21·1	− 47·2	− 33·1	− 79·2	− 68·9	− 89·7	− 97·5
16	− 77·1	− 69·0	− 110·6	− 89·5	− 123·1	− 115·8	− 128·7	− 135·6
17	− 117·4	− 115·4	− 182·0	− 158·6	− 173·6	− 172·9	− 170·8	− 175·1
18	− 142·0	− 148·0	− 242·0	− 222·0	− 216·0	− 224·0	− 203·0	− 204·0
19	− 143·4	− 157·4	− 271·6	− 260·4	− 235·4	− 251·7	− 213·0	− 210·3
20	− 119·1	− 139·0	− 257·4	− 258·5	− 220·3	− 242·8	− 192·1	− 186·2
21	− 72·8	− 94·9	− 196·0	− 210·1	− 166·8	− 191·3	− 137·9	− 130·1
22	− 13·4	− 33·1	− 95·1	− 119·9	− 79·8	− 101·5	− 56·1	− 47·7
23	+ 47·2	+ 34·1	+ 27·9	− 2·6	+ 27·6	+ 13·2	+ 41·4	+ 48·6

IV. *Comparison of Observed and Calculated Vertical Forces.*

The complete potential can be written down from the value at its surface in the usual manner, either on the supposition that the potential is zero at an infinite distance, or that it is zero at the centre of the Earth, the first supposition corresponding to the hypothesis that the seat of the magnetic variation is outside the Earth. If Y_n is a surface harmonic of degree n occurring in the expansion of V/a, the solid harmonic will either be $Y_n(r/a)^n$ or $Y_n(r/a)^{-(n+1)}$. The vertical force is given by $\partial V/\partial r$, as the force is considered positive when it acts downwards. At the surface, therefore, we have a term for the vertical force which is either nY_n or $-(n+1)Y_n$.

Before we proceed to discuss the comparison between the observed and calculated values of the vertical force, a few words are necessary regarding the available observations.

The only station for which we have complete records for 1870 is Lisbon. It is therefore impossible to obtain a satisfactory series for the vertical force which would give us, if our information was more complete, directly the two terms, one due to outside, the other due to inside, effects. But I shall show that even from the existing data we can draw important conclusions.

At Bombay no vertical force determinations are published, as far as I know, before 1873, when the magnetograph came into operation; but we have complete records between 1873 and 1878. During these years the general type of the vertical force remained practically the same, only the range varying. Figs. 5 and 6, for instance, are tracings of the curves given by Mr. CHAMBERS in the years 1873 and 1877, the two years which differ most in range. The other years show curves varying between these two extremes. As far as the general form of the curve is concerned, we cannot go far wrong, therefore, if we make use of the 1873 observations, especially as the horizontal components show no marked difference (except as regards range) in 1873 and 1870.

Fig. 5.

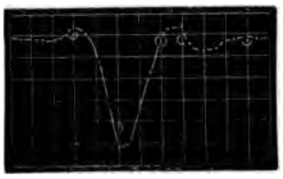
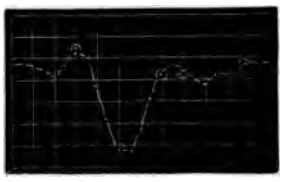


Fig. 6.



Similar remarks apply to Greenwich. Although published records exist for 1870, there is an uncertainty about the temperature correction, which makes the vertical force observations previous to 1882 useless for our present purpose.

Table XI. will give an idea of the changes which the vertical force variation is

TABLE XII.—Comparison between the observed and calculated coefficients for the vertical force, a_1, b_1, a_2, b_2 on the hypothesis that the disturbing force is inside the Earth.

	a_1.		b_1.		a_2.		b_2.	
	Calculated.	Observed.	Calculated.	Observed.	Calculated.	Observed.	Calculated.	Observed.
Bombay	+ 219	− 42	− 57	+ 9	+ 79	− 21	− 152	+ 28
Lisbon, summer . . .	+ 580	− 196	− 206	+ 66	+ 381	− 201	− 87	+ 26
„ winter	+ 336	− 135	− 149	+ 55	+ 277	− 103	− 19	+ 1
Greenwich, 1884 . . .	+ 350	− 42	− 190	+ 49	+ 139	− 51	− 34	+ 2
St. Petersburg, 1870. .	+ 177	+ 114	− 154	+ 125	+ 62	− 59	− 45	− 38
„ 1878. .		− 8	..	+ 29	..	− 24	..	− 2

TABLE XIII.—Comparison between the observed and calculated coefficients a_3, b_3, a_4, b_4 for the vertical force on the hypothesis that the disturbing force is inside the Earth.

	a_3.		b_3.		a_4.		b_4.	
	Calculated.	Observed.	Calculated.	Observed.	Calculated.	Observed.	Calculated.	Observed.
Bombay	− 73·4	− 10	− 42·0	+ 35	− 20·4	+ 3	− 29·0	+ 16
Lisbon	+156·1	− 81	−171·4	+ 12	+ 9·2	− 21	− 22·1	+ 11
Greenwich	+ 85·5	− 24	− 88·0	+ 3	+ 28·1	− 8	− 38·2	+ 3
St. Petersburg. . . .	+ 12·0	− 12	− 17·4	− 8	+ 22·3	− 18	− 31·3	− 1

TABLE XIV.—Comparison between the observed and calculated coefficients a_1, b_1, a_2, b_2 on the assumption that the disturbing force is outside the Earth.

	a_1.		b_1.		a_2.		b_2.	
	Calculated.	Observed.	Calculated.	Observed.	Calculated.	Observed.	Calculated.	Observed.
Bombay	− 141	− 42	+ 31	+ 9	− 53	− 21	+ 121	+ 28
Lisbon, summer . . .	− 398	− 196	+ 137	+ 66	− 315	− 201	+ 52	+ 26
„ winter	− 248	− 135	+ 108	+ 55	− 235	− 103	+ 15	+ 1
Greenwich, 1884 . . .	− 235	− 42	+ 132	+ 49	− 110	− 51	+ 21	+ 2
St. Petersburg, 1870. .	− 97	+ 114	+ 104	+ 125	− 40	− 59	+ 35	− 38
„ 1878. .	..	− 8	..	+ 29	..	− 24	..	− 2

TABLE XV.—Comparison between the observed and calculated coefficients a_3, b_3, a_4, b_4 on the assumption that the disturbing force is outside the Earth.

	a_3.		b_3.		a_4.		b_4.	
	Calculated.	Observed.	Calculated.	Observed.	Calculated.	Observed.	Calculated.	Observed.
Bombay	+ 66	− 10	+ 22	+ 35	+ 18	+ 3	+ 24	+ 16
Lisbon	− 134	− 81	+ 125	+ 12	− 8	− 21	+ 18	+ 11
Greenwich	− 70	− 24	+ 38	+ 3	− 25	− 8	+ 33	+ 3
St. Petersburg	− 7	− 12	+ 16	− 8	− 20	− 18	+ 27	− 1

Confining ourselves in the first place to the first four coefficients, we find that out of twenty coefficients eighteen have the wrong sign on the hypothesis of an internal cause, while only two have the wrong sign on the hypothesis of an external cause, and those two belong to St. Petersburg, to which station, as was pointed out, we cannot attach much value. If instead of the numbers given for 1870 we take the numbers given at the same station for 1878, the agreement becomes better, even for St. Petersburg. The coefficients a_3, b_3, a_4, b_4 are of course more uncertain; but even here the evidence is strongly in favour of the external cause. Out of twenty coefficients seventeen agree in sign with that hypothesis.

A better comparison can perhaps be obtained in a different way: the two terms

$$a_n \cos nt + b_n \sin nt$$

can be written

$$r_n \cos n (t - t_n),$$

where r_n is the amplitude of the oscillation, and t_n the time at which the maximum elongations take place.

Tables XVI. and XVII. contain the results for t_n, and from these tables I think it will clearly appear that the phase of the vertical force completely agrees with the assumption of an *external* cause and completely disagrees with the assumption of an *internal* cause.

For Lisbon, our principal station of comparison, the phase in Table XVI. agrees for both the diurnal and semi-diurnal variation within four minutes of time. For Bombay the diurnal variation agrees within three minutes, and the semi-diurnal variation within 36 minutes. For Greenwich, the semi-diurnal variation, which we have seen differs little from year to year, agrees closely, while the diurnal variation shows a greater difference. In all these cases the phases, as calculated in Table XV., are in as great a disagreement as possible. St. Petersburg gives less decisive results, but they still go in the same direction, especially if we take the observations of 1878 to represent the type of vertical variation.

The amplitudes of the variation are given in Table XVIII. where the second column gives the calculated values on the hypothesis which we must reject, while the third column gives the same numbers calculated on the assumption of an external cause.

The calculated numbers are much larger than the observed ones, which is a fact requiring explanation. But, the range of vertical force differing from year to year, we must confine ourselves in our reasoning principally to the Lisbon observations.

The changes in the range of different years, as far as we have any observations, are not so considerable, however, as to account for the discrepancies at Bombay or Greenwich, and we must conclude, I believe, that also for those stations a considerably smaller range is obtained in the observed than in the calculated forces.

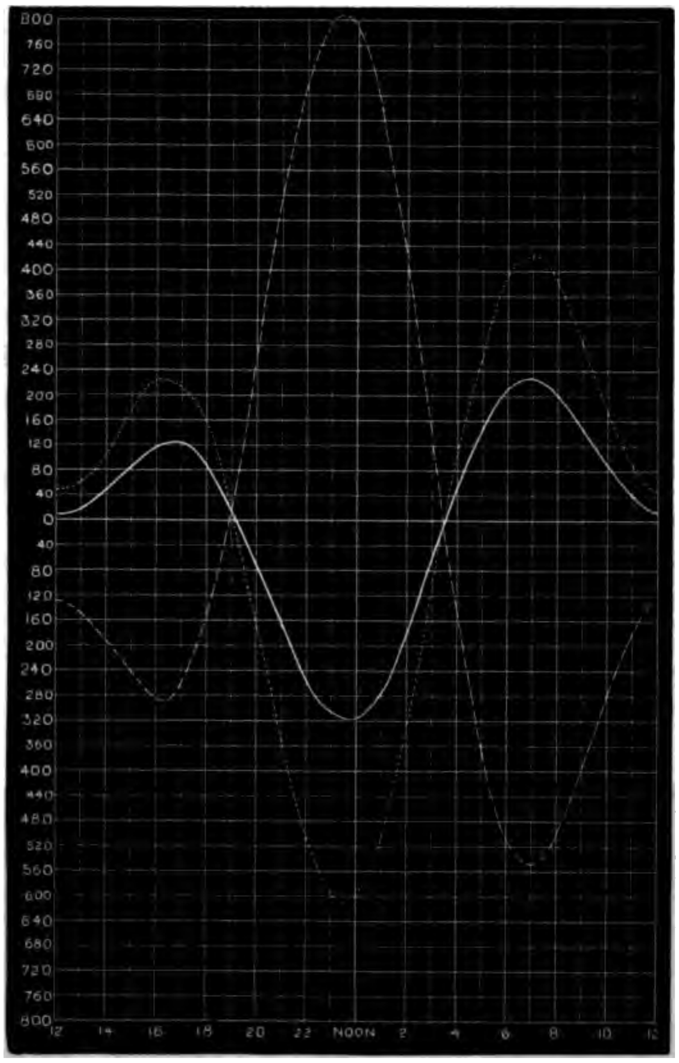
TABLE XVI.—Observed and calculated values of the coefficients t_1 and t_2 of vertical force, when expressed in the form $r_1 \cos (t - t_1) + r_2 \cos 2 (t - t_2)$ on the supposition that the disturbing force is inside the Earth.

	t_1.			t_2.		
	Calculated.	Observed.	Difference.	Calculated.	Observed.	Difference.
	h. m.	h. m.	h. m.	h. m.	h. m.	h. m.
Bombay	23 02	11 13	+ 11 49	9 55	4 23	+ 5 32
Lisbon	22 35	10 40	+ 11 58	11 42	5 50	+ 5 52
Greenwich	22 06	8 42	− 11 57	11 32	5 56	+ 5 36
St. Petersburg, 1870. .	21 16	3 10	− 5 54	10 48	7 05	+ 3 43
„ 1878. .	..	7 05	− 9 49	..	6 12	+ 4 36

TABLE XVII.—Observed and calculated values of the coefficients t_1 and t_2 when expressed in the form $r_1 \cos (t - t_1) + r_2 \cos 2 (t - t_2)$ on the supposition that the disturbing force is outside the Earth.

	t_1.			t_2.		
	Calculated.	Observed.	Difference.	Calculated.	Observed.	Difference.
	h. m.	h. m.	h. m.	h. m.	h. m.	h. m.
Bombay	11 10	11 13	− 0 03	3 47	4 23	− 0 36
Lisbon	10 37	10 40	− 0 03	5 46	5 50	− 0 04
Greenwich	10 03	8 42	+ 1 21	5 38	5 56	− 0 18
St. Petersburg, 1870. .	8 52	3 10	+ 5 42	4 38	7 05	− 2 27
„ 1878. .	..	7 05	− 1 47	..	6 12	− 1 34

Fig. 8.

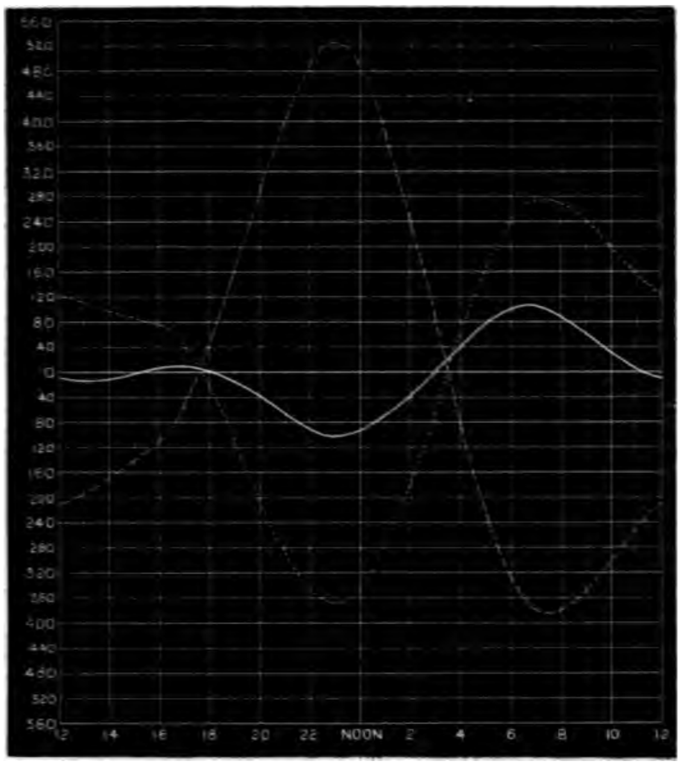


LISBON.

Comparison between calculated and observed curve of vertical force. The abscissæ denote astronomical time, the ordinates vertical force, the unit of force being C.G.S. 10^{-6}.

Observed curve, white line. Curve calculated on hypothesis of outside force, dotted line. Curve calculated on hypothesis of inside force, white and black line.

Fig. 9.

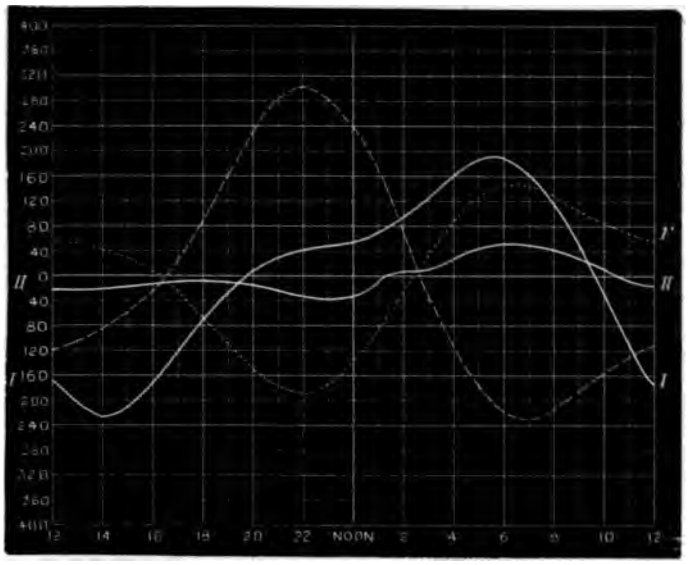


GREENWICH.

Comparison between calculated and observed curve of vertical force. The abscissæ denote astronomical time, the ordinates vertical force, the unit of force being C.G.S. 10^{-6}.

Observed curve, white line. Curve calculated on hypothesis of outside force, dotted line. Curve calculated on hypothesis of inside force, white and black line.

Fig. 10.



ST. PETERSBURG.

Comparison between calculated and observed curve of vertical force. The abscissæ denote astronomical time, the ordinates vertical force, the unit of force being C.G.S. 10^{-6}.

Observed curves, white line—I. for 1870, II. for 1878. Curve calculated on hypothesis of outside force, dotted line. Curve calculated on hypothesis of inside force, white and black line.

TABLE XIX.—Comparison between calculated and observed variation of vertical force. The unit of force is C.G.S. 10^{-6}. The first four terms of the series $a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t$ have been taken into account.

Astronomical time.	Bombay.			Lisbon.			Greenwich.			St. Petersburg.			
	Observed.	Calculated from outside.	Calculated from inside.	Observed.	Calculated from outside.	Calculated from inside.	Observed.	Calculated from outside.	Calculated from inside.	Observed 1870.	Calculated from outside.	Calculated from inside.	Observed 1878.
	a_1 −42	−141	+219	a_1 −165	−323	+458	a_1 −42	−235	+350	+114	− 97	+177	− 8
	b_1 + 9	+ 31	− 57	b_1 + 60	+122	−177	b_1 +49	+132	−190	+125	+104	−154	+29
	a_2 −21	− 53	+ 79	a_2 −152	−275	+329	a_2 −51	−110	+139	− 59	− 40	+ 62	−24
	b_2 +28	+121	−152	b_2 + 13	+ 34	− 53	b_2 + 2	+ 21	− 34	− 38	+ 35	− 45	− 2
Hour.													
0	−63·0	−194·0	+298·0	−317·0	−598·0	+787·0	− 93·0	−345·0	+489·0	+ 55·0	−137·0	+239·0	−32·0
1	−42·5	−113·6	+189·2	−269·0	−501·6	+655·0	− 71·1	−277·6	+392·3	+ 72·4	− 83·9	+162·3	−22·0
2	−18·2	− 28·3	+ 69·1	−177·6	−326·8	+426·7	− 35·7	−174·3	+248·2	+ 98·8	− 21·7	+ 68·3	− 6·1
3	+ 4·7	+ 43·2	− 37·5	− 61·3	−108·1	+145·7	+ 6·9	− 51·8	+ 79·2	+131·0	+ 39·9	− 28·8	+12·8
4	+21·5	+ 87·6	−111·0	+ 56·8	+111·1	−134·7	+ 48·6	+ 70·0	− 88·4	+161·9	+ 91·9	−114·9	+31·4
5	+30·0	+ 99·8	−142·8	+153·4	+289·4	−363·8	+ 81·6	+172·4	−230·2	+182·4	+127·5	−179·2	+45·7
6	+30·0	+ 84·0	−136·0	+212·0	+397·0	−506·0	+100·0	+242·0	−329·0	+184·0	+144·0	−216·0	+53·0
7	+23·8	+ 51·8	−104·2	+225·8	+422·8	−548·0	+101·4	+273·2	−377·6	+161·4	+142·7	−225·8	+51·9
8	+15·1	+ 19·0	− 66·8	+199·2	+375·3	−500·9	+ 87·2	+268·6	−379·6	+113·7	+128·3	−213·9	+42·8
9	+ 8·1	+ 0·6	− 43·1	+146·1	+280·7	−395·9	+ 62·3	+238·4	−347·8	+ 45·8	+107·1	−189·0	+28·2
10	+ 6·2	+ 6·3	− 47·1	+ 85·6	+173·8	−274·7	+ 33·7	+196·3	−299·2	− 32·8	+ 85·7	−160·3	+11·1
11	+10·7	+ 37·8	− 82·0	+ 36·8	+ 88·4	−176·8	+ 8·1	+155·4	−249·9	−109·8	+ 68·5	−134·7	− 4·6
12	+21·0	+ 88·0	−140·0	+ 13·0	+ 48·0	−129·0	− 9·0	+125·0	−211·0	−173·0	+ 57·0	−115·0	−16·0
13	+34·1	+142·8	−204·4	+ 18·8	+ 59·2	−138·2	− 15·3	+108·0	−185·5	−212·6	+ 49·7	− 99·9	−21·6
14	+45·6	+184·9	−253·3	+ 48·2	+110·6	−189·5	− 11·9	+100·7	−168·0	−223·6	+ 42·3	− 84·3	−21·3
15	+51·3	+198·8	−266·5	+ 87·3	+176·1	−251·7	− 2·9	+ 93·8	−147·2	−207·0	+ 30·1	− 61·2	−16·8
16	+47·9	+175·0	−231·2	+117·8	+222·7	−286·1	+ 5·8	+ 76·4	−109·4	−168·7	+ 8·7	− 25·1	−10·8
17	+34·4	+113·0	−146·0	+122·8	+221·0	−259·0	+ 8·8	+ 39·2	− 44·6	−118·2	− 23·3	+ 26·8	− 6·1
18	+12·0	+ 22·0	− 22·0	+ 92·0	+153·0	−152·0	+ 2·0	− 22·0	+ 51·0	− 66·0	− 64·0	+ 92·0	− 5·0
19	−15·4	− 81·0	+119·4	+ 24·4	+ 19·6	+ 31·2	− 15·0	−103·6	+170·8	− 21·2	−108·5	+163·4	− 8·3
20	−42·5	−175·6	+251·0	− 69·8	−159·1	+263·7	− 39·6	−195·0	+299·4	+ 11·1	−148·9	+229·9	−15·4
21	−64·1	−242·6	+347·1	−172·1	−348·7	+501·9	− 66·3	−280·4	+415·8	+ 30·2	−177·1	+279·0	−24·2
22	−75·6	−268·9	+389·3	−260·2	−507·6	+695·5	− 88·1	−342·7	+497·0	+ 39·6	−186·3	+300·3	−31·7
23	−75·1	−250·6	+370·8	−313·0	−598·8	+799·6	− 98·5	−367·0	+524·7	+ 45·6	−172·7	+287·1	−35·0

If we, then, take it as proved that the primary cause of the variation comes to us from outside the Earth's surface, we are led to consider that a varying magnetic potential must cause induced currents within the Earth, if that body is a sufficiently good conductor. These induced currents might be the cause of the apparent reduction in amplitude. As my colleague, Professor LAMB, had given considerable attention to the problem of currents inside a conducting sphere, I consulted him, and he gave me the formulæ by means of which the induced currents can be calculated. His investigation is added as an Appendix to this paper.

V. *Discussion of effects due to Currents Induced in the Inside of the Earth.*

I shall assume, then, for the present, that there is a periodic magnetic disturbance having its cause outside the Earth, and being probably due to electric currents in our atmosphere. Currents will be induced within the Earth, and we must now discuss what the effect of these currents will be, and whether they will account for the reduction in amplitude of the vertical forces which the observations show.

The varying potential can be expressed as a sum of terms of the form

$$\Omega_n \cos (pt + \lambda),$$

where Ω is a solid harmonic of degree n. Professor LAMB's formulæ allow us to calculate for each value of n, and for each value of p, the magnetic effect due to the induced currents, on the supposition that the specific resistance ρ of the Earth is uniform. The forces due to these currents will have a different amplitude and a different phase from the original forces, and it is the resultant effect which we observe in the diurnal variations. The general effect will be to increase the horizontal components and to diminish the vertical component. The difference of phase will be the same for all components, provided we give a different sign to the amplitude of the vertical components of the inside and outside currents respectively. Otherwise the difference of phase of the vertical component will be greater by two right angles than the difference of phase of the horizontal components.

If one of the horizontal forces and the vertical force due to the solid harmonic of positive degree n are written

$$a \cos (pt + \lambda) \quad \text{and} \quad b \cos (pt + \lambda + \epsilon),$$

the corresponding components due to the induced potential of negative degree will be of the form

$$c'a \cos (pt + \lambda + \alpha) \quad \text{and} \quad cb \cos (pt + \lambda + \epsilon + \alpha).$$

Table XX. gives the coefficients c', c, and α for given values of the specific resistance ρ, if $n = 2$. The value of δ has the same meaning as in Professor LAMB's paper, and is connected with ρ by the equation

$$\rho\delta = 4\pi p \mathrm{R}^2$$

where p is equal to $2\pi m/\mathrm{T}$, and m is equal to 1 for the diurnal variation, and equal to 2 for the semidiurnal variation.

Table XXI. gives the same quantities for $n = 4$.

An example may render the use of the Table more intelligible. Let the Earth, for instance, have a uniform specific resistance, which in C.G.S. units is $1{\cdot}23 \times 10^{13}$, and consider that term of the magnetic potential which, on the surface of the Earth, has the form

$$\mathrm{A} \sin u \cos u \cos \frac{2\pi t}{\mathrm{T}},$$

where T is the time of the revolution of the Earth. This term alone represents fairly well the characteristic features of the diurnal variation. Here, as $n = 2$, we may use Table XX. The value of δ corresponding to the assumed resistance is 30, and we find $c = -{\cdot}5$ approximately, which means that the vertical force due to the induced currents has half the amplitude of the vertical force due to the primary currents; also, the difference in phase is 41°; as the sign of the amplitude is changed, this means that the *minimum* of the vertical force due to induced currents takes place not quite three hours after the *maximum* of the corresponding primary force. Similarly the horizontal force due to the inside current is only one-third of the corresponding horizontal force due to the outside potential, and here the *maximum* due to the secondary currents takes place nearly three hours after the *maximum* due to the secondary currents. To get similar numbers for the semidiurnal variation we should have to put $m = 2$, and find c, c', and α for $\delta = 60$, because $\rho = 1{\cdot}23 \times 10^{13} = 2 \times 6{\cdot}15 \times 10^{12}$, and looking up $6{\cdot}15 \times 10^{12}$ in the last column we should find the corresponding number in the first column to be 60.

TABLE XX.—Comparison of the magnetic forces due to a system of varying electric currents outside the Earth, with the forces due to the currents induced inside the Earth. $n = 2$.

δ.	c. Ratio of normal forces due to secondary and primary variation.	c'. Ratio of tangential forces due to secondary and primary variation.	α. Difference of phase. ° ′	Corresponding value of ρ.
1	$-{\cdot}02854$	$+{\cdot}01903$	87 27½	$3{\cdot}70 \times 10^{14} \times m.$
2	·05688	·03792	84 55½	$1{\cdot}85 \times 10^{14}$
3	·08484	·05656	82 25	$1{\cdot}23 \times 10^{14}$
4	·11226	·07484	79 57	$9{\cdot}25 \times 10^{13}$
5	·13895	·09263	77 32½	$7{\cdot}40 \times 10^{13}$
6	·16479	·10986	75 10½	$6{\cdot}17 \times 10^{13}$
7	·18968	·12645	72 53	$5{\cdot}29 \times 10^{13}$
8	·21354	·14236	70 41	$4{\cdot}63 \times 10^{13}$
9	·23633	·15755	68 32	$4{\cdot}11 \times 10^{13}$
10	·25799	·17199	66 29	$3{\cdot}70 \times 10^{13}$
20	·41771	·27847	50 32	$1{\cdot}85 \times 10^{13}$
30	·50520	·33680	41 03	$1{\cdot}23 \times 10^{13}$
40	·56158	·37439	35 11	$9{\cdot}25 \times 10^{12}$
50	·59864	·39909	31 11	$7{\cdot}40 \times 10^{12}$
100	·69949	·46632	21 33	$3{\cdot}70 \times 10^{12}$

TABLE XXI.—Comparison of the magnetic force due to a varying potential represented by a solid harmonic of degree 4 with the corresponding forces due to currents induced inside the Earth.

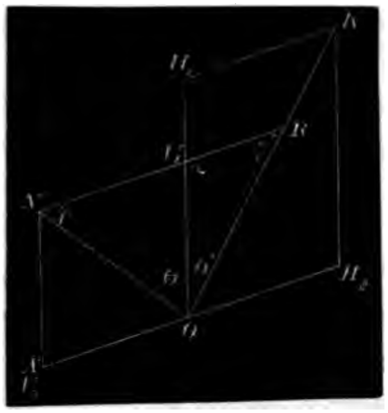
δ.	c. Ratio of normal forces due to secondary and primary variation.	c'. Ratio of tangential forces due to secondary and primary variation.	a. Difference of phase.	Corresponding value of ρ.
			° ′	
1	− ·0101	+ ·0081	89 01	$3{\cdot}70 \times 10^{14} \times m$.
10	·0992	·0794	80 21	$3{\cdot}70 \times 10^{13}$
100	·5149	·4119	37 50	$3{\cdot}70 \times 10^{13}$

Tables XX. and XXI. cannot, however, be used to compare our calculated and observed results, but form only an intermediate step.

We observe on the Earth the resultant of the outside and inside effect, and we have calculated the vertical force on the assumption that the whole horizontal force is due to outside effect.

In fig. 11, let OH_1 represent that part of the horizontal force which is due to the outside effect, and OH_2 the corresponding force in phase and amplitude which is due to the induced effect. The observed horizontal force will be the resultant OK. Let OV_1 be the magnitude of the vertical force due to the outside currents, and OV_2 the vertical force in phase and amplitude due to the induced effect. The observed vertical force will be ON. If we calculate the vertical force on the assumption that the resultant OK is due entirely to the outside, we should obtain a force OR, such that OR : OK :: $OV_1 : OH_1$. Our calculated value of vertical force will be OR, and our observed value ON. We require the ratio of lengths of these lines and the angle between them.

Fig. 11.



In the triangle ONR we know, from Table XX., the ratios

$$V_1R : OV_1 = H_1K : OH_1 = OH_2 : OH_1 = c',$$
$$V_1N : OV_1 = OV_2 : OV_1 = c,$$

also the angle

$$NV_1O = V_1OH_2 = \alpha.$$

Write

$$NOV_1 = \theta;\ ROV_1 = \theta';\ ONV_1 = \phi;\ ORV_1 = \phi'.$$

The triangles OV_1N and OV_1R give us the equations

$$\tan \tfrac{1}{2}(\phi - \theta) = \frac{1 - c}{1 + c} \cot \frac{\alpha}{2},$$

$$\tan \tfrac{1}{2}(\phi' - \theta') = \frac{1 - c'}{1 + c'} \tan \frac{\alpha}{2},$$

$$(\phi + \theta) = \pi - \alpha,$$

$$(\phi' + \theta') = \alpha.$$

These equations determine θ and θ', and hence the required angle $\gamma = \theta + \theta'$; also $OR : ON = \sin \phi : \sin \phi'$.

In Table XXII. the angle γ and the ratio $ON : OR : = r$ have been calculated for $n = 2$. Table XXIII. gives the corresponding quantities if the inducing solid harmonic is of degree 4.

TABLE XXII.—Comparison between resultant vertical force as regards magnitude and phase when induced currents are taken into account and vertical force calculated on the assumption that the whole is due to an outside effect. The inducing potential is a solid harmonic of degree 2.

$\hat{c}$.	Reduction in amplitude. r.	Change of phase. γ.	ρ.
1	·9981	$2^\circ\ 43'$	$3{\cdot}70 \times 10^{14}$
5	·9565	13 02	$7{\cdot}40 \times 10^{13}$
10	·8589	23 10	$3{\cdot}70 \times 10^{13}$
20	·6705	34 03	$1{\cdot}85 \times 10^{13}$
30	·5516	38 12	$1{\cdot}23 \times 10^{13}$
40	·4762	40 16	$9{\cdot}25 \times 10^{12}$
50	·4261	41 10	$7{\cdot}40 \times 10^{12}$
100	·3004	43 08	$3{\cdot}70 \times 10^{12}$

TABLE XXIII.—Comparison between resultant vertical force, as regards magnitude and direction when induced currents are taken into account, and vertical force calculated on the assumption that the whole is due to an outside effect. The inducing potential is a solid harmonic of degree 4.

δ.	Reduction in amplitude.	Change of phase.	ρ.
		° ′	
1	·9997	1 06	$3{\cdot}70 \times 10^{14}$
10	·9723	10 06	$3{\cdot}70 \times 10^{13}$
100	·4982	38 49	$3{\cdot}70 \times 10^{12}$

The observed amplitude of the vertical force at Lisbon is about one half of its calculated value. If the conductivity of the Earth was such as to produce this reduction in amplitude, it is seen from Tables XXII. and XXIII. that the phase would be altered about 40°, while in reality there is a remarkable agreement in phase. If the conductivity is so small as to leave the resultant phase practically unaltered, as observation tends to show, the amplitude also should not be sensibly altered. There is, therefore, no uniform conductivity of the Earth which can make the observations agree with the calculation. Such an agreement, however, can be easily brought about, as Professor LAMB has suggested to me, if the conductivity of the inside of the Earth is larger than the conductivity of the upper layers. It is extremely probable that this is really the case. The bulk of the outside layer of the Earth, except in so far as it is water, is made up of material which in its ordinary condition is non-conducting; but we know that some of the silicates begin to conduct at temperatures above 200° C., and, generally speaking, insulators lose their insulating powers at high temperatures. Without regard even to the quantities of metallic matter that may be stored inside the Earth, there is nothing improbable in the supposition that its conductivity increases towards the inside. If the bulk of the observed induced effect is due to currents in a fairly conducting inner sphere, the calculated phase would be that due to good conducting matter, and would not differ from the observed value, while the reduction in amplitude might yet be sufficient to account for the observed facts. In order to give a better idea of the kind of conductivity which is required to produce a certain change of phase, it may be stated that for the purest distilled water obtained by KOHLRAUSCH ρ would be about $1{\cdot}4 \times 10^{15}$. Such water is, as is well known, a very bad conductor, and, according to our Tables, if the whole Earth was made up of matter which conducts as badly, there would be no currents in the Earth induced by the diurnal variation of sufficient intensity to affect our magnetic needle sensibly. Ordinary rain water, however, has a specific resistance of about 6×10^{13}. A conducting sphere of the same resistance would already produce a retardation in phase of about an hour for the diurnal variation if the solid harmonic is of degree 2. For salt water the resistance may get as low as 4×10^9. A whole sphere made up of such water would very considerably reduce the amplitude of the observed vertical force, and alter the resultant phase by 45° nearly. The average conductivity of the Earth, as seen from these examples, must be small, although it may be considerable over limited areas. Such limited areas would principally affect the harmonic terms of higher degrees, and we should not consequently expect for them such a good agreement between theory and observation. Table XXII. shows that as the resistance of the sphere diminishes the retardation of the resultant phase seems to approach a constant value of 45°. This can be proved to be quite generally the case. It follows directly from a formula given by Professor LAMB in the Appendix, and can also be seen as follows:—When the conductivity is good the angle between OH_2 and OH_1 in fig. 11 will steadily diminish, and ultimately vanish. OK will ultimately coincide with OH_1, and OR with OV_1; but the angle between OV_2 and OV_1 will increase towards 180°, and the sides will tend towards equality. Two very nearly equal and very nearly opposite forces may have a resultant which is inclined by a finite angle to the forces. To find the angle in the limit between OV_1 and OV_2 we deduce, in the first place, an expression for the ratio of the vertical force due to the outside effect to the vertical force due to induced currents for good conductivities.

This ratio depends, as shown by Professor LAMB, on the function

$$\chi_n(\kappa R) \quad \text{where} \quad \kappa^2 = \frac{4\pi p i}{\rho},$$

and

$$\chi_n(\zeta) = 3 \cdot 5 \ldots (2n+1) \left(\frac{d}{\zeta d\zeta}\right)^n \frac{\sinh \zeta}{\zeta}.$$

If n is odd,

$$\left(\frac{d}{\zeta d\zeta}\right)^n \sinh \zeta = \frac{\cosh \zeta}{\zeta^{n+1}} - \frac{n.n+1}{2}\frac{\sinh \zeta}{\zeta^{n+2}} + \frac{n-1.n.n+1.n+2}{2.4}\frac{\cosh \zeta}{\zeta^{n+3}} - \frac{(n-2)\ldots(n+3)}{2.4.6}\frac{\sinh \zeta}{\zeta^{n+4}}.$$

If n is even, we must interchange $\sinh \zeta$ and $\cosh \zeta$.

If ζ is larger, so that $e^{-\zeta}$ can be neglected, compared to e^{ζ}, $\cosh \zeta = \sinh \zeta = e^{\zeta}$, and

$$\frac{\kappa^2 R^2}{2n+1.(2n+3)} \frac{\chi_{n+1}(\kappa R)}{\chi_{(n-1)}(\kappa R)}$$

which is the ratio of the vertical forces due to the induced and inducing potential will become

$$\frac{1 - \frac{n+1.n+2}{2}\zeta^{-1} + \frac{n.n+1.n+2.n+3}{2.4}\zeta^{-2} - \ldots}{1 - \frac{n-1.n}{2}\zeta^{-1} + \frac{n-2.n-1.n.n+1}{2.4}\zeta^{-2} - \ldots},$$

where

$$\zeta^2 = \frac{4\pi p}{\rho} R^2 i = i\delta ,$$

$$\zeta^{-2} = -\frac{i}{\delta}, \text{ and } \zeta^{-1} = \pm \frac{\delta^{-\frac{1}{2}}}{\sqrt{2}} (1 - i) = \delta^{-\frac{1}{2}} \left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right).$$

Generally in the above expansion we may put therefore

$$\zeta^{-q} = \delta^{-q/2}\left(\cos\frac{\pi q}{4} - i\sin\frac{\pi q}{4}\right);$$

so that the ratio of the vertical forces will be

$$\frac{X + iY}{X' + iY'},$$

where

$$X = 1 - \frac{n+1.n+2}{2}\cos\frac{\pi}{4} \quad \delta^{-\frac{1}{2}} + \frac{n.n+1.n+2.n+3}{2.4}\cos\frac{2\pi}{4} \quad \delta^{-1}$$
$$- \frac{(n-1)\ldots(n+4)}{2.4.6}\cos\frac{3\pi}{4} \quad \delta^{-\frac{3}{2}},$$

$$Y = \frac{n+1.n+2}{2}\sin\frac{\pi}{4} \quad \delta^{-\frac{1}{2}} - \frac{n.n+1.n+2.n+3}{2.4}\sin\frac{2\pi}{4} \quad \delta^{-1}$$
$$+ \frac{(n-1)\ldots(n+4)}{2.4.6}\sin\frac{3\pi}{4} \quad \delta^{-\frac{3}{2}},$$

$$X' = 1 - \frac{n-1.n}{2}\cos\frac{\pi}{4} \quad \delta^{-\frac{1}{2}} + \frac{n-2.n-1.n.n+1}{2.4}\cos\frac{2\pi}{4} \quad \delta^{-1} - \ldots,$$

$$Y' = \frac{n-1.n}{2}\sin\frac{\pi}{4} \quad \delta^{-\frac{1}{2}} - \frac{n-2.n-1.n.n+1}{2.4}\sin\frac{2\pi}{4} \quad \delta^{-1} + \ldots$$

The ratio of the amplitudes, if δ is large, becomes

$$\frac{\sqrt{(X^2 + Y^2)}}{\sqrt{(X'^2 + Y'^2)}} = 1 - (2n+1)(2\delta)^{-\frac{1}{2}},$$

and the angle between the two vertical forces

$$\tan^{-1}\frac{Y}{X} - \tan^{-1}\frac{Y'}{X'} = (2n+1)(2\delta)^{-\frac{1}{2}}.$$

The resultant of these two forces, resolved in a direction parallel to either of them, is therefore equal to the component which is at right angles, and the resultant will consequently be at an inclination of 45°.

The result will not hold, of course, if conduction in the Earth's crust takes place chiefly at some distance below the surface, as in that case the vertical force due to the induced currents will not tend to become equal to the vertical force due to the primary variation. For a bad conductivity we shall always have a resultant vertical force sensibly equal to the primary force. If the conductivity increases, the resultant will have a different phase from the primary variation, tending towards a difference of 45°, if the conductivity is uniform. If the conductivity is not uniform, a maximum difference of phase will be reached, which, if the conductivity is still further increased, diminishes indefinitely.

VI. *The Magnetic Potential on the Surface of the Earth.*

As it seemed interesting to trace the equipotential lines on the surface of the Earth as far as they depend on the diurnal variation, I have calculated the potential from the equations [A] and [B].

It is necessary, for this purpose, to compute the tesseral harmonics for definite points on the Earth's surface. It would seem most natural to choose these points, so that they lie on equidistant circles of latitudes, but as tables* exist for the zonal harmonics in terms of the *cosine* of the colatitude, I have selected values of these cosines so that the corresponding angles should differ as nearly as possible by 10°.

The values of u and $\mu = \cos u$, for which the potential is computed on the Northern hemisphere, are given in Table XXIV., u being the colatitude.

TABLE XXIV.

$\mu = \cos u =$	·98	·94	·87	·77	·64	·50	·34	·17	·00
$u =$	11° 29′	19° 57′	29° 32′	39° 39′	50° 12′	60° 00′	70° 07′	80° 13′	90° 00′

Symmetrical circles of latitude were taken on the Southern hemisphere.

Taking from Mr. GLAISHER'S Table the values of P_i corresponding to each of the above values of μ, we obtain the differential coefficients of zonal harmonics by a successive application of the formula

$$\frac{dP_i}{d\mu} - \frac{dP_{i-2}}{d\mu} = (2i - 1) P_{i-1}.$$

The first and second differential coefficients thus calculated are given in Tables XXV. and XXVI.

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* 'Report of the British Association' (Sheffield, 1879).

TABLE XXV. Values of $\frac{dP_i}{d\mu}$ where P_i is the zonal harmonic of degree i and μ the cosine of the colatitude.

$\mu =$	1·00.	0·98.	0·94.	0·87.	0·77.	0·64	0·50.	0·34.	0·17.	0·00.
$\frac{dP_1}{d\mu}$	+ 1·000	+ 1·000	+ 1·000	+ 1·000	+ 1·000	+ 1·000	+ 1·000	+ 1·000	+ 1·000	+ 1·000
$\frac{dP_2}{d\mu}$	+ 3·000	+ 2·940	+ 2·820	+ 2·610	+ 2·310	+ 1·920	+ 1·500	+ 1·020	+ 0·510	0·000
$\frac{dP_3}{d\mu}$	+ 6·000	+ 5·705	+ 5·125	+ 4·175	+ 2·945	+ 1·570	+ 0·375	+ 0·635	− 1·285	− 1·500
$\frac{dP_4}{d\mu}$	+ 10·000	+ 9·121	+ 7·482	+ 4·997	+ 2·212	− 0·215	− 1·559	− 1·864	− 1·191	0·000
$\frac{dP_5}{d\mu}$	+ 15·000	+ 12·988	+ 9·418	+ 4·562	+ 0·155	− 2·273	− 2·226	− 0·635	+ 1·145	+ 1·875
$\frac{dP_6}{d\mu}$	+ 21·000	+ 17·041	+ 10·496	+ 2·731	− 2·307	− 2·043	0·569	+ 1·755	+ 1·856	0·000
$\frac{dP_7}{d\mu}$	+ 28·000	+ 21·046	+ 10·393	− 0·170	− 4·174	− 1·428	+ 1·973	+ 2·030	0·662	2·181

TABLE XXVI.—Values of $\frac{\partial^2 P_i}{\partial \mu^2}$ where P_i is the zonal harmonic of degree i and μ the colatitude.

$\mu =$	1·00.	0·98	0·94.	0·87.	0·77.	0·64.	0·50.	9·34.	0·17.	0·00.
$\frac{d_2P_2}{d\mu_2}$	+ 3·000	+ 3·000	+ 3·000	+ 3·000	+ 3·000	+ 3·000	+ 3·000	+ 3·000	+ 3·000	+ 3·000
$\frac{d_2P_3}{d\mu_2}$	+ 15·000	+ 14·700	+ 14·100	+ 13·050	+ 11·550	+ 9·600	+ 7·500	+ 5·100	+ 2·550	0·000
$\frac{d_2P_4}{d\mu_2}$	+ 45·000	+ 42·935	+ 38·875	+ 32·225	+ 23·615	+ 13·990	+ 5·525	− 1·445	− 5·995	− 7·500
$\frac{d_2P_5}{d\mu_2}$	+ 105·000	+ 96·789	+ 81·438	+ 58·023	+ 31·458	+ 7·665	− 6·531	− 11·676	− 8·269	0·000
$\frac{d_2P_6}{d\mu_2}$	+ 210·000	+ 185·781	+ 142·473	+ 82·407	+ 25·320	− 11·013	− 18·861	− 8·430	+ 6·600	+ 13·125
$\frac{d_2P_7}{d\mu_2}$	+ 378·000	+ 318·322	+ 217·886	+ 93·526	+ 0·297	− 30·594	− 13·928	+ 11·139	+ 15·859	0·000
$\frac{d_2P_8}{d\mu_2}$	+ 630·000	+ 501·471	+ 298·368	+ 79·857	− 37·290	− 32·433	+ 10·734	+ 22·020	− 3·330	− 19·590

Introducing these quantities in the equations for the potential, and taking proper account of the change of sign of μ in the Southern hemisphere, I have obtained Table XXVII., in which the potentials are given for 24 equidistant meridian circles. In order to reproduce the daily variations, we must imagine the whole system of equipotential lines to revolve round the Earth from East to West; the time for which the potential is given is mean noon for the zero meridian. It will be remembered that the equations for the potential have been derived from the mean summer values in the Northern, and mean winter values in the Southern hemisphere. If we want to get a symmetrical potential in both hemispheres, we must take the average variation for the whole year, or, what comes to the same thing, we may in Table XXVII. write down the mean values for two corresponding circles of latitude, one in each hemisphere. This has been done in Table XXVIII., where the values are only given for the Northern hemisphere. The mean equipotential lines for the year are drawn in fig. 12. If we imagine the variable part of the magnetic force to be produced by a system of surface currents in a conducting sphere concentric with the Earth, and surrounding it, we may, if the potential is known, calculate the distribution of the lines of flow.

If the magnetic surface potential is of the form Ω_n, when Ω_n is a harmonic of degree n, the current function ϕ_n is given by

$$4\pi\phi_n = -\frac{2n+1}{n+1}\Omega_n,^*$$

so that the lines of flow are the same as the equipotential lines. This is no longer true when the magnetic potential is made up of a number of terms corresponding to harmonics of different degrees, for the factor $(2n+1)/(n+1)$ will vary for different terms, and the resultant current function will therefore no longer be proportional to the resultant magnetic potential.

In our own case, taking the mean values for the whole year, the series begins with Ω_2, and the factor $(2n+1)/(n+1)$ will vary, therefore, only between 5/3 and 2. We may then, as an approximation, still take the equipotential lines to give us the general form of the lines of flow. We conclude that we may imagine the daily variation of the Earth's magnetic force to be produced by a system of electric currents in a sphere surrounding the Earth, in which the lines of flow are roughly represented in fig. 12, the direction being such that at longitude 60° East the flow is away from the equator.

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* MAXWELL, 'Electricity and Magnetism,' vol. 2, p. 281.

TABLE XXVII.—Values of the variable part of the magnetic potential on the cosine of the colatitude) and 24 equidistant meridian circles, reckoned from is that of the mean summer months. The time is Greenwich noon.

$\mu = \cos u$.	$\lambda = 0$.	15.	30.	45.	60.	75.	90.	105.	120.	135.	150.
+ ·98	− 21·7	− 15·8	− 9·5	− 3·3	+ 1·6	+ 5·4	+ 7·9	+ 9·8	+ 11·6	+ 13·5	+ 15·9
+ ·94	− 61·8	− 46·6	− 28·6	−10·6	+ 5·1	+17·1	+ 25·4	+ 30·7	+ 34·1	+ 37·4	+ 41·0
+ ·87	−135·2	−106·9	− 69·5	−28·8	+ 9·1	+40·3	+ 62·3	+ 75·3	+ 81·3	+ 83·2	+ 84·1
+ ·77	−212·4	−171·6	−113·3	−47·0	+17·0	+70·3	+107·6	+127·7	+133·4	+129·6	+122·5
+ ·64	−247·3	−197·5	−126·4	−45·9	+30·7	+93·0	+134·2	+153·2	+153·7	+143·1	+129·6
+ ·50	−221·2	−167·3	− 96·3	−21·2	+46·1	+96·3	+125·3	+133·7	+127·3	+114·6	+103·3
+ ·34	−158·1	−105·7	− 45·3	+11·8	+56·3	+83·4	+ 92·5	+ 88·0	+ 76·9	+ 67·0	+ 64·1
+ ·17	− 93·5	− 49·5	− 5·1	+31·2	+54·2	+62·4	+ 58·2	+ 47·2	+ 36·0	+ 30·8	+ 34·7
00	− 42·4	− 13·4	+ 13·1	+32·2	+41·6	+41·5	+ 34·8	+ 24·9	+ 17·0	+ 14·4	+ 18·5
− ·17	+ 6·4	+ 17·1	+ 24·1	+26·8	+24·7	+19·6	+ 13·0	+ 6·8	+ 2·9	+ 1·4	+ 2·5
− ·34	+ 62·3	+ 56·2	+ 43·3	+26·3	+ 8·4	− 7·1	− 18·5	− 24·5	− 26·0	− 24·5	− 22·3
− ·50	+111·2	+ 94·3	+ 65·6	+30·4	− 5·2	−35·6	− 56·7	− 67·1	− 67·2	− 61·0	− 52·2
− ·64	+129·2	+107·8	+ 72·3	+28·1	−17·3	−56·7	− 84·9	− 98·7	− 98·9	− 88·7	− 73·1
− ·77	+106·8	+ 85·4	+ 52·2	+11·7	−29·7	−65·9	− 92·1	−105·1	−104·9	− 93·9	− 76·2
− ·87	+ 63·0	+ 44·2	+ 19·1	− 9·2	−37·2	−61·1	− 78·0	− 85·9	− 84·8	− 76·5	− 61·7
− ·94	+ 26·3	+ 12·1	− 3·8	−20·1	−35·1	−47·1	− 55·0	− 58·1	− 56·3	− 50·1	− 40·4
− ·98	+ 8·6	− 0·2	− 9·2	−17·5	−24·6	−29·9	− 33·1	− 33·7	− 32·0	− 28·1	− 22·2

TABLE XXVIII.—Values of the variable part of the magnetic potential on the cosine of the colatitude) and 24 equidistant meridian circles, reckoned from is that corresponding to the mean values for the year. The time is Green-

$\mu = \cos u$.	$\lambda = 0$.	15.	30.	45.	60.	75.	90.	105.	120.	135.	150.
·98	− 15·1	− 7·8	− 0·1	+ 7·1	+13·1	+17·6	+ 20·5	+ 21·7	+ 21·8	+ 20·8	+ 19·0
·94	− 44·0	− 29·3	− 12·4	+ 4·7	+20·1	+32·1	+ 40·2	+ 44·4	+ 45·2	+ 43·7	+ 40·7
·87	− 99·1	− 75·5	− 44·3	− 9·8	+23·1	+50·7	+ 70·1	+ 80·6	+ 83·0	+ 79·8	+ 72·9
·77	−159·6	−128·5	− 82·7	−29·3	+23·3	+68·1	+ 99·8	+116·4	+119·1	+111·7	+ 99·3
·64	−188·2	−152·6	− 99·3	−37·0	+24·0	+74·8	+109·5	+125·9	+126·3	+115·9	+101·3
·50	−166·2	−130·8	− 80·9	−25·8	+25·6	+65·9	+ 91·0	+100·4	+ 97·2	+ 87·8	+ 77·7
·34	−110·2	− 80·9	− 44·3	− 7·2	+23·9	+45·2	+ 55·5	+ 56·2	+ 51·4	+ 45·7	+ 43·2
·17	− 49·9	− 33·3	− 14·6	+ 2·2	+14·7	+21·4	+ 22·6	+ 20·2	+ 16·5	+ 14·7	+ 16·1
·00	00·0	00·0	00·0	00·0	00·0	00·0	00·0	00·0	00·0	00·0	00·0

Earth's surface for 17 latitude circles corresponding to different values of μ (the Greenwich towards the East. The period of the year to which the Table refers

165.	180.	195.	210.	225.	240.	255.	270.	285.	300.	315.	330.	345.
+18·4	+20·3	+21·0	+19·7	+16·1	+10·2	+2·2	−6·5	−15·0	−21·8	−26·3	−27·7	−26·0
+45·2	+48·8	+50·4	+48·4	+40·8	+27·5	+9·1	−12·4	−34·5	−53·9	−67·6	−73·6	−71·4
+85·7	+87·6	+98·3	+85·4	+74·2	+54·1	+23·0	−14·7	−56·7	−96·9	−128·6	−147·3	−149·5
+116·6	+114·0	+113·4	+110·9	+101·0	+79·0	+41·9	−9·2	−69·5	−131·0	−183·6	−218·5	−228·8
+119·7	+116·3	+118·2	+119·8	+113·9	+93·5	+54·4	−3·2	−73·8	−147·1	−211·1	−253·8	−267·1
+98·9	+102·6	+111·1	+118·0	+114·2	+93·7	+53·7	−6·7	−77·5	−148·7	−207·6	−243·2	−247·9
+70·7	+84·5	+99·9	+109·1	+104·4	+81·1	+38·4	−18·9	−82·2	−140·7	−183·2	−201·5	−192·5
+47·5	+64·9	+80·9	+88·3	+81·2	+57·4	+18·6	−29·6	−78·6	−119·2	−143·2	−146·3	−128·5
+27·8	+39·4	+48·6	+50·9	+43·4	+25·2	−1·1	−31·5	−60·1	−81·0	−90·0	−85·3	−68·2
+5·1	+7·4	+7·9	+5·1	−0·8	−9·1	−18·6	−26·8	−31·8	−32·1	−27·4	−18·1	−6·1
−21·4	−22·7	−26·6	−31·5	−35·5	−36·2	−31·7	−21·1	−5·1	+14·2	+33·7	+50·1	+60·2
−44·7	−41·0	−41·5	−44·6	−46·6	−43·8	−33·2	−13·5	+14·2	+46·2	+77·2	+101·2	+113·5
−57·2	−45·0	−38·2	−35·7	−34·5	−30·1	−19·3	+0·7	+29·1	+62·3	+95·1	+120·5	+133·2
−56·6	−38·8	−25·4	−16·8	−9·1	−1·9	+8·5	+24·1	+44·9	+68·5	+91·3	+107·8	+114·0
−44·8	−27·6	−12·2	+0·9	+12·0	+22·0	+31·9	+42·6	+53·9	+64·8	+73·2	+76·8	+74·0
−28·7	−15·7	−2·7	+9·4	+20·5	+29·9	+38·1	+44·4	+48·7	+50·7	+49·7	+45·6	+37·7
−15·0	−6·8	+1·4	+9·6	+17·1	+23·4	+28·3	+31·3	+32·5	+31·6	+28·5	+23·4	+16·6

Earth's surface for 17 latitude circles corresponding to different values of μ (the Greenwich towards the East. The period of the year to which the Table refers wich noon.

165.	180.	195.	210.	225.	240.	255.	270.	285.	300.	315.	330.	345.
+16·7	+13·5	+9·8	+5·0	−0·5	−6·6	−13·0	−18·9	−23·7	−26·7	−27·4	−25·5	−21·3
+36·9	+32·2	+26·5	+19·5	+10·1	−1·2	−14·5	−28·4	−41·6	−52·3	−58·6	−59·6	−54·5
+65·2	+57·6	+55·2	+42·2	+31·1	+16·0	−4·4	−28·6	−55·3	−80·8	−100·9	−112·0	−111·7
+86·6	+76·4	+69·4	+63·8	+55·0	+40·4	+16·7	−16·6	−57·2	−99·7	−137·4	−163·1	−171·4
+88·4	+80·6	+78·2	+77·7	+74·2	+61·8	+36·8	−1·9	−51·4	−104·7	−153·1	−187·1	−200·1
+71·8	+71·8	+76·3	+81·3	+80·4	+68·7	+43·4	+3·4	−45·8	−97·4	−142·4	−172·2	−180·7
+46·0	+53·6	+63·2	+70·3	+69·9	+58·6	+35·0	+1·1	−38·5	−77·4	−108·4	−125·8	−126·3
+21·2	+28·7	+36·5	+41·6	+41·0	+33·2	+18·6	−1·4	−23·4	−43·5	−57·9	−64·1	−61·2
00·0	00·0	00·0	00·0	00·0	00·0	00·0	00·0	00·0	00·0	00·0	00·0	00·0

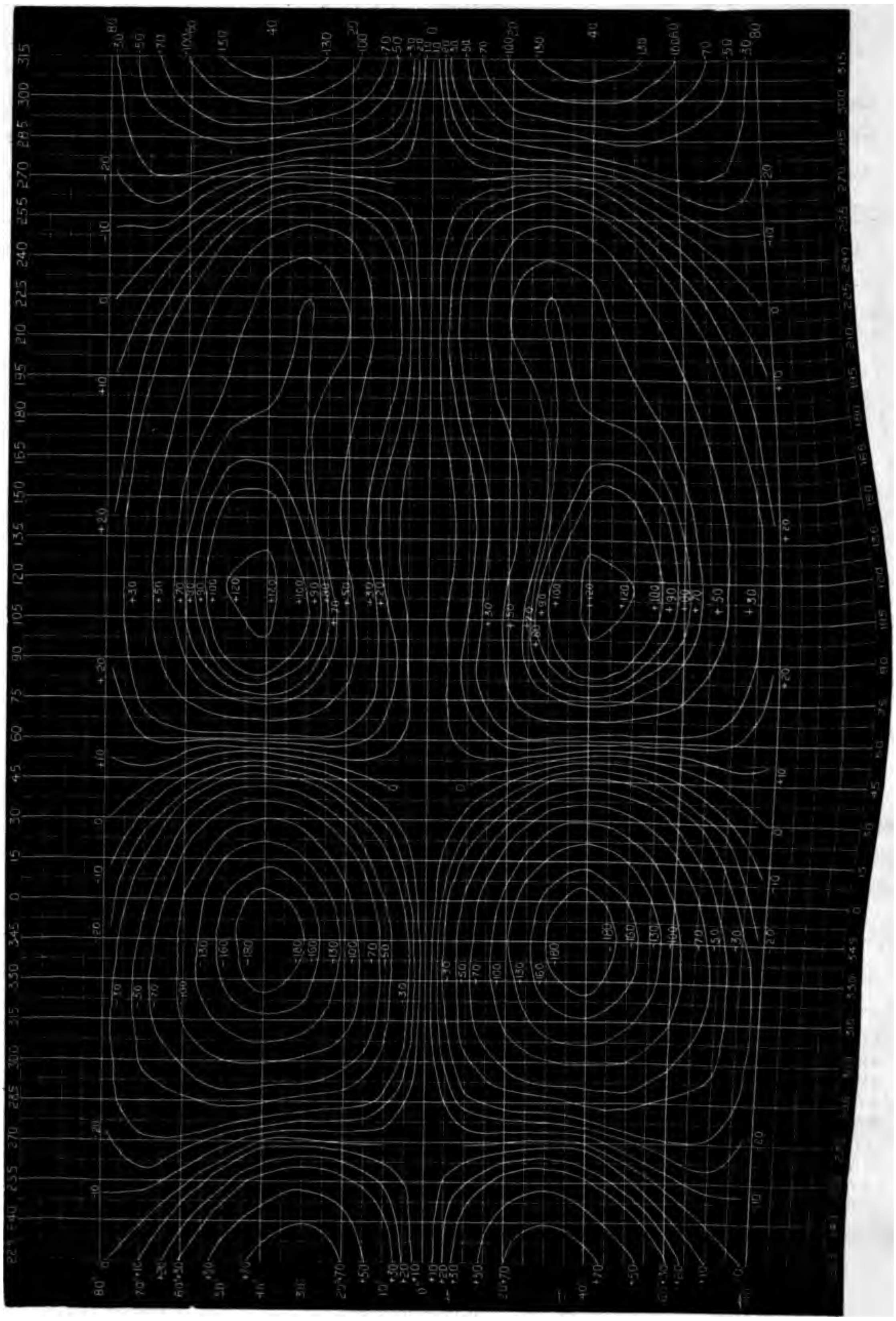
315
300
285
270
255
240
225
210
195
180
165
150
135
120
105
90
75
60
45
30
15
0
345
330
315
300
285
270
255
240
225
80°
70°
60°
50°
40°
30°
20°
10°
0°
+120
+100
+90
+70
+50
+30
+20
+10
0
-10
-20
-30
-50
-70
-100
-130
-150
-160
-180

Fig. 12.

VII. *Concluding Remarks.*

Faraday, in the year 1850, discussed the diurnal variation of the magnetic needle. He showed that the changes which took place during daytime could be accounted for by supposing two magnetic poles—namely, a North pole in the Southern hemisphere, and a South pole in the Northern hemisphere—to be carried round with the Sun in our atmosphere. A glance at fig. 13 will show that our result entirely agrees with **Faraday's**. The proof that the principal part of the Earth's magnetism is due to causes outside its surface would have been almost as complete in the year 1850 as it is now, if **Faraday** had added the remark that, if all three components of the variation can be completely accounted for by hypothetical changes taking place outside the Earth's surface, they *cannot* be accounted for by changes taking place in the inside.

I cannot agree, however, with **Faraday** in the explanation which he gives of the variation. He imagines that the solar radiation, heating up the air, produces a sufficient change in its magnetic permeability to account for the observed deflection of the lines of magnetic force.

The magnetic susceptibility of oxygen at the atmospheric pressure and temperature is about $\cdot 5, 10^{-7}$, and for air it is smaller still. This would give the magnetic permeability as 1·0000006. If the air was entirely removed the change of magnetic force would be so small that we could not detect it. I have tried in various ways to find how a partial removal of the atmosphere as a magnetic medium could affect the needle in any appreciable way, but have failed to do so. **Faraday** suggests that the oxygen in the higher regions of the atmosphere might, owing to the greater cold, be much more magnetic than what we observe it to be. But, on the other hand, owing the smaller density, the permeability would be diminished; so that I do not think we are at present justified in ascribing any material part of the daily variation to a change of the magnetic permeability of air due to the heating effect of the Sun. The effect of the Moon suggests a tidal action as the cause, and we may inquire whether such a tidal action could produce the observed effects. The late Professor **Balfour Stewart** has suggested that the Earth's magnetic force might induce electric currents in the convection currents which flow in the upper regions of the atmosphere. One difficulty of this hypothesis was removed by an experimental investigation, by means of which I have proved that the air can be thrown into a sensitive state in which small electromotive forces will produce sensible electric currents. To bring the air into that sensitive state it is only necessary to send an electric current through it from some independent source of high potential. It is very likely that the air in the upper regions of our atmosphere is in such a sensitive state, and it is quite possible, therefore, that the induced electric currents suggested by Professor **Balfour Stewart** really exist.

The symmetry of the diurnal variation in both hemispheres shows that, if it is due to the assumed cause, the vertical component of the magnetic force is the important factor, as that component changes sign on crossing the magnetic equator. In order that electric currents should be induced which could account for the observed movement of the magnetic needle, it is only necessary to imagine convection currents in the upper regions from East to West during certain parts of the day, and from West to East at other times. Judging from the analogy of the theory of waves in shallow water, a horizontal motion of considerable velocity might be produced by a tidal action due to solar and lunar attraction. It is true that no periodic change of the barometer has been traced with certainty to a tidal action; but I suppose that a tidal wave must nevertheless exist, and that its horizontal flow might be considerable, while the changes of pressure might escape our attention. As regards the effect of the Sun we have, indeed, a daily period of the barometer which is probably due to thermal effects. It is curious and suggestive that the horizontal motion which must accompany the change in pressure is just such as would account for the daily variation of the magnetic needle.. In the tropics the principal minimum of the barometer takes place about 3.40 o'clock in the afternoon, and the principal maximum about 9 o'clock in the morning. According to the theory of waves, there would be a horizontal movement from West to East in the afternoon, and from East to West in the morning. The direction of the induced electric currents would be away from the equator in both hemispheres in the afternoon, and towards the equator in the morning. This is exactly the system of currents we have been led to, starting from the observed magnetic variation. The only difficulty I feel in suggesting that the cause of the diurnal variation of the magnetic needle is the diurnal variation of the barometer lies in the fact that it would oblige us to place the electric currents into the lower regions of the atmosphere, as these only will be much affected by the thermal radiation of the Sun. The phase of the barometric oscillation has been found to be reversed on the top of mountains, and it would be interesting to see whether the magnetic variations show any peculiarities at great heights.*

The region of the atmosphere which other considerations lead me to consider as the most sensitive to electromotive forces is that of the cirrus clouds, and I should be inclined to look to that region for a solution of the question. The lunar action seems, according to the researches of Mr. CHAMBERS, to be a modification of the solar action rather than an independent effect. This might be accounted for if we suppose that the conductivity of the air depends on the position of the Sun, while the electromotive forces depend on the combined positions of the Sun and Moon.

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* [Note added October 11, 1889.—Since writing the above I have become acquainted with HANN'S recent work on the diurnal oscillation of the barometer ('Wien, Denkschriften,' vol. 55, 1889.) It appears from the regularity of the semidiurnal period in different altitudes and latitudes that its cause must lie in atmospheric movements in higher regions of the atmosphere. The reversal of phase mentioned in the text is due to local effects and has nothing to do with the regular oscillation. It seems to be exceedingly probable in the light of these researches that the daily variation of the magnetic needle is connected with the daily oscillation of the barometer in the way described in the text.]

It will be interesting to follow out in future researches the field which this investigation has opened, especially in order to trace the effect of the Sunspot variation; but for this purpose it is absolutely necessary that different observatories should follow a more uniform plan in reducing their observations. It has been found by experience that if the hourly readings of the magnetic needle are collected together, and their mean taken, that mean is different according as the disturbed days are taken into account or rejected. In other words, the disturbances are not irregularly distributed, but have a daily period which is mixed up with the regular daily variation. If we want to separate the investigation concerning the regular variation and the disturbance variation, we must adopt some plan of obtaining the one without the other. I need not here describe SABINE'S well-known method of doing this. Grave objections have been urged against it, but it is still adopted in many observatories. A discussion of the various methods of reduction which have been proposed will be found in recent Reports of the British Association, and amongst them that adopted by Mr. WILD at St. Petersburg seems to me to be the only one which can be justified on strict scientific principles. It consists in selecting the curves for the quiet days, of which there are always a sufficient number in each month, and not to take account, as SABINE'S method does, of any reading at all during the disturbed days. We get in this way something perfectly definite, namely, the mean variation of the magnetic needle during certain specified days. It seems to me that if the heads of different observatories could adopt some system of intercommunication, by which they could select those days which are most quiet all over the world, and if the elements are reduced for those days solely at the different stations, we should obtain a series of values for different points of the Earth which are strictly comparable with each other. The labour of reduction, as far as I can judge, would thereby be seriously diminished. The method hitherto adopted at Greenwich is very similar to that of WILD, and will not, probably, lead to results which are sensibly different.

The reduction of the observations by spherical harmonic analysis would be a very simple matter according to the method which I have followed, if the results of different stations were published in a manner which would lend itself easily to the work. The method of publication adopted at Greenwich is very convenient, and might serve as a model to other observatories. Much labour is, however, involved in reducing variations in horizontal force and declination to variations in force towards the geographical West and North respectively. If all observatories could publish the coefficients of the harmonic series of the elements as at Greenwich, but reduced to the geographical instead of the magnetic co-ordinates, the progress of magnetic science would be much assisted, as every scientific investigation must take the geographical components for its starting point.

I have tried to form an idea as to the degree of accuracy reached in the determination of such quantities as the daily variation of declination; the result is not altogether satisfactory. Mr. WHIPPLE, in the 'British Association Report' (Birmingham, 1886, page 71), says :—

"Contrasting the Kew results with those of Greenwich, we may fairly consider the difference to be due in some measure to instrumental causes, the construction of the magnetographs being dissimilar at the two observatories. The slight difference in position of the two observatories may likewise have some influence."

The difference amounts to about 15 per cent., and it seems as if the question whether such a difference can be due to instrumental causes deserves a careful examination. Mr. CHAMBERS, at Bombay, has found similarly that the results of the magnetographs differ from those obtained by the old magnetometers; and he seems to ascribe the difference to an "influence of height above or below the ground level." The height of the magnetometer was 6 feet above ground, and that of the magnetograph 7½ feet below; the former gives ranges greater by 7 per cent. for the declination variation, and the difference is greater still for the horizontal force component. That there should be a real difference of that magnitude in the two positions seems excessively unlikely, and we must conclude that at present the results given by magnetographs are doubtful to the extent of about 10 per cent.

In conclusion, we may sum up the principal results obtained in this paper as follows :—

1. The principal part of the diurnal variation is due to causes outside the Earth's surface, and probably to electric currents in our atmosphere.

2. Currents are induced in the Earth by the diurnal variation which produce a sensible effect chiefly in reducing the amplitude of the vertical component and increasing the amplitude of the horizontal components.

3. As regards the currents induced by the diurnal variation, the Earth does not behave as a uniformly-conducting sphere, but the upper layers must conduct less than the inner layers.

4. The horizontal movements in the atmosphere which must accompany a tidal action of the Sun or Moon or any periodic variation of the barometer such as is actually observed would produce electric currents in the atmosphere having magnetic effects similar in character to the observed daily variation.

5. If the variation is actually produced by the suggested cause, the atmosphere must be in that sensitive state in which, according to the author's experiments, there is no lower limit to the electromotive force producing a current.

In conclusion, the author begs to return his thanks to Mr. WILLIAM ELLIS for help given in some of the calculations, and also to his assistant, Mr. ARTHUR STANTON, for much labour bestowed on making and checking numerical calculations.

APPENDIX.

On the Currents Induced in a Spherical Conductor by Variation of an External Magnetic Potential.

By HORACE LAMB, *M.A., F.R.S.*

THE general formulæ for the currents induced in a sphere of uniform conductivity by any electric or magnetic disturbances outside it have been given in the 'Phil. Trans.,' 1883, pp. 526 *et seq.* I here reproduce (with some further developments) so much of the investigation as is required for the discussion in Part V of the foregoing paper.

Suppose that, the origin being taken at the centre of the Earth, we have an external disturbing force, whose magnetic potential near the Earth's surface is

$$\Omega_n e^{ipt},$$

where Ω_n is a solid harmonic of positive degree n, and $i = \sqrt{-1}$.

The corresponding values of the components of electric momentum are

$$F = [illegible]$$

$$G = [illegible]$$

$$H = [illegible]$$

the time-factor being omitted here and elsewhere for brevity. These values make

$$[illegible] = 0$$

and also give for the components of magnetic force the values

$$a = [illegible]$$

$$[illegible]$$

and similarly

$$b = [illegible]$$

$$c = [illegible]$$

[illegible]

If u, v, w be the components of electric current, the equations of induced currents are

$$\rho u = -\dot{\mathrm{F}}, \qquad \rho v = -\dot{\mathrm{G}}, \qquad \rho w = -\dot{\mathrm{H}},$$

where ρ denotes the specific resistance. Eliminating u, v, w by means of the relations

$$\nabla^2\mathrm{F} = -4\pi u, \qquad \nabla^2\mathrm{G} = -4\pi v, \qquad \nabla^2\mathrm{H} = -4\pi w,$$

we find that at all points within the sphere F, G, H must satisfy the equations

$$(\nabla^2 - k^2)\,\mathrm{F} = 0, \qquad (\nabla^2 - k^2)\,\mathrm{G} = 0, \qquad (\nabla^2 - k^2)\,\mathrm{H} = 0,$$

$$\frac{d\mathrm{F}}{dx} + \frac{d\mathrm{G}}{dy} + \frac{d\mathrm{H}}{dz} = 0,$$

where

$$k^2 = \frac{4\pi p}{\rho}\cdot\iota.$$

The appropriate solution of these is

$$\left.\begin{aligned} \mathrm{F} &= \chi_n(kr).\left(y\frac{d}{dz} - z\frac{d}{dy}\right)\omega_n, \\ \mathrm{G} &= \chi_n(kr).\left(z\frac{d}{dx} - x\frac{d}{dz}\right)\omega_n, \\ \mathrm{H} &= \chi_n(kr).\left(x\frac{d}{dy} - y\frac{d}{dx}\right)\omega_n, \end{aligned}\right\}$$

where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, ω_n denotes a solid harmonic of degree n, and

$$\begin{aligned} \chi_n(\zeta) &= 1 + \frac{\zeta^2}{2.2n+3} + \frac{\zeta^4}{2.4.2n+3.2n+5} + \ldots \\ &= 3.5\ldots 2n+1.\left(\frac{d}{\zeta d\zeta}\right)^n \frac{\sinh\zeta}{\zeta}. \end{aligned}$$

The total magnetic potential outside the sphere will be

$$\Omega_n + \Omega_{-n-1},$$

where Ω_{-n-1} is the part due to the induced currents. The values of F, G, H at external points will then be

$$\begin{aligned} \mathrm{F} &= \left(y\frac{d}{dz} - z\frac{d}{dy}\right)\left(\frac{1}{n+1}\Omega - \frac{1}{n}\Omega_{-n-1}\right), \\ \mathrm{G} &= \left(z\frac{d}{dx} - x\frac{d}{dz}\right)\left(\frac{1}{n+1}\Omega_n - \frac{1}{n}\Omega_{-n-1}\right), \\ \mathrm{H} &= \left(x\frac{d}{dy} - y\frac{d}{dx}\right)\left(\frac{1}{n+1}\Omega_n - \frac{1}{n}\Omega_{-n-1}\right). \end{aligned}$$

It remains to introduce the conditions to be satisfied at the surface of the sphere ($r = \mathrm{R}$). The continuity of electromotive force, *i.e.*, of $\dot{\mathrm{F}}$, $\dot{\mathrm{G}}$, $\dot{\mathrm{H}}$, requires

$$\chi_n(k\mathrm{R}) \, . \, \omega_n = \frac{1}{n+1}\,\Omega_n - \frac{1}{n}\,\Omega_{-n-1}. \quad [r = \mathrm{R}]$$

The continuity of the magnetic force involves the continuity of the space-derivatives of F, G, H, and, therefore, of $d\mathrm{F}/dr$, $d\mathrm{G}/dr$, $d\mathrm{H}/dr$. Hence

$$\{k\mathrm{R}\chi'_n(k\mathrm{R}) + n\chi_n(k\mathrm{R})\}\,\omega_n = \frac{n}{n+1}\,\Omega_n + \frac{n+1}{n}\,\Omega_{-n-1}. \quad [r = \mathrm{R}]$$

We thence find

$$\left.\begin{aligned} k\mathrm{R} \, . \, \chi'_n(k\mathrm{R}) \, . \, \omega_n &= \frac{2n+1}{n}\,\Omega_{-n-1}, \\ \{k\mathrm{R}\chi'_n(k\mathrm{R}) + (2n+1)\chi_n(k\mathrm{R})\}\,\omega_n &= \frac{2n+1}{n+1}\,\Omega_n, \end{aligned}\right\} \quad [r = \mathrm{R}]$$

which are equivalent to

$$\left.\begin{aligned} \frac{k^2\mathrm{R}^2}{2n+1 . 2n+3}\,\chi_{n+1}(k\mathrm{R}) \, . \, \omega_n &= \frac{1}{n}\,\Omega_{-n-1}, \\ \chi_{n-1}(k\mathrm{R}) \, . \, \omega_n &= \frac{1}{n+1}\,\Omega_n. \end{aligned}\right\} \quad [r = \mathrm{R}]$$

Hence

$$\frac{\Omega_{-n-1}}{\Omega_n} = \frac{n}{n+1} \, . \, \frac{k^2\mathrm{R}^2}{2n+1 . 2n+3} \, . \, \frac{\chi_{n+1}(k\mathrm{R})}{\chi_{n-1}(k\mathrm{R})}.$$

This gives the ratio of surface potentials, and, therefore, of horizontal forces, due to internal and external influences respectively. Since this ratio is "complex," there will be a difference of phase, as well as of amplitude. The corresponding ratio of vertical forces is

$$\left[\frac{-\dfrac{d\Omega_{-n-1}}{dr}}{-\dfrac{d\Omega_n}{dr}}\right]_{r=\mathrm{R}} = -\frac{n+1}{n} \, . \, \frac{\Omega_{-n-1}}{\Omega_n} = -\frac{k^2\mathrm{R}^2}{2n+1 . 2n+3} \, . \, \frac{\chi_{n+1}(k\mathrm{R})}{\chi_{n-1}(k\mathrm{R})}.$$

To interpret these results it is necessary to calculate the function

$$\frac{\zeta^2}{2n+1 . 2n+3} \, . \, \frac{\chi_{n+1}(\zeta)}{\chi_{n-1}(\zeta)},$$

where

$$\zeta^2 = k^2\mathrm{R}^2 = \frac{4\pi p\mathrm{R}^2}{\rho} \cdot i = i\delta, \text{ say.}$$

For moderate values of δ we may use the form

$$\frac{i\delta}{2n+1.2n+3}\cdot\frac{A_{n+1}+iB_{n+1}}{A_{n-1}+iB_{n-1}},$$

where

$$A_n = 1 - \frac{\delta^2}{2.4.2n+3.2n+5} + \frac{\delta^4}{2.4.6.8\,2n+3.2n+5.2n+7.2n+9} - \dots,$$

$$B_n = \frac{\delta}{2.2n+3} - \frac{\delta^3}{2.4.6.2n+3.2n+5.2n+7} = \dots.$$

The following Table gives the values of A_1, B_1, A_3, B_3, A_5, B_5, for various values of δ. It may possibly be of service in other investigations.

TABLE XXIX.

δ.	A_1.	B_1.	A_3.	B_3.	A_5.	B_5.
1	+ 0·996429	+ 0·099934	+ 0·998738	+ 0·055539	+ 0·999359	+ ·038456
2	+ 0·985726	+ 0·199471	+ 0·994952	+ 0·110982		
3	+ 0·967918	+ 0·298216	+ 0·988647	+ 0·116230		
4	+ 0·943050	+ 0·395773	+ 0·979832	+ 0·221187		
5	+ 0·911184	+ 0·491751	+ 0·968518	+ 0·275757		
6	+ 0·872401	+ 0·585759	+ 0·954720	+ 0·329843		
7	+ 0·826800	+ 0·677412	+ 0·938455	+ 0·383350		
8	+ 0·774498	+ 0·766327	+ 0·919714	+ 0·436182		
9	+ 0·715628	+ 0·852126	+ 0·898610	+ 0·488246		
10	+ 0·650341	+ 0·934439	+ 0·875083	+ 0·539448	+ 3·936309	+ ·3784
20	− 0·310366	+ 1·489206	+ 0·516311	+ 0·984135		
30	− 1·628644	+ 1·351846	− 0·029614	+ 1·248628		
40	− 2·912855	+ 0·337304	− 0·688914	+ 1·265555		
50	− 3·716054	− 1·563371	− 1·366602	+ 0·993199		
100	+ 12·514840	− 10·691120	− 1·308236	− 3·811231	− 1·9434	− ·6646

If we write

$$A_n + iB_n = S_n e^{i\phi_n},$$

the above fraction becomes

$$\frac{\delta}{2n+1.2n+3}\cdot\frac{S_{n+1}}{S_{n-1}}\cdot e^{i(\phi_{n+1}-\phi_{n-1}+\frac{1}{2}\pi)}.$$

If we prefix the *minus* sign this gives the ratio (k') of the vertical forces. To get the ratio (k) of the horizontal forces we must multiply by $n/(n+1)$.

For large values of δ we may make use of the second form of χ_n. Thus

$$\chi_1 = 3\left\{\frac{\cosh\zeta}{\zeta^2} - \frac{\sinh\zeta}{\zeta^3}\right\},$$

$$\chi_3 = 3.5.7\left\{\left(\frac{1}{\zeta^4}+\frac{15}{\zeta^6}\right)\cosh\zeta - \left(\frac{6}{\zeta^5}+\frac{15}{\zeta^7}\right)\sinh\zeta\right\},$$

whence

$$\frac{\zeta^2}{5.7}\frac{\chi_3(\zeta)}{\chi_1(\zeta)} = \frac{(1 + 15\zeta^{-2})\cosh\zeta - (6\zeta^{-1} + 15\zeta^{-3})\sinh\zeta}{\cosh\zeta - \zeta^{-1}\sinh\zeta}.$$

Here

$$\zeta = (i\delta)^{\frac{1}{2}} = (1 + i)\,\delta^{\frac{1}{2}}/\sqrt{2} = (1 + i)\,\beta,\ \text{say.}$$

If β is moderately large we may put $\cosh\zeta/\sinh\zeta = 1$ approximately. The error thus committed is of the order $e^{-2\beta}$. Since the value of e^{-14} has six cyphers after the decimal point, this approximation is amply sufficient for $\beta > 7$, or say for $\delta > 100$. The above fraction is then

$$= \frac{1 - 6\zeta^{-1} + 15\zeta^{-2} - 15\zeta^{-3}}{1 - \zeta^{-1}} = \frac{(1 - 3\beta^{-1} + \frac{15}{4}\beta^{-3}) + i(3\beta^{-1} - \frac{15}{2}\beta^{-2} + \frac{15}{4}\beta^{-3})}{(1 - \frac{1}{2}\beta^{-1}) + i.\frac{1}{2}\beta^{-1}}.$$

It is by these methods that Tables XX. and XXI. above were calculated. The values of ρ, the specific conductivity, given in the fifth columns, were obtained from the formula

$$\rho = 4\pi p\mathrm{R}^2/\delta$$

by putting

$$2\pi\mathrm{R} = 4\cdot10^9\ \text{cm.},\quad 2\pi/p = 86{,}400/m\ \text{secs.},$$

where m denotes the number of complete periods in a day, and is therefore $= 1$ for the diurnal and $= 2$ for the semidiurnal variations.

As the resistance diminishes, the difference of phase tends to zero, and the ratio of normal forces to the value $- 1$; *i.e.*, the total normal force at the surface tends to zero, in accordance with the theory of electromagnetic screens.

The ratio of the total vertical to the total horizontal force in any assigned direction is

$$\frac{-\dfrac{d}{dr}(\Omega_n + \Omega_{-n-1})}{-\dfrac{d}{\mathrm{R}d\eta}(\Omega_n + \Omega_{-n-1})}$$

where $\mathrm{R}d\eta$ denotes a linear element drawn in the proper direction on the Earth's surface. By means of the preceding results this can be put in the form

$$\frac{n(n+1)\,\chi_n(k\zeta)}{k\mathrm{R}\,\chi'_n(k\mathrm{R}) + (n+1)\,\chi_n(k\mathrm{R})}\cdot\Omega_n\Big/\frac{d\Omega_n}{d\eta}.$$

The coefficient may be calculated independently, by a proper adaptation of the previous methods, or we may deduce its value from the results already obtained, in the manner explained by Professor SCHUSTER. The function actually computed by Professor SCHUSTER in Tables XXII. and XXIII. is the ratio

$$\frac{-\frac{d}{dr}(\Omega_n + \Omega_{-n-1})}{-\frac{d\Omega_n}{dr}},$$

which

$$= \frac{(n+1)\,\chi_n(k\mathrm{R})}{k\mathrm{R}\,\chi_n'(k\mathrm{R}) + (n+1)\,\chi_n(k\mathrm{R})}.$$

For large values of δ, *i.e.* for sufficiently small values of ρ, we may put $\cosh\zeta = \sinh\zeta$, whence, keeping only the most important term in $\chi_n(\zeta)$, the fraction written becomes

$$= (n+1)/\zeta = \frac{n+1}{\sqrt{\delta}} \cdot e^{-i\pi/4}.$$

The difference of phase therefore tends to the limit 45°, as remarked by Professor SCHUSTER. For $\delta = 100$, this formula gives for the reduction of amplitude the value ·3 and ·5 in the cases $n = 2$ and $n = 4$ respectively.

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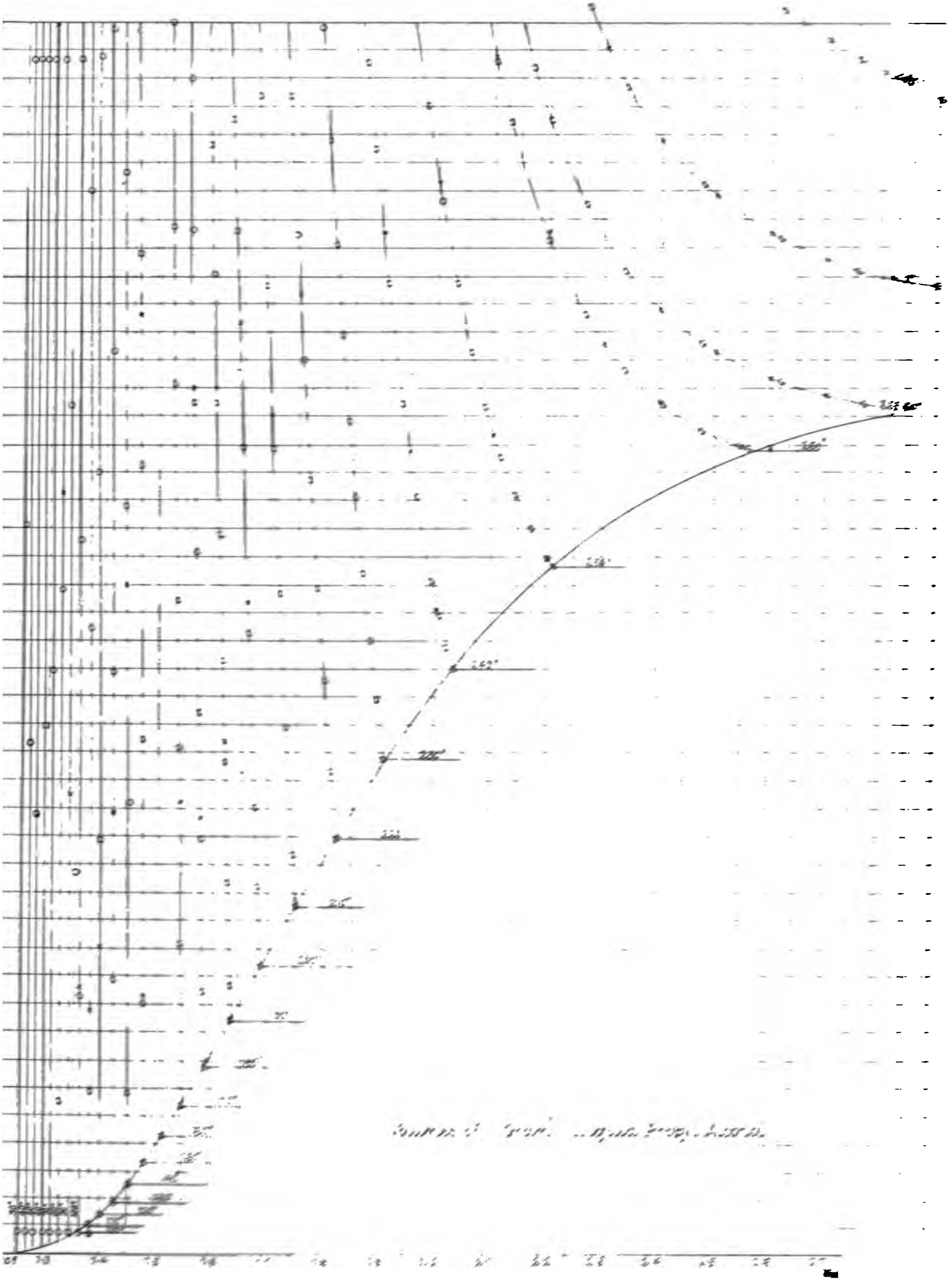

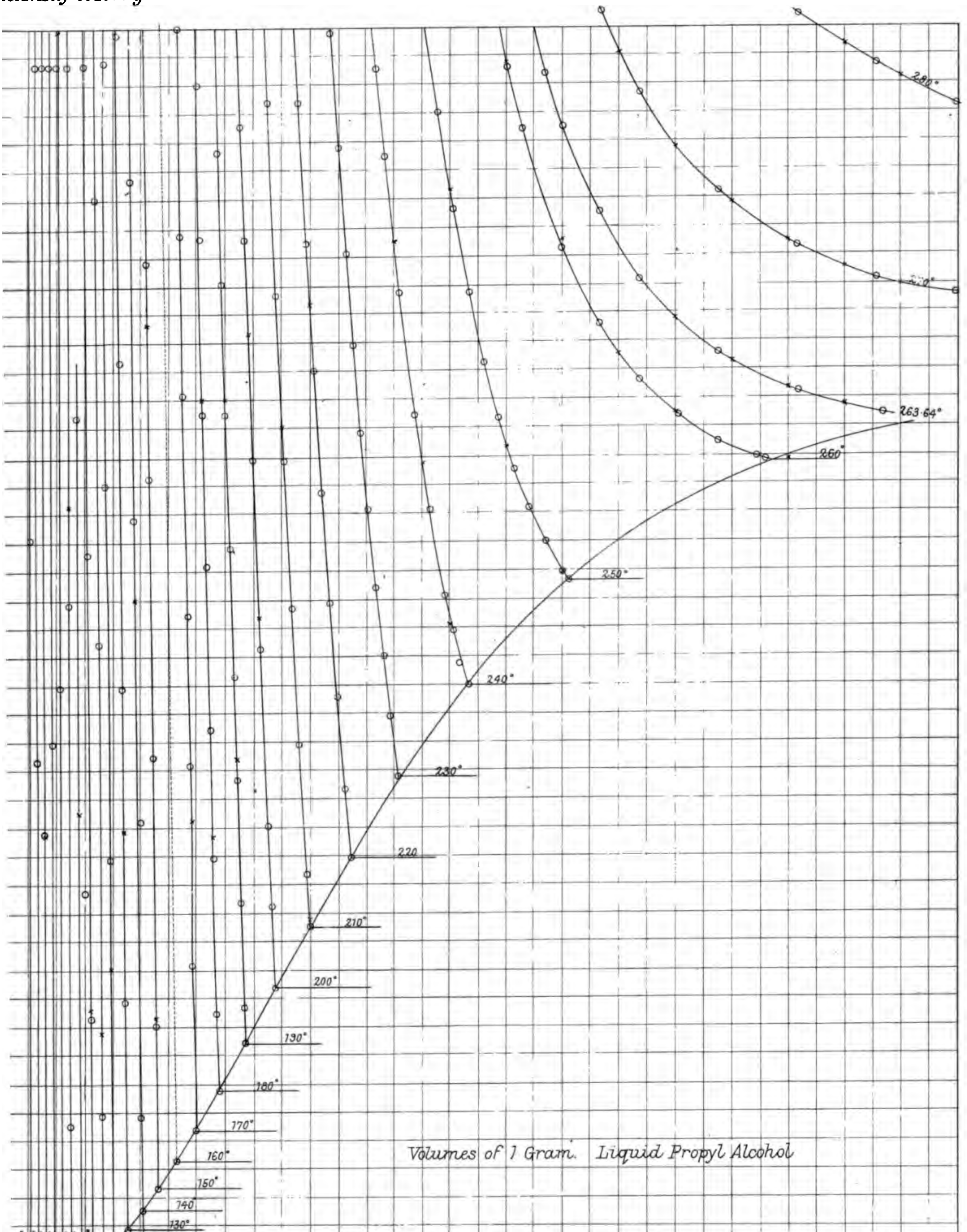

Volumes of 1 Gram. Liquid Propyl Alcohol

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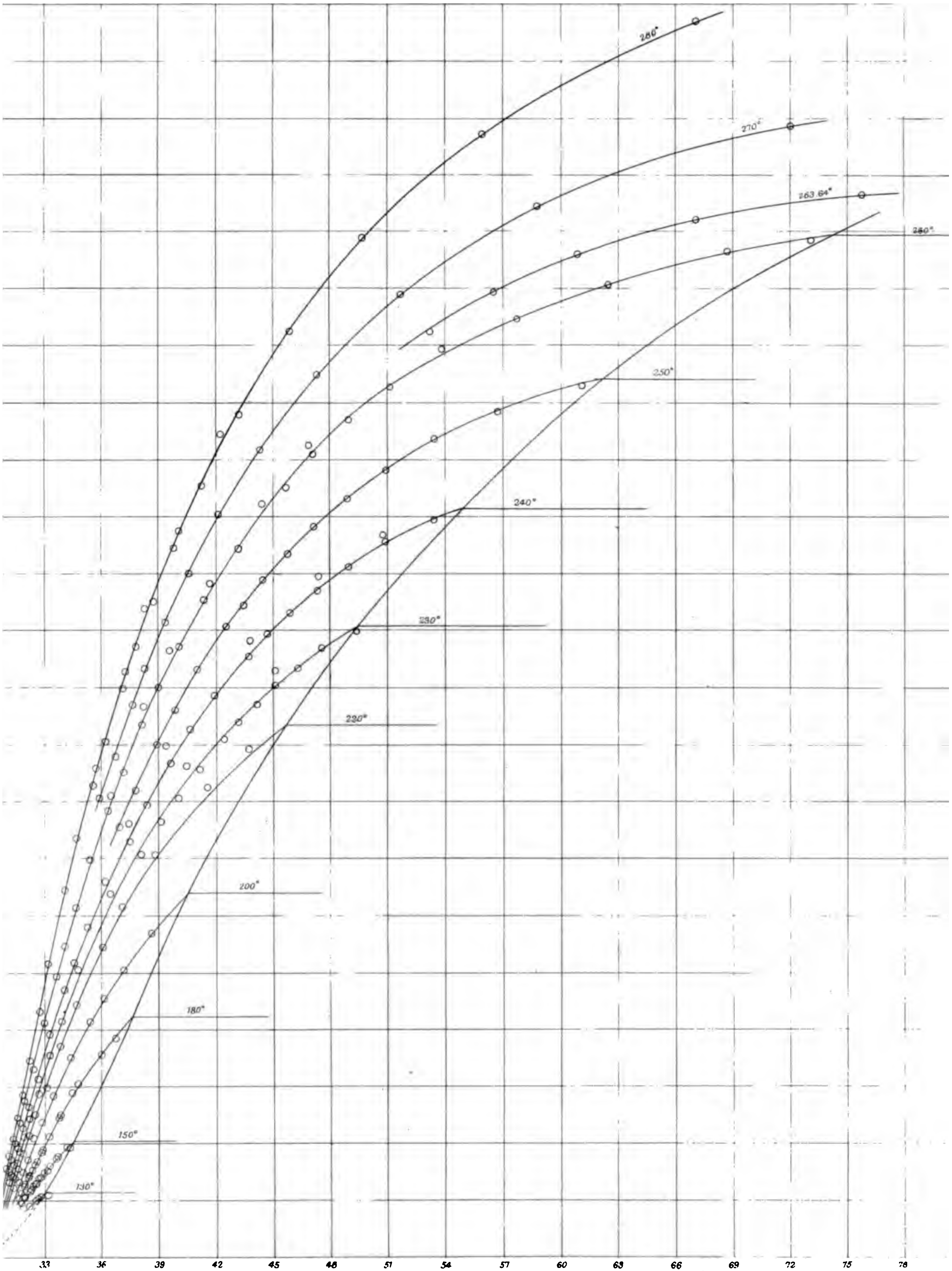
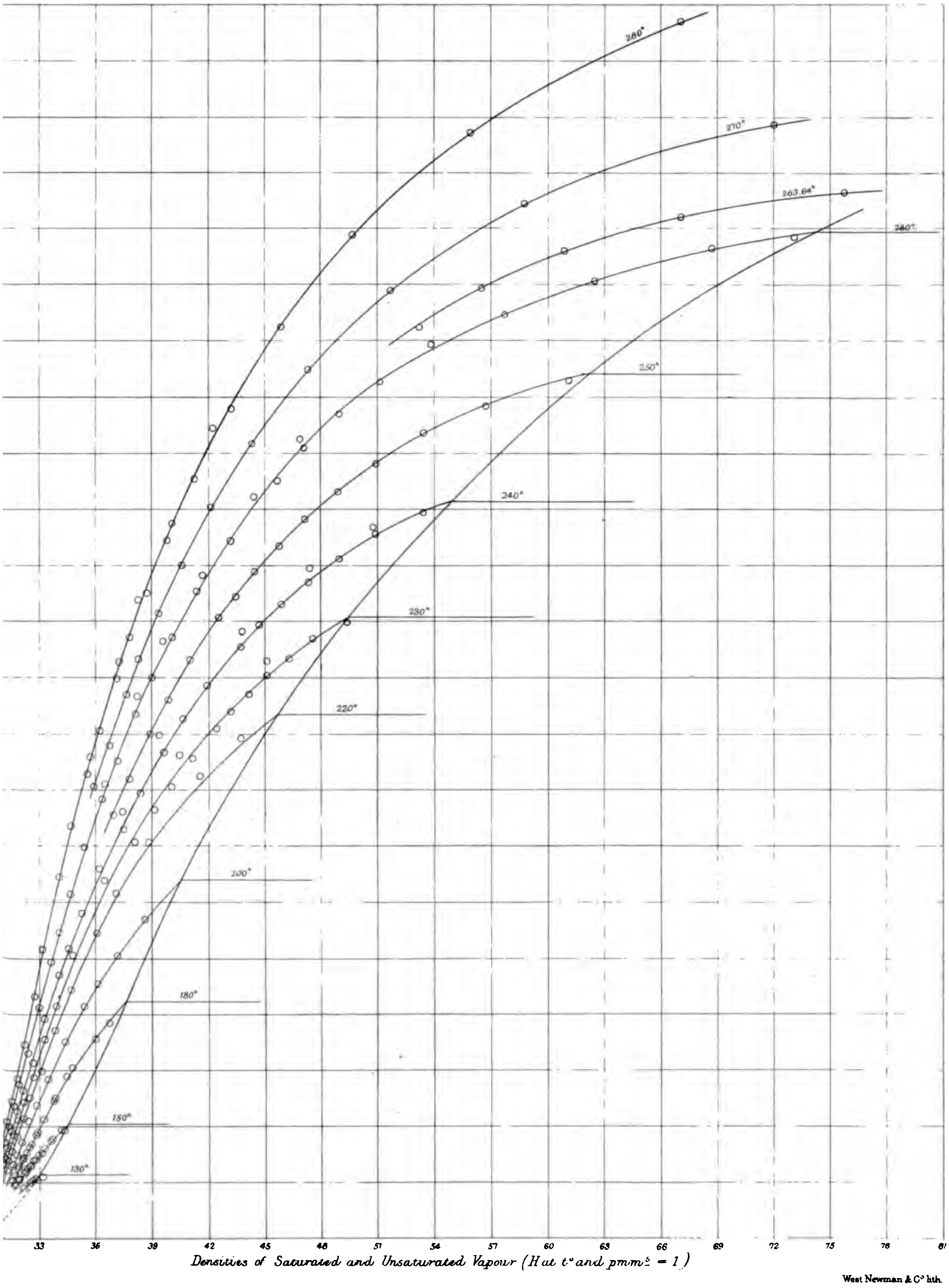
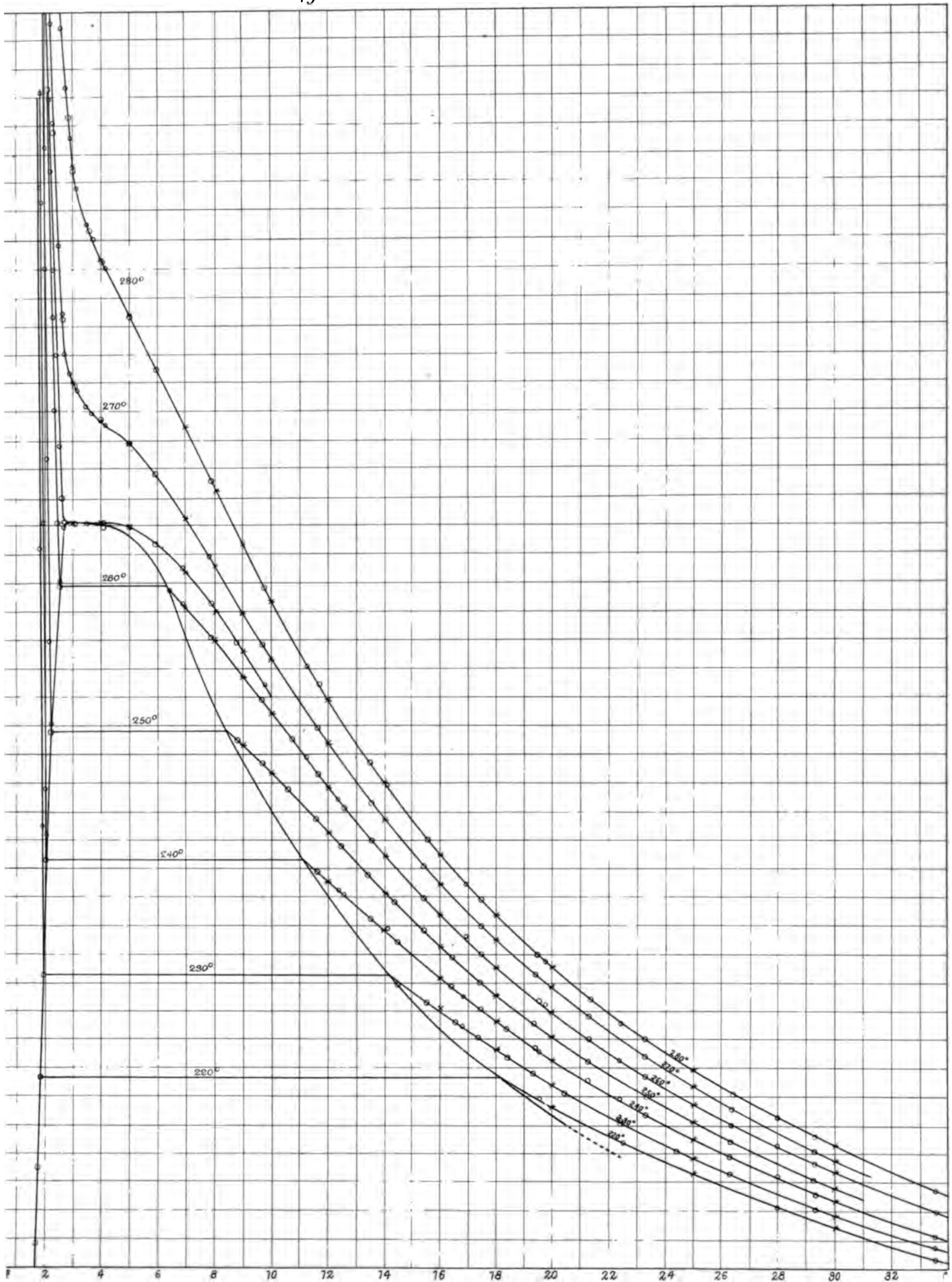


West Newman & C°.



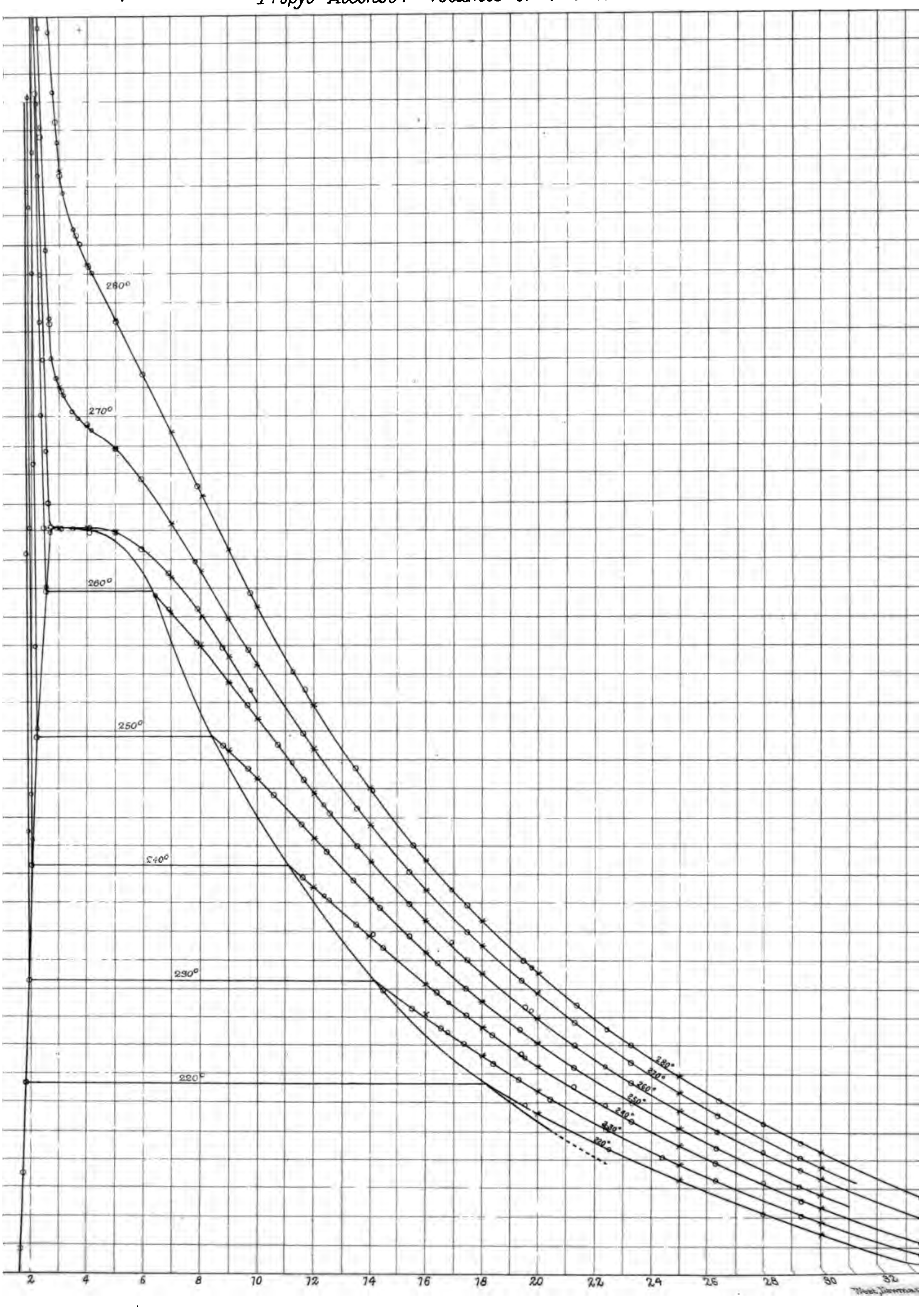
West Newman & Co. lith.

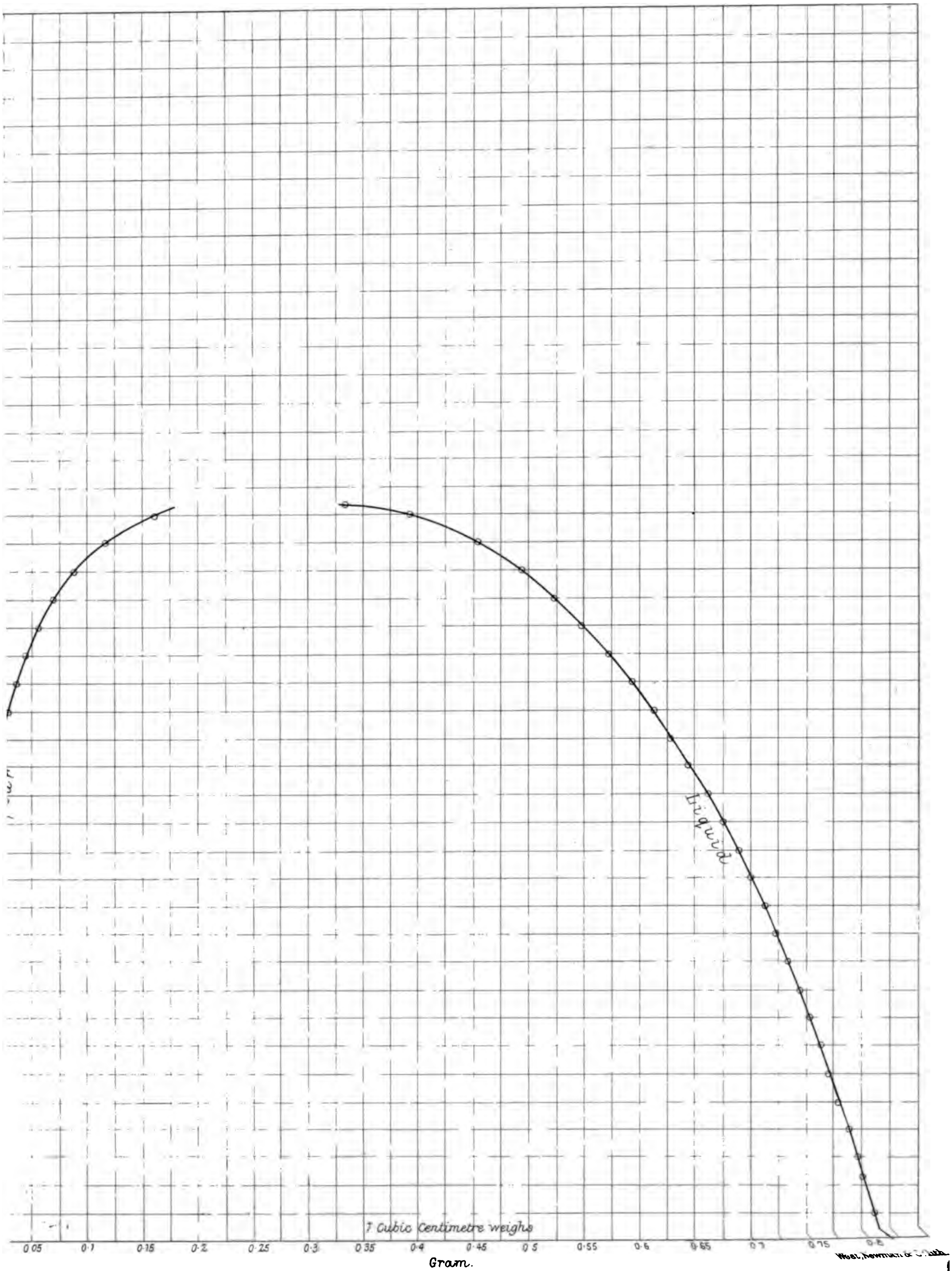
Propyl Alcohol. Volumes of 1 Gram.



West, Newman & Co.

Propyl Alcohol. Volumes of 1 Gram.





West, Newman & C° lith.

Fig. 1.

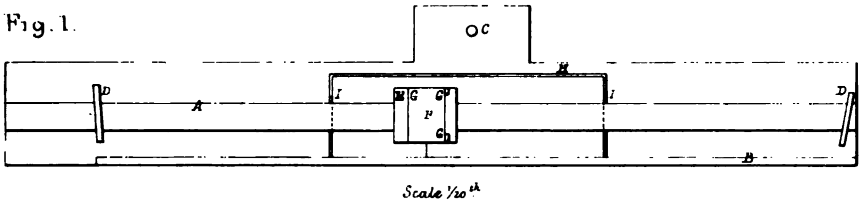


Fig. 3

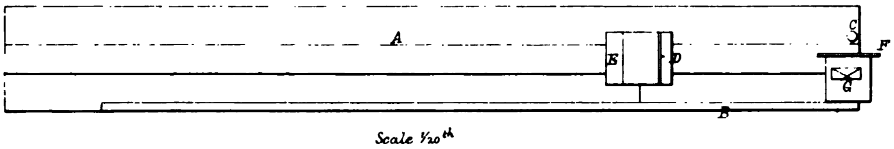


Fig. 1 *a*

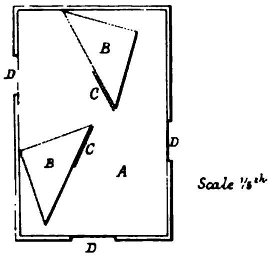


Fig. 2

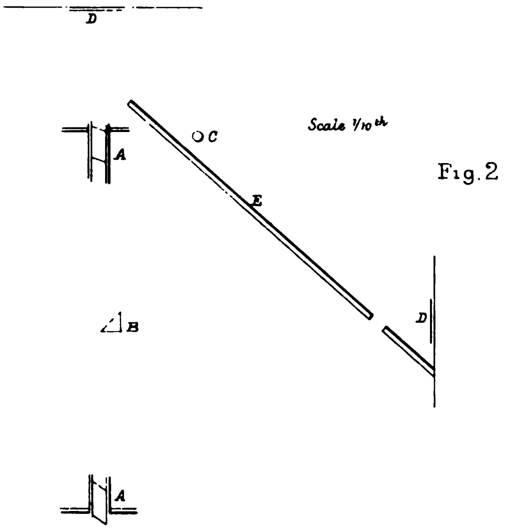


Fig. 3 *a*

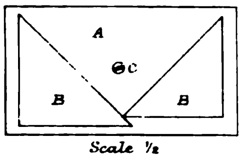
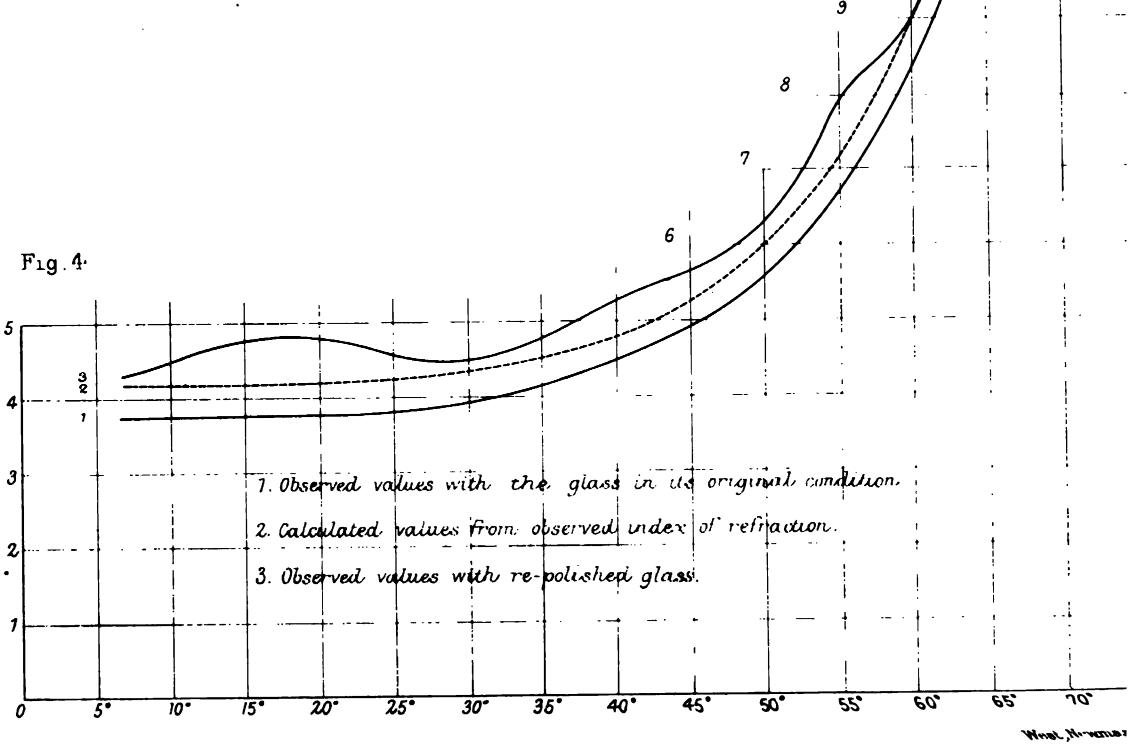
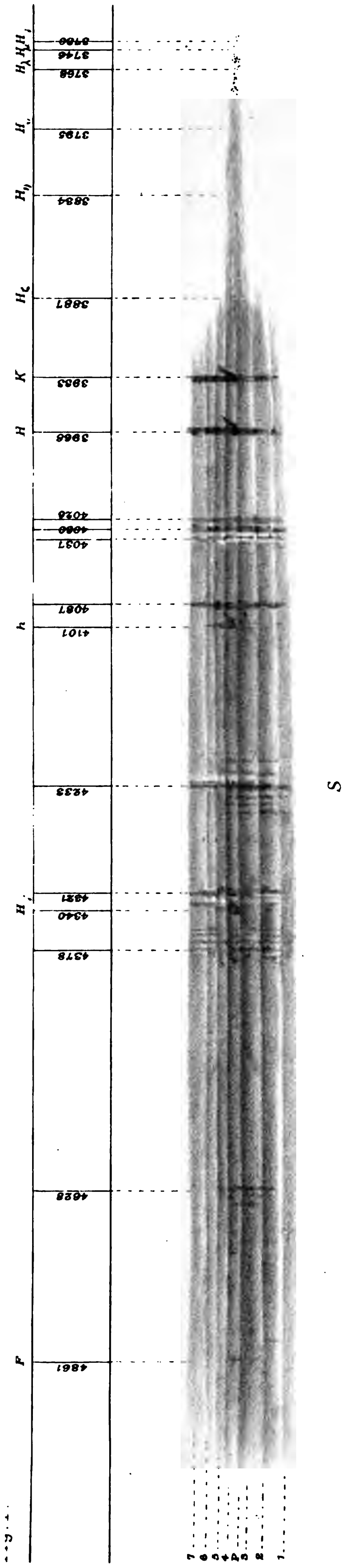


Fig. 4.

Per centage amount of light reflected by crown glass at various angles of incidence.



West, Newman



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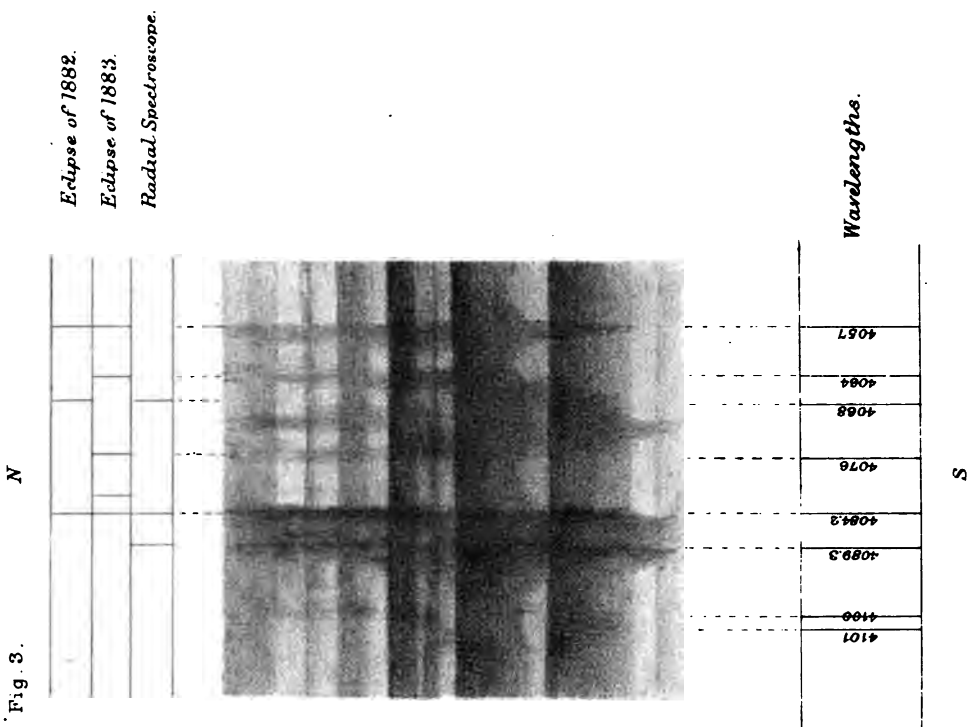


Fig. 3.

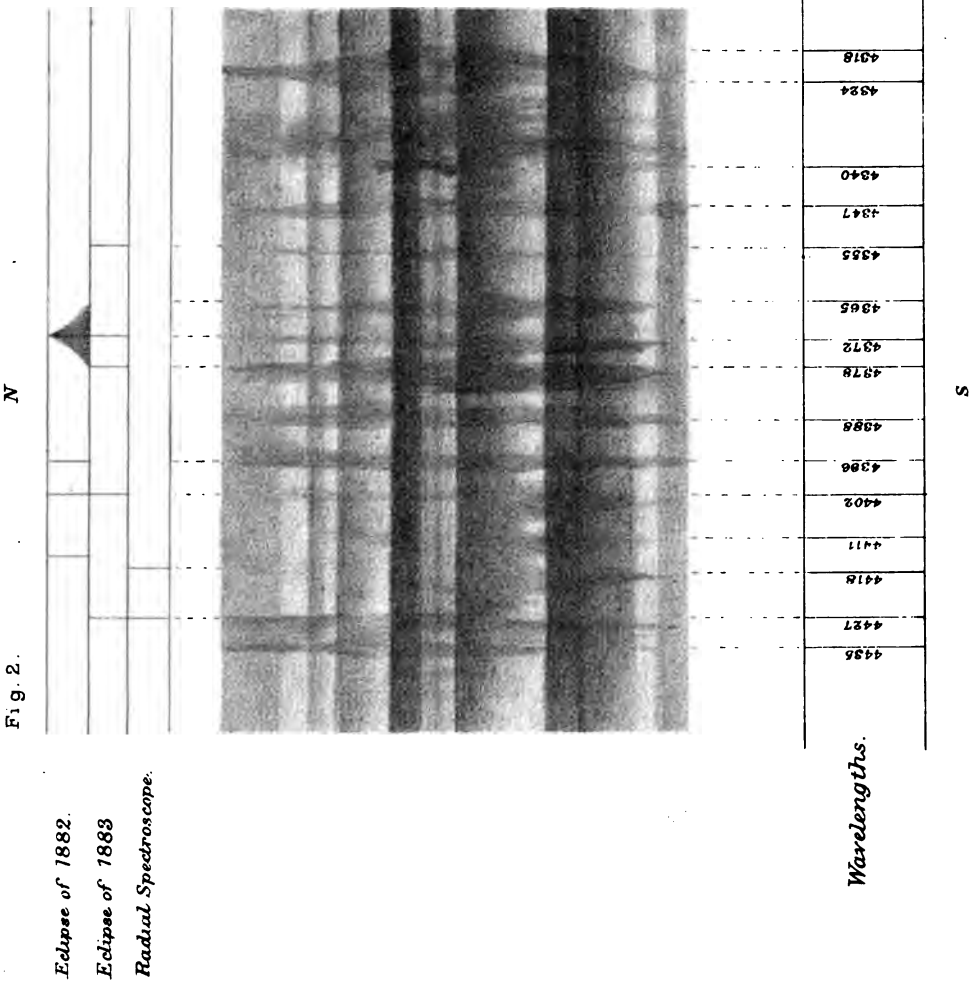
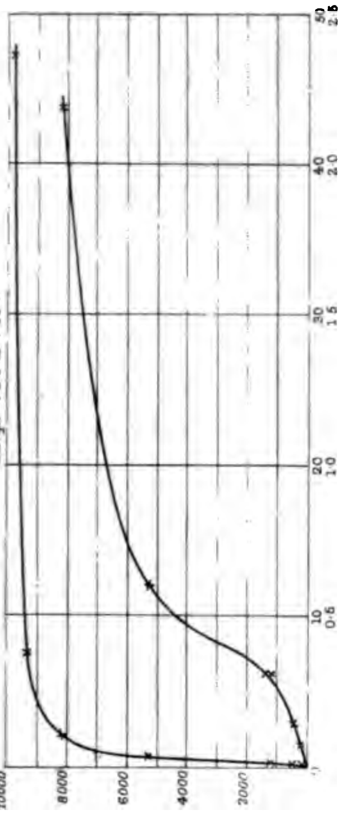


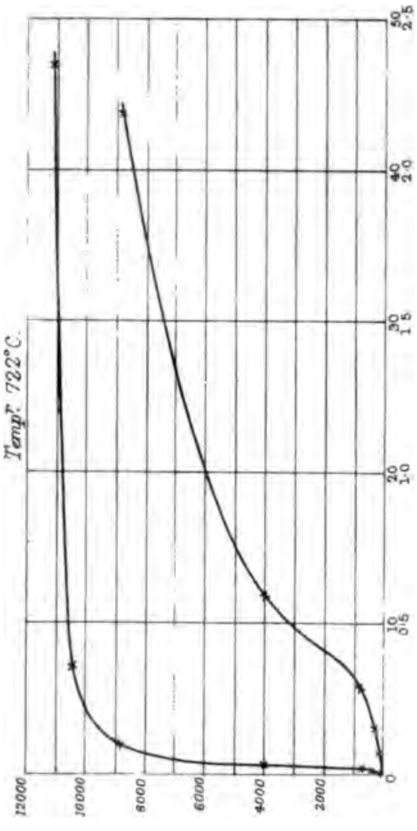
Fig. 2.

West, Newman lith.

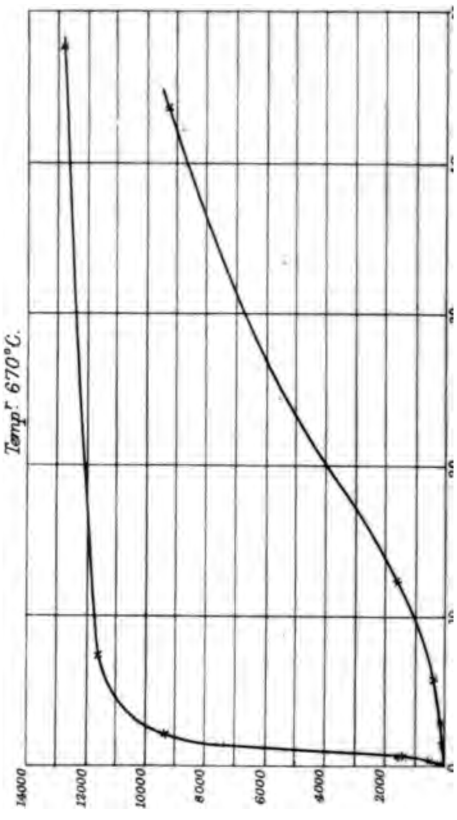
Temp^r 744 to 741°C.



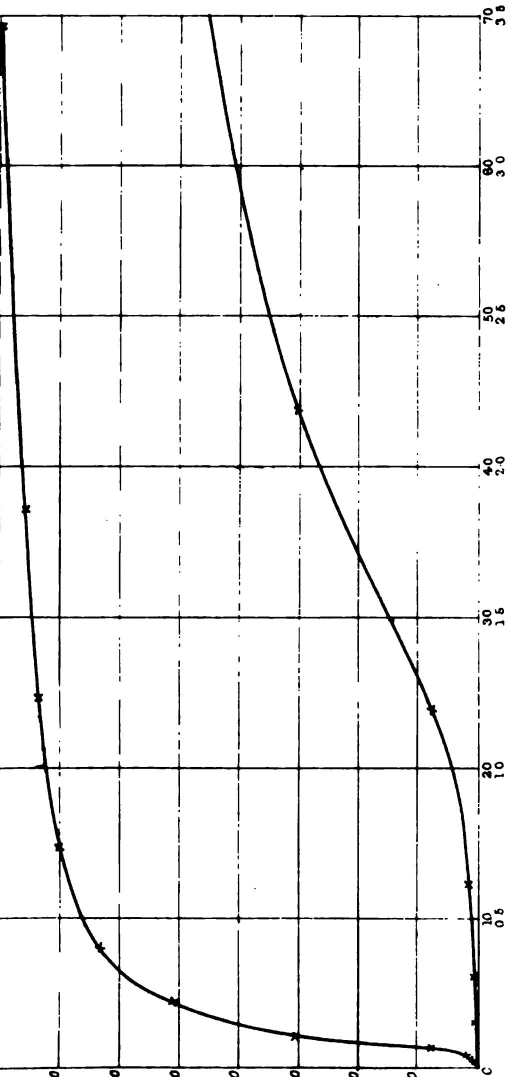
Curve VI.
Temp^r 722°C.



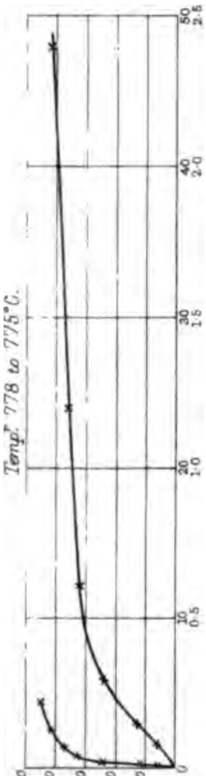
Curve VII.
Temp^r 670°C.



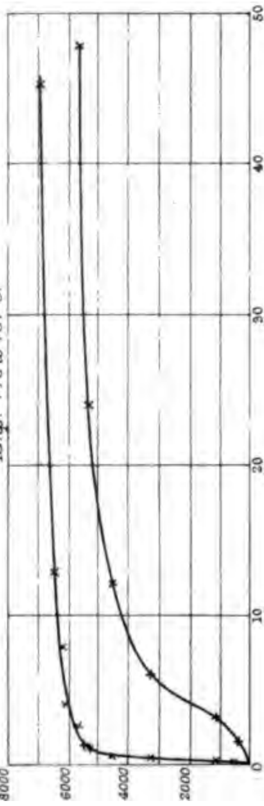
Temp^r 8·3°C.



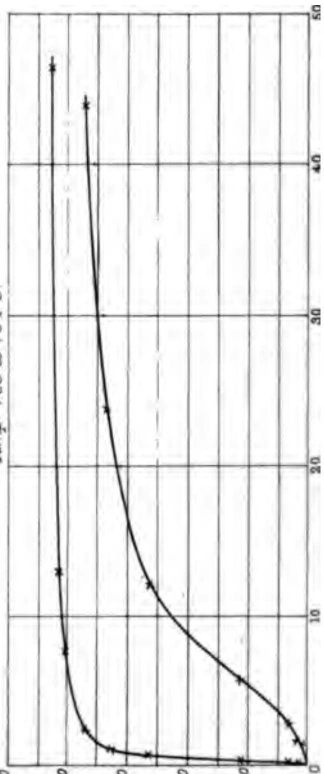
Curve II.
Temp^r 778 to 775°C.



Curve III.
Temp^r 775 to 767°C.



Curve IV.
Temp^r 763 to 754°C.

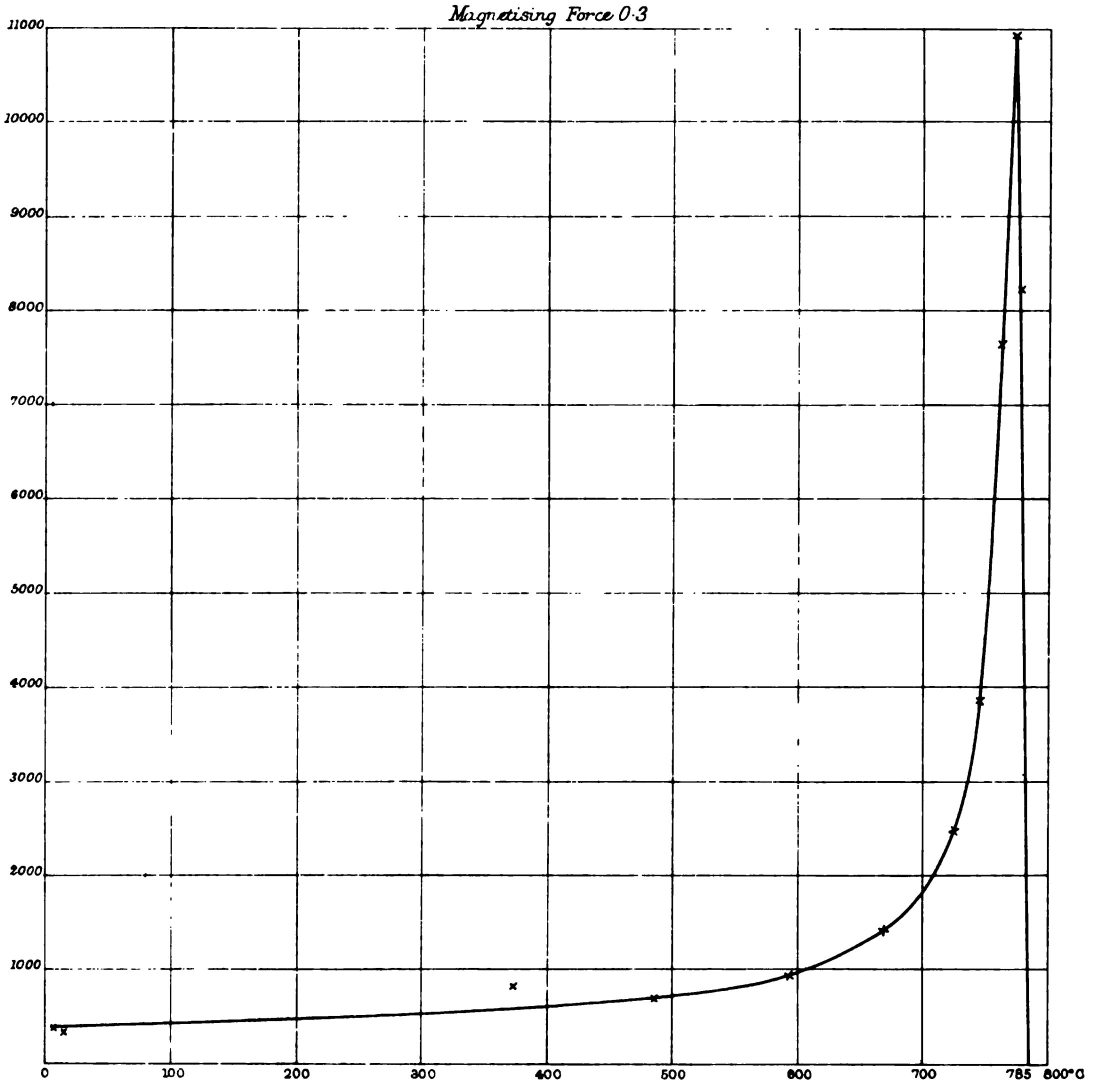


West, Newman & Co lith.

Wrought Iron.

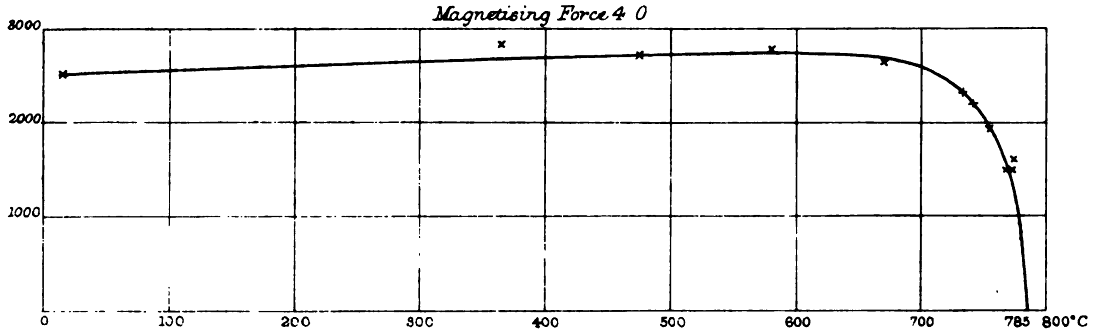
Curve XII.

Magnetising Force 0·3



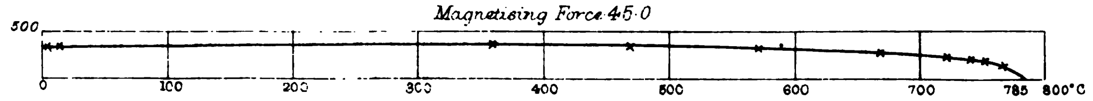
Curve XIII.

Magnetising Force 4 0



Curve XIV.

Magnetising Force 45·0

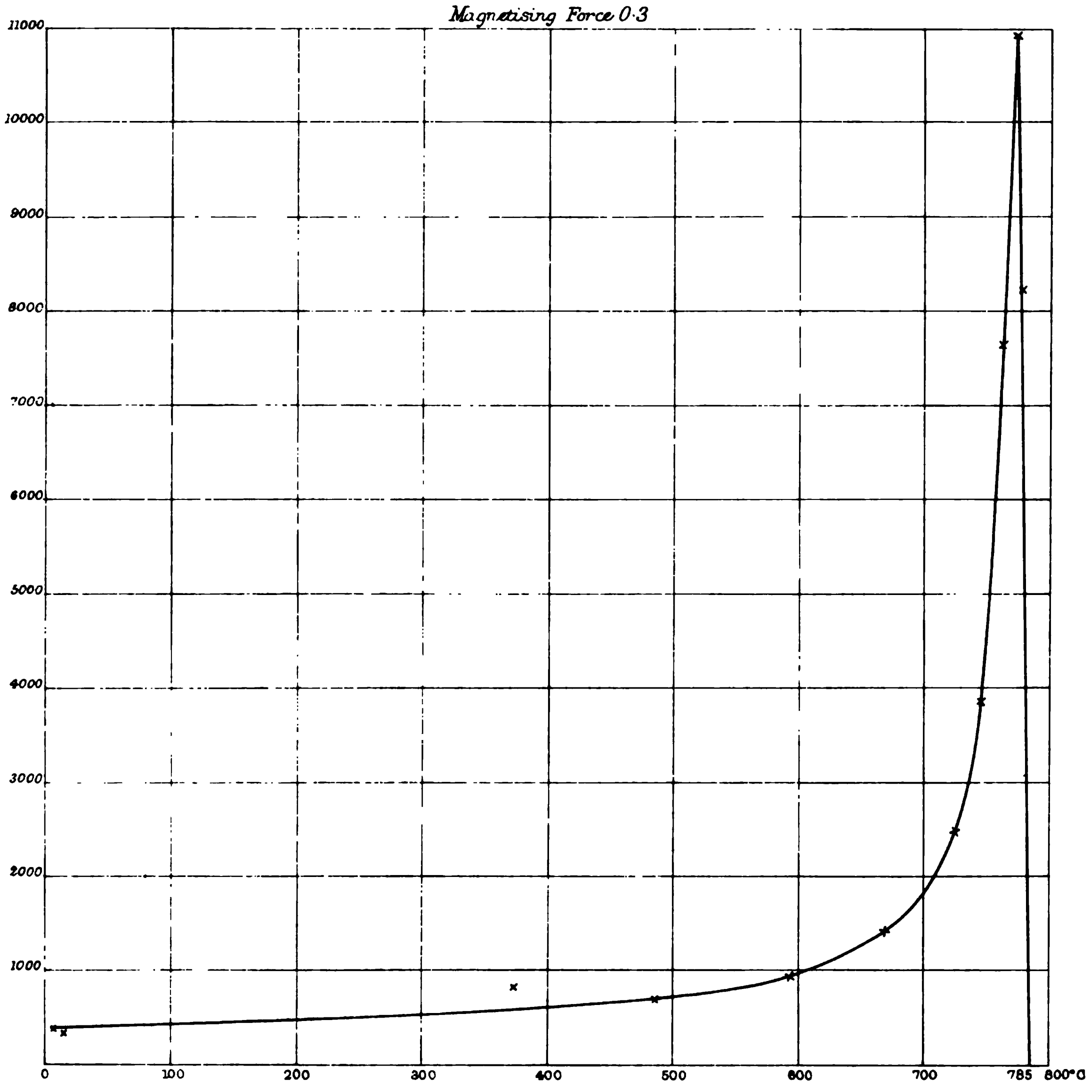


West, Newman & Co

Wrought Iron.

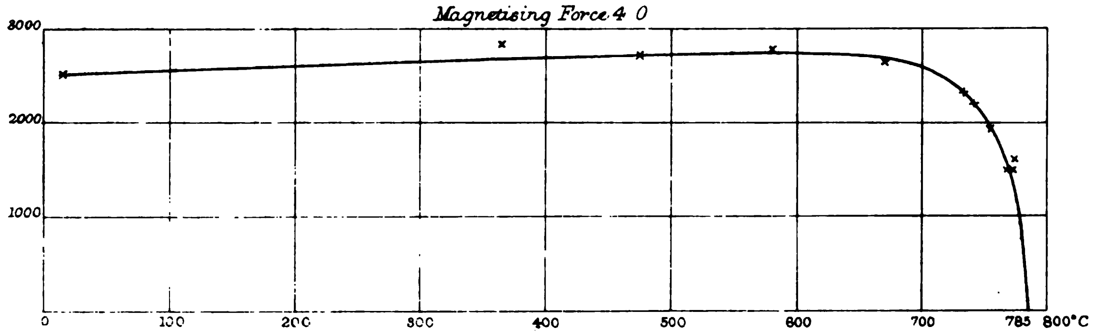
Curve XII.

Magnetising Force 0·3



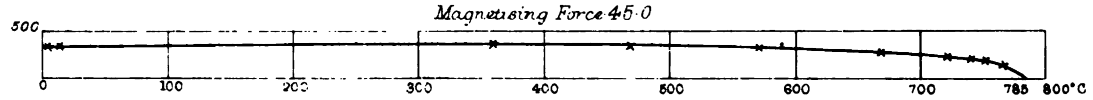
Curve XIII.

Magnetising Force 4 0



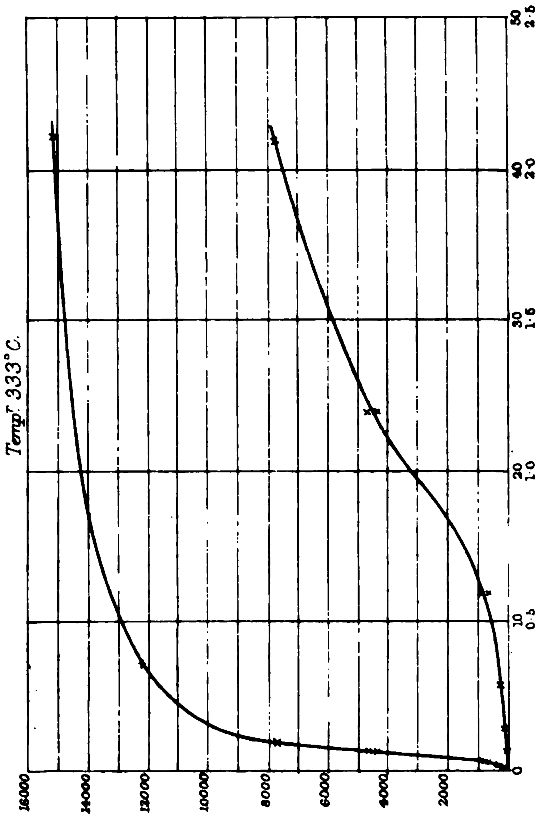
Curve XIV.

Magnetising Force 45·0

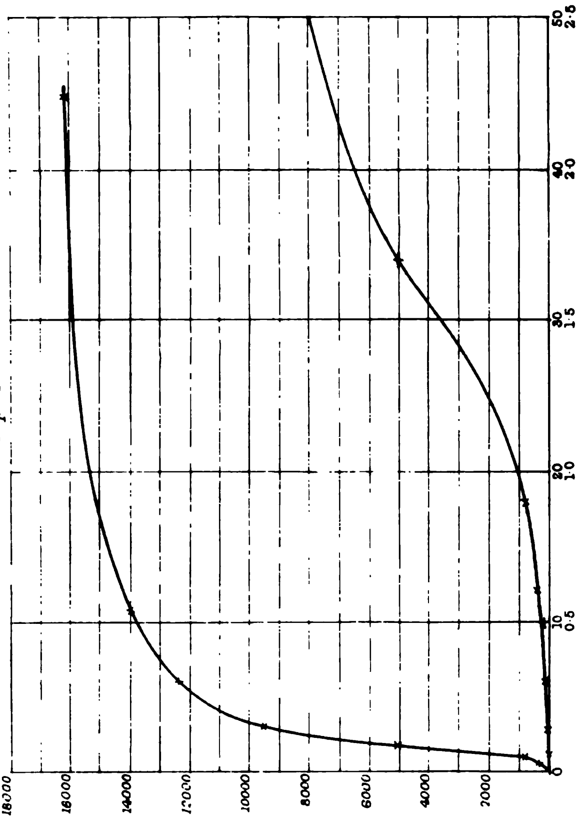


West, New-

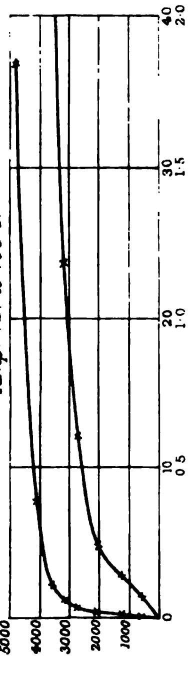
Whitworth's Mild Steel.



Curve XVIII.
Tempr 333°C.



Curve XIX.
Tempr 12°C.



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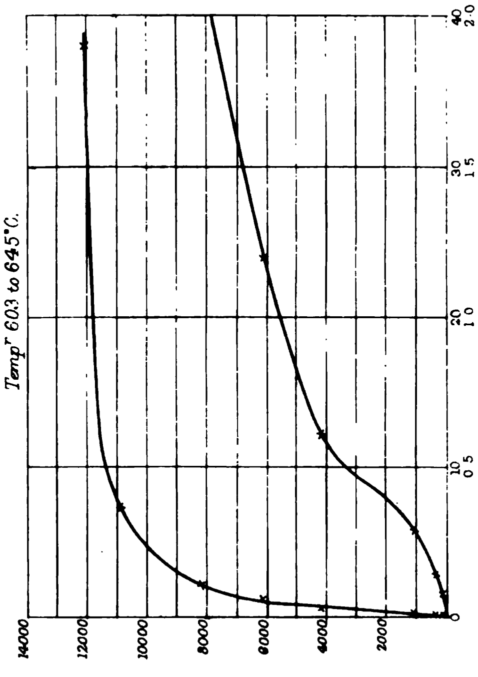
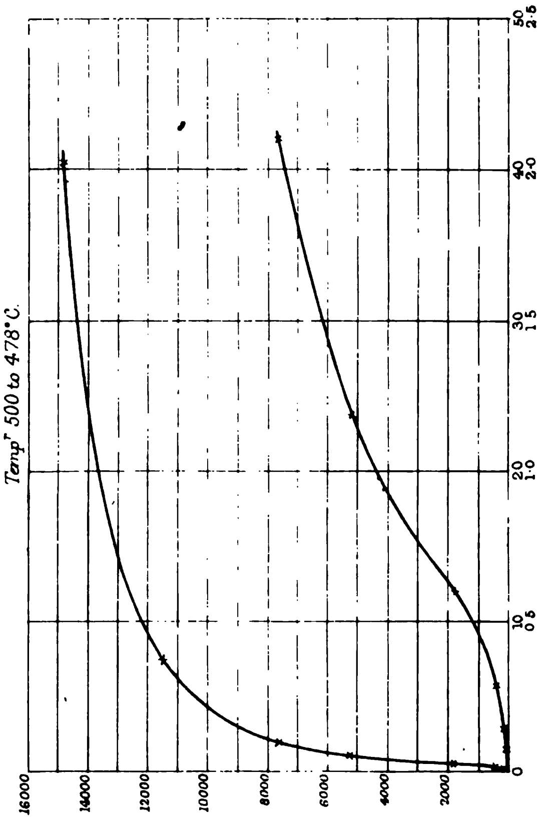


Curve XVI.
Tempr 603 to 645°C.



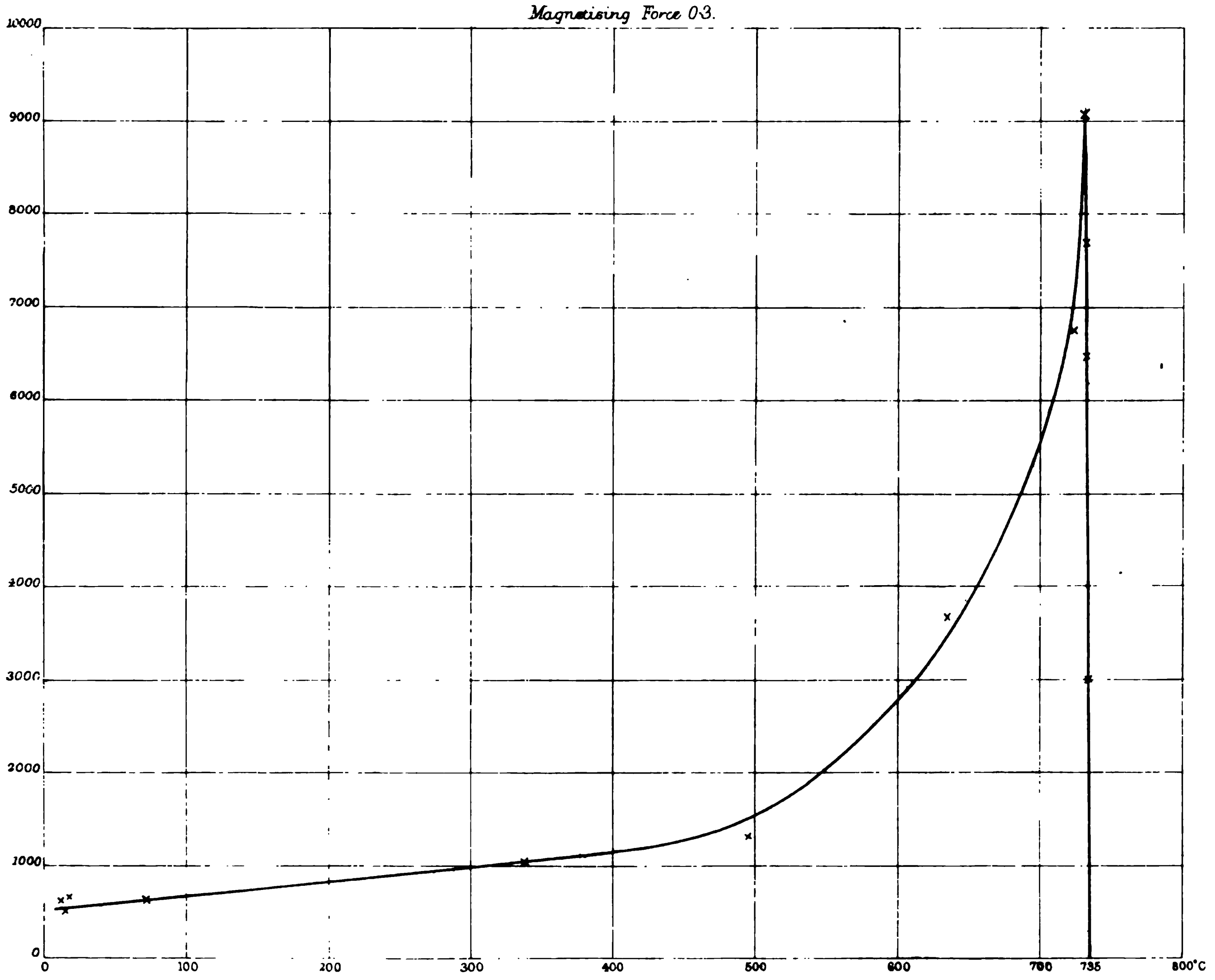
Curve XVII.
Tempr 500 to 478°C.

West, Newman & Cº lith.

Whitworth's Mild Steel.

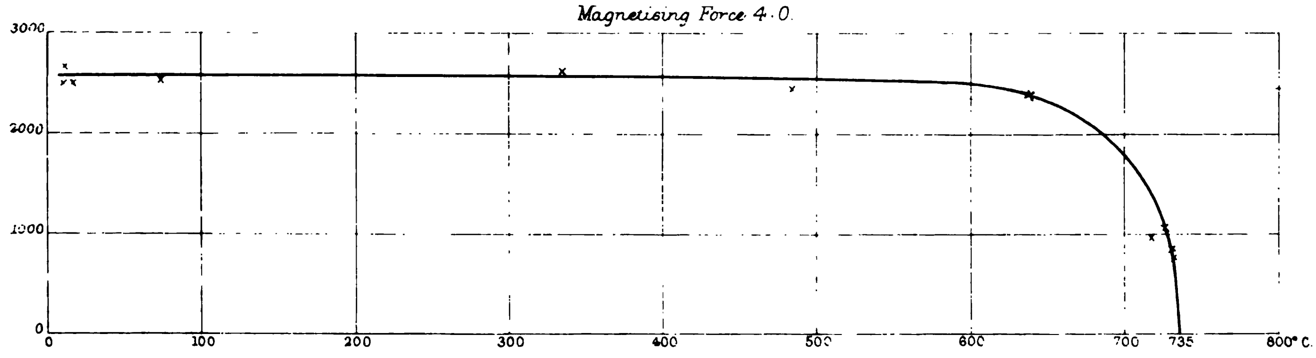
Curve XX.

Magnetising Force 0·3.



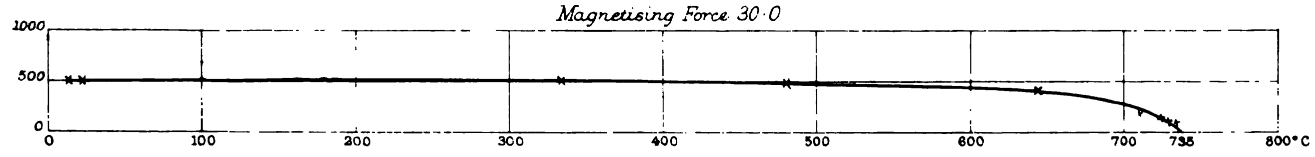
Curve XXI.

Magnetising Force 4·0.



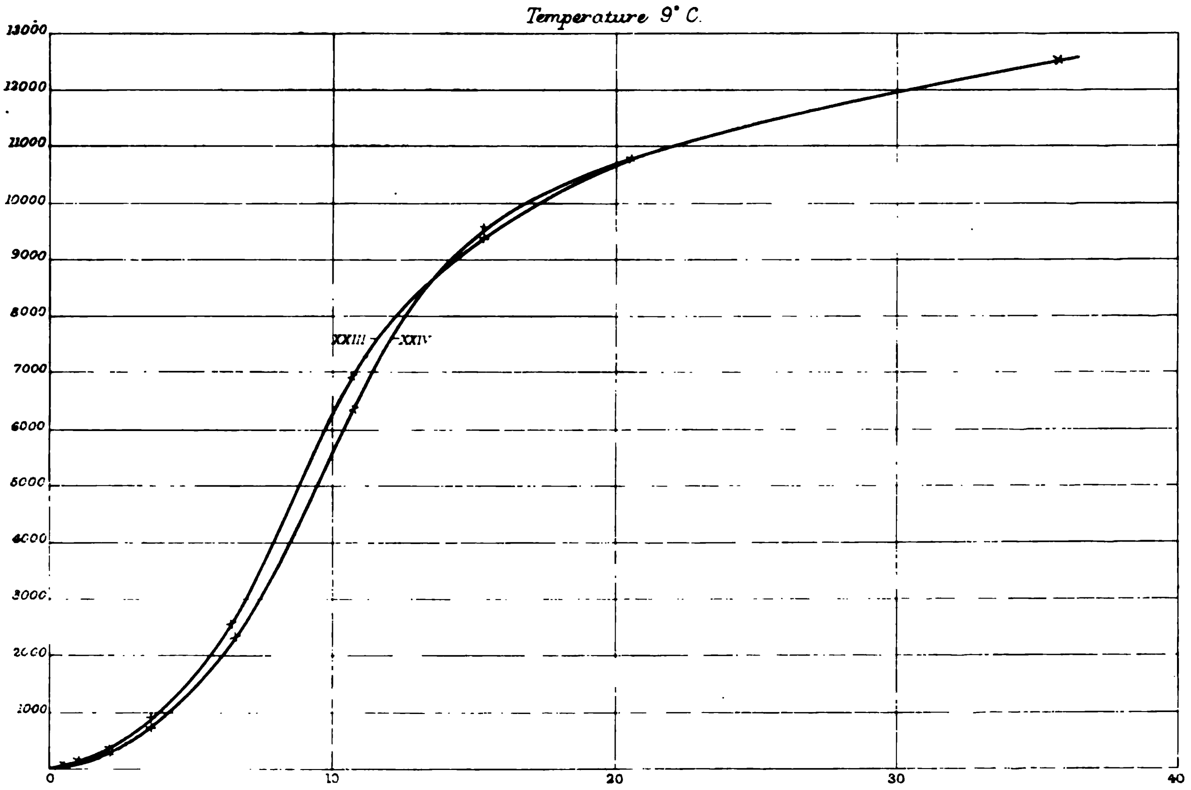
Curve XXII.

Magnetising Force 30·0

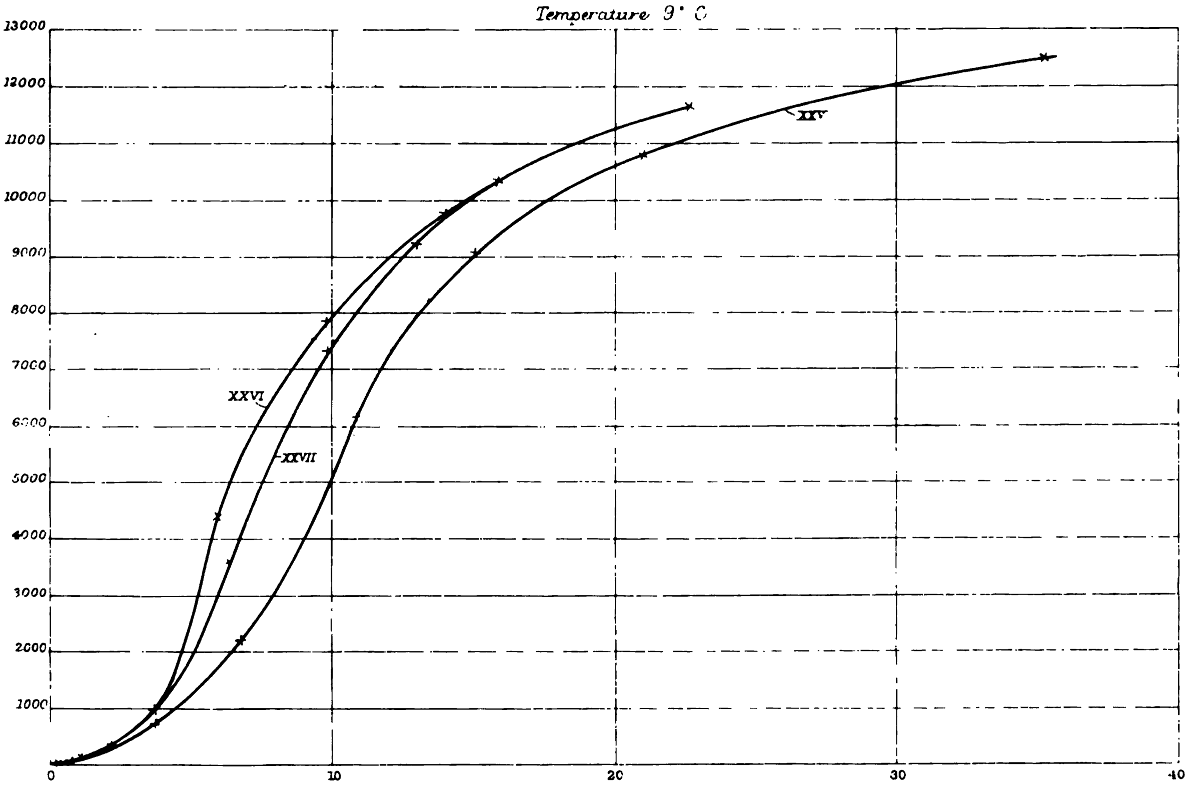


West, Newman & C° lith.

Whitworth's Hard Steel.
Curves XXIII & XXIV.
Temperature 9° C.



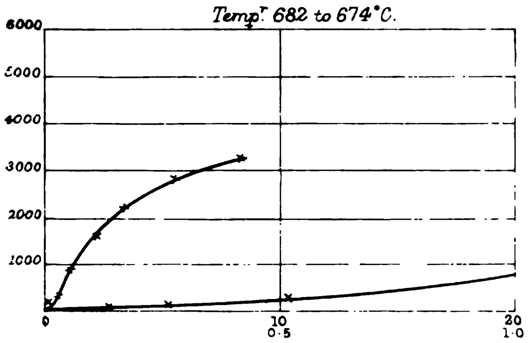
Curves XXV, XXVI, XXVII.
Temperature 9° C.



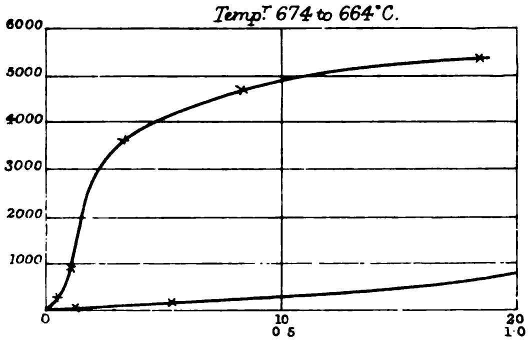
West, Newman & Co. lith.

Whitworth's Hard Steel.

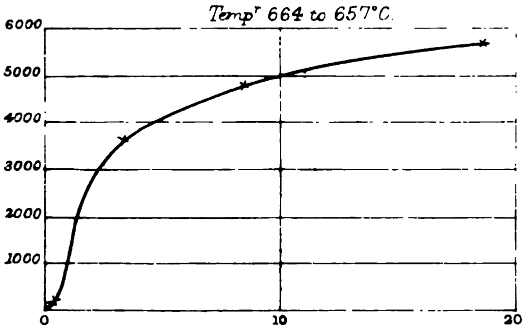
Curve XXVIII.
Temp^{r} 682 to 674°C.



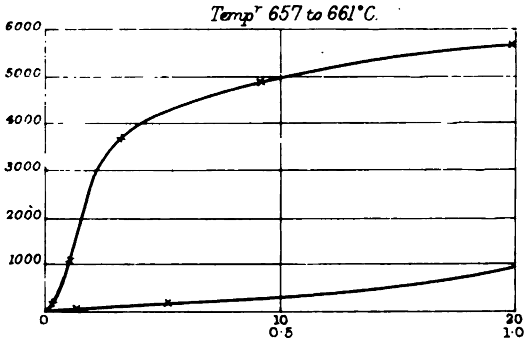
Curve XXIX.
Temp^{r} 674 to 664°C.



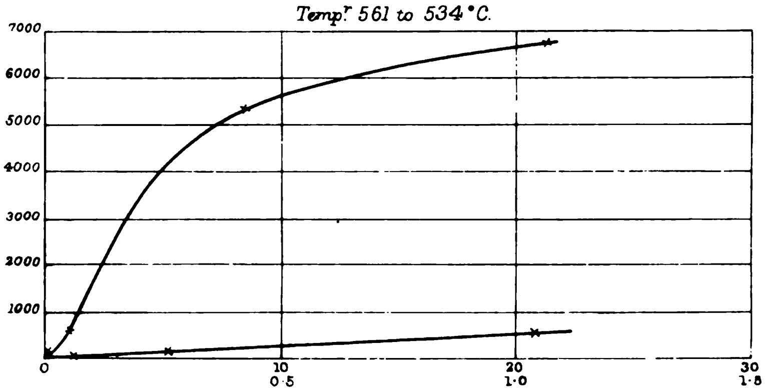
Curve XXX.
Temp^{r} 664 to 657°C.



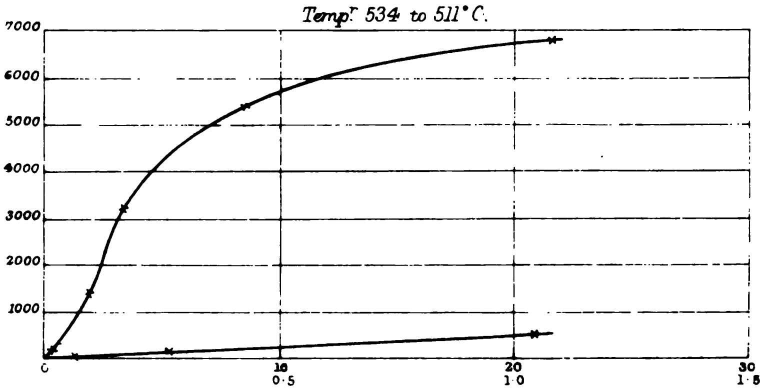
Curve XXXI.
Temp^{r} 657 to 661°C.



Curve XXXII.
Temp^{r} 561 to 534°C.



Curve XXXIII.
Temp^{r} 534 to 511°C.

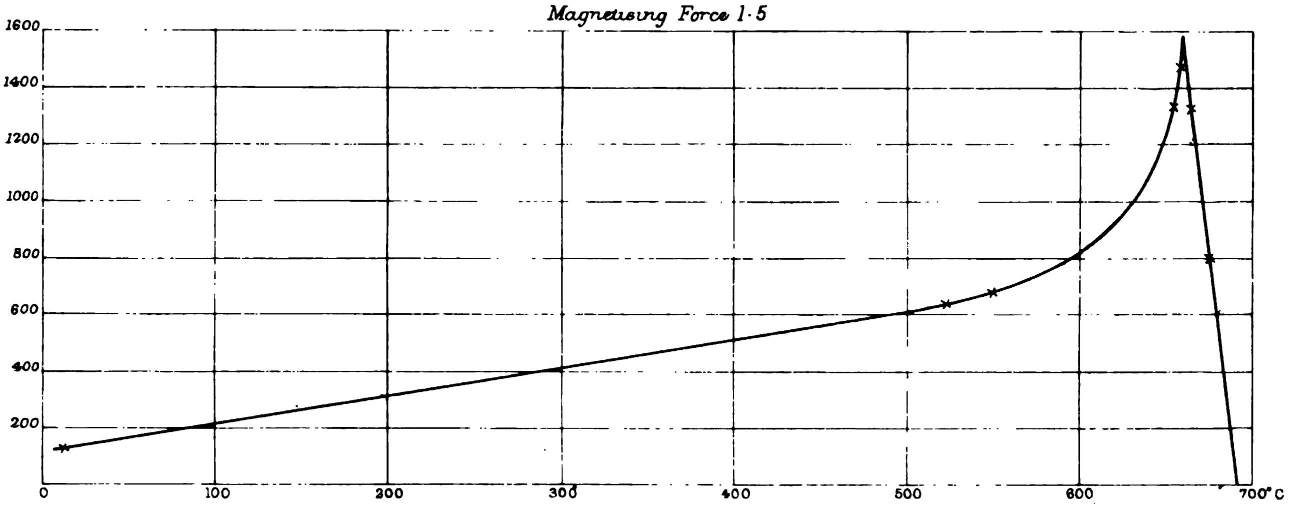


[illegible] Newman

Whitworth's Hard Steel.

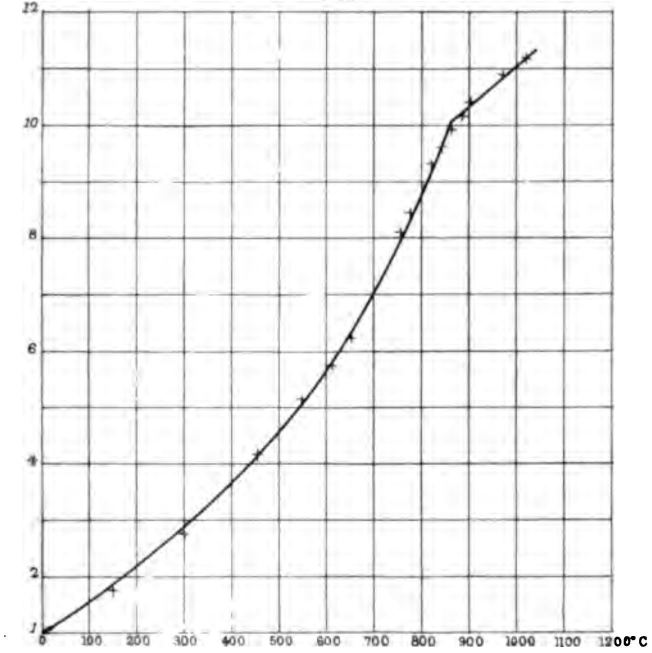
Curve XXXIV.

Magnetising Force 1·5



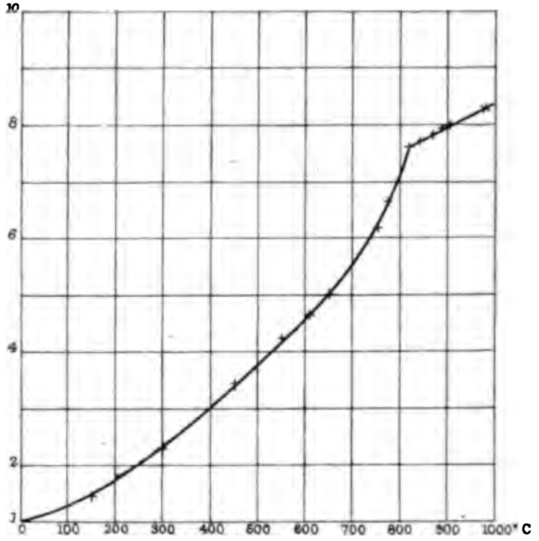
Soft iron wire.

Curve XXXV.



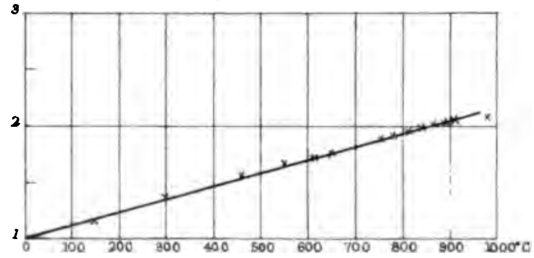
Pianoforte wire.

Curve XXXVI.

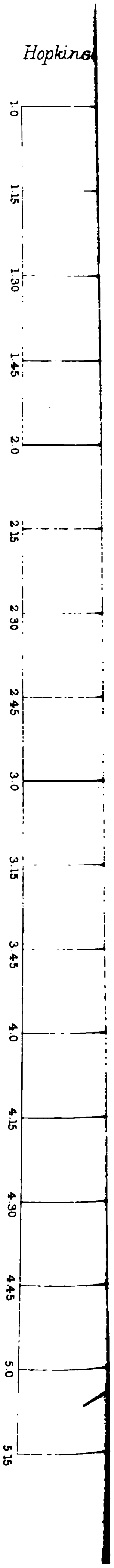


Manganese Steel Wire.

Curve XXXVII.



West, Newman & Cº l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